Compensation Problem in Finite Dimension
Linear Dynamical Systems

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Abstract
In this work, we consider a class of finite observation disturbed dynamical systems. We study with respect to the observation, the possibility of finite time or asymptotic compensation of known or unknown disturbances. Under convenient hypothesis, we show how to find the optimal control ensuring the compensation. This concept is also examined as minimization problem with a decent cost function. A comparison between the two approaches is given and the relation with the other notions of controllability, stability and stabilizability is studied. Various situations are also examined.

Keywords: Dynamical systems, remediability, observation, control, disturbance

1 Introduction
The importance of disturbed systems and pollution problems is increasing, particularly in last decades. Several works on this area are developed. Con-
cerning inverse and detection problems, one can see for example [1], [2], [6], [7], [10], [13], [19], [21], [23], [25] ... 

The detection, or even if the knowledge, of a disturbance is not generally sufficient. In order to compensate, or at least to limit, the effect of the disturbance (pollution), one must act decently on the system. Some works on disturbances rejection or decoupling problems are developed ([14], [15], [20], [22], [24] ... ).

With the same worry, but with a different approach, the notions of remediability and efficient actuators are introduced and studied first for a class of parabolic systems in the case of a finite time horizon, and then for other situations and systems (discrete systems, hyperbolic systems, regional and asymptotic cases, internal or boundary actions or disturbances), one can see [3], [4], [5], [8], [9] ... 

In these works, we study with respect to the output (observation), the existence of an input operator (actuators) ensuring the compensation of any disturbance. We also show that the remediability is weaker and more supple than the controllability of the system.

In the case of finite dimension linear systems, the space variable, and hence the notions of actuators and sensors does not exist ([11], [12], [16], [17], [18], ... ).

The purpose of this paper is to study this problem for such systems. We give a sufficient condition, and then a necessary and sufficient one for the remediability. We show that equally in the finite dimension case, this concept remain weaker than the controllability. Then, using the observation only, we give the optimal control ensuring the finite time compensation of a known or unknown disturbance. An extension to the asymptotic case is also presented and its relationship with the notions of stability and stabilizability is studied.

The finite time and asymptotic compensation is equally studied as a more general and decent minimization problem. We show that this problem admits a unique solution depending naturally on the observation, on the other parameters of the considered system and cost function.

A comparison between the two approaches is given and illustrative examples are presented.
2 Finite time compensation

2.1 Problem statement and characterizations
We consider a class of finite dimension linear and disturbed dynamical systems described by the following state equation:

\[
\begin{align*}
\dot{z}(t) &= Az(t) + f(t) + Bu(t) ; \ 0 < t < T \\
z(0) &= z_0
\end{align*}
\]  

(1)

where \( A \in M_n(R) \), \( B \in M_{n,p}(R) \), \( u \in L^2(0,T;R^p) \) and \( f \in L^2(0,T;R^n) \)

The system (1) is augmented by the output equation:

\[y(t) = Cz(t) ; \ 0 < t < T\]

(2)

with \( C \in M_{q,n}(R) \), we have

\[z(t) = e^{At}z_0 + H_tu + G_tf\]
then

\[y(t) = Ce^{At}z_0 + CH_tu + CG_tf\]

where \( H_t \) and \( G_t \) are the operators defined by

\[
H_t : L^2(0,t;R^p) \rightarrow R^n
\]
\[
u \rightarrow \int_0^t e^{A(t-s)}Bu(s)ds
\]

(3)

and

\[
G_t : L^2(0,t;R^n) \rightarrow R^n
\]
\[
f \rightarrow G_tf = \int_0^t e^{A(t-s)}f(s)ds
\]

(4)

Definition 2.1
(1) + (2) is remediable on \([0,T]\), if for any \( f \in L^2(0,T;R^n) \), there exists a control \( u \in L^2(0,T;R^p) \) such that:

\[CH_Tu + CG_Tf = 0\]
We have the following characterization result.

**Proposition 2.2** The following properties are equivalent:

i) $(1)+(2)$ is remediable on $[0,T]$

ii) $\text{Im}(CG_T) \subset \text{Im}(CH_T)$

iii) $\text{Ker}(H_T^*C^*) \subset \text{Ker}(G_T^*C^*)$

iv) $\exists \gamma > 0$ such that for any $\theta \in \mathbb{R}^q$, we have

\[
\| e^{A^*(T-::)C^*\theta} \|_{L^2(0,T;\mathbb{R}^n)} \leq \gamma \| B^*e^{A^*(T-::)C^*\theta} \|_{L^2(0,T;\mathbb{R}^p)}
\]

**Proof:** Derive from the definition, the fact that

$\text{Ker}(H_T^*C^*) = \text{Ker}(B^*e^{A^*(T-::)C^*})$

$\text{Ker}(G_T^*C^*) = \text{Ker}(e^{A^*(T-::)C^*})$

and also the following well known result.

**Lemma 2.3**

Let $X$, $Y$ and $Z$ be Banach reflexive spaces, $P \in \mathcal{L}(X,Z)$ and $Q \in \mathcal{L}(Y,Z)$. We have

$$\text{Im}(P) \subset \text{Im}(Q)$$

if and only if

$$\exists \gamma > 0 \text{ such that for any } z^* \in Z', \text{ we have } \|P^*z^*\|_{X'} \leq \gamma \|Q^*z^*\|_{Y'}$$

In order to study the relationship between the notions of controllability and remediability, let us first recall the controllability and its rank characterization.

**Definition 2.4**

The system

\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bu(t) \quad 0 < t < T \\
\quad z(0) &= z_0
\end{align*}
\]
Compensation problem in finite dimension linear dynamical systems

is controllable on \([0, T]\), if for any initial state \(z_0\) and desired state \(z_d\) in \(R^n\), there exists a control \(u \in L^2(0, T; R^n)\) such that:

\[
z(T) = z_d
\]

This is equivalent to

\[
\text{Im}(H_T) = R^n
\]

or (using Cayley-Hamilton Theorem)

\[
\text{rank} \left( \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \right) = n
\]

We have the following result.

**Proposition 2.5**

i) If the system (6) is controllable on \([0, T]\), then (1)+ (2) is remediable on \([0, T]\).

ii) The converse is not true.

**Proof:**

i) (6) is controllable on \([0, T]\) \iff \(\text{Im} H_T = R^n\), then

\[
\text{Im} \left( CH_T \right) = \text{Im} C
\]

and

\[
\text{Im} \left( CG_T \right) \subset \text{Im} \left( CH_T \right)
\]

consequently (1)+ (2) is remediable on \([0, T]\).

ii) Counter example: We consider the case where \(n = 2, p = q = 1\) and

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}
\]

we have

\[
e^{A^*(T-s)}C^*\theta = e^{(T-s)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \theta = \begin{pmatrix} e^{(T-s)}\theta \\ 0 \end{pmatrix}
\]

and
\[ B^* e^{A^*(T-s)} C^* \theta = \begin{pmatrix} 1 & 0 \\ e^{(T-s)} & 0 \end{pmatrix} \theta = e^{(T-s)} \theta \]
	hen
\[
\| e^{A^*(T-s)} C^* \theta \|_{L^2(0,T;R^2)} = \| B^* e^{A^*(T-s)} C^* \theta \|_{L^2(0,T;R^2)}
\]

The inequality (5) is then true for \( \gamma = 1 \). Consequently (1)+ (2) is remediable on \([0,T]\), but

\[
\text{rank} \left( \begin{array}{cc} B & AB \\ \end{array} \right) = \text{rank} \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) = 1 < 2
\]

Then (6) is not controllable on \([0,T]\).

We give hereafter a sufficient condition ensuring the remediability of (1)+ (2) on \([0,T]\).

**Proposition 2.6 If**

\[
\text{rank} \left( \begin{array}{cccc} C & B & CAB & \ldots & CA^{n-1}B \end{array} \right) = q
\]

then (1)+ (2) is remediable on \([0,T]\).

**Proof:** Using Caylay-Hamilton theorem, we have

\[
\text{rank} \left( \begin{array}{cccc} C & B & CAB & \ldots & CA^{n-1}B \end{array} \right) = q
\]

\[
\iff \forall y \in Y^q, \left( \begin{array}{c} B^* C^* \\ B^* A^* C^* \\ \vdots \\ B^*(A^*)^{n-1} C^* \end{array} \right)_{(np,q)} y = 0 \implies y = 0
\]

\[
\iff \text{Ker}(H_T)^* C^* = \{0\}
\]

Hence, if \( \text{Ker} [(H_T)^* C^*] = \{0\} \), then \( \text{Ker} [(H_T)^* C^*] \subset \text{Ker} [(G_T)^* C^*] \) and then, (1)+ (2) is remediable on \([0,T]\). \( \square \)

**Remark 2.7**

i) One can has

\[
\text{rank} \left( \begin{array}{cccc} C & B & CAB & \ldots & CA^{n-1}B \end{array} \right) = q
\]

even if the system (6) is not controllable on \([0,T]\).
ii) (1)+ (2) can be remediable on [0, T] without having
\[ \text{rank}( CB \quad CAB \quad \ldots \quad CA^{n-1}B ) = q \]
This is illustrated in the following example.

Example 2.8

i) We consider the case where \( n = 2, \ p = q = 1 \) and
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad C = ( 1 \ 0 )
\]
The controllability matrix is given
\[
\begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]
and its rank is \( 1 < 2 \). Consequently, the corresponding system is not controllable on \([0, T]\). On the other hand
\[
( CB \quad CAB ) = ( 1 \ 1 )
\]
its rank is \( 1 = q \), then (1)+ (2) is remediable on \([0, T]\).

ii) Now, for \( n = 2, \ p = 1, \ q = 2 \) and
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}
\]
we have
\[
e^{A^*(T-s)}C^*\theta = e^{(T-s)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}
= e^{(T-s)} \begin{pmatrix} \theta_1 + \theta_2 \\ \theta_1 + \theta_2 \end{pmatrix}
\]
and \( B^*e^{A^*(T-s)}C^*\theta = e^{(T-s)}2(\theta_1 + \theta_2) \), then
\[
\| e^{A^*(T-s)}C^*\theta \|^2_{L^2(0,T;\mathbb{R}^2)} = \int_0^T e^{2(T-s)} \| \begin{pmatrix} \theta_1 + \theta_2 \\ \theta_1 + \theta_2 \end{pmatrix} \|^2 \, ds
= 2 \int_0^T e^{2(T-s)}(\theta_1 + \theta_2)^2 \, ds
\]
and
\[ \| B^* e^{A^* (T-s)} C^* \theta \|_{L^2(0,T;R)}^2 = 4 \int_0^T e^{2(T-s)} (\theta_1 + \theta_2)^2 \, ds \]

hence

\[ \| e^{A^* (T-s)} C^* \& \|_{L^2(0,T;R^2)} \leq \sqrt{2} \| B^* e^{A^* (T-s)} C^* \theta \|_{L^2(0,T;R)} \]

and consequently, (1) + (2) is remediable on \([0, T]\), even if

\[ \text{rank} \left( CB \ CAB \ldots CA^{n-1}B \right) = \text{rank} \left( \begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right) = 1 \neq 2 \]

In the following result, we give a necessary and sufficient rank condition for the remediability.

**Proposition 2.9**

(1) + (2) is remediable on \([0, T]\) if and only if

\[ \text{rank} \left( CB \ CAB \ldots CA^{n-1}B \right) = \text{rank} \left( C \right) \]

**Proof:** Using the characterization proposition, we have (1) + (2) is remediable on \([0, T]\) if and only if

\[ \text{Ker} \left( H_T^* C^* \right) = \text{Ker} \left( G_T^* C^* \right) \]

then from Caylay-Hamilton theorem, we deduce that

\[ y \in \text{Ker} \left( H_T^* C^* \right) \iff \begin{pmatrix} B^* C^* \\ B^* A^* C^* \\ \vdots \\ B^*(A^*)^{n-1} C^* \end{pmatrix}_{(m,p,q)} y = 0 \]

Since

\[ \text{Ker} \left[ (G_T^*)^* \right] = \text{Ker} \left( C^* \right) \]

then (1) + (2) is remediable on \([0, T]\) if and only if

\[ \text{Ker} \begin{pmatrix} B^* C^* \\ B^* A^* C^* \\ \vdots \\ B^*(A^*)^{n-1} C^* \end{pmatrix} = \text{Ker} C^* \]
or equivalently

$$\text{Im} \left( CB \begin{bmatrix} CAB & \ldots & CA^{n-1}B \end{bmatrix} \right) = \text{Im}(C)$$

\[ \Box \]

### 2.2 Minimum energy problem: The optimal control

In this part, we consider the following minimum energy problem:

For $z_0$ in $\mathbb{R}^n$ and $f \in L^2(0,T;\mathbb{R}^n)$, we examine the existence and the unicity of the optimal control $u \in L^2(0,T;\mathbb{R}^p)$ such that

$$CH_Tu + CG_Tf = 0$$

For this, we use an extension of the Hilbert Uniqueness Method. Indeed, for $\theta \in \mathbb{R}^q$, let us note:

$$\| \theta \|_* = \left( \int_0^T \left\| (H_T)C^*\theta \right\|^2_{\mathbb{R}^p} ds \right)^{\frac{1}{2}} = \left( \int_0^T \left\| B^*e^{A^*(T-s)}C^*\theta \right\|^2_{\mathbb{R}^p} ds \right)^{\frac{1}{2}}$$

$\| \theta \|_*$ is a semi-norm on $\mathbb{R}^q$.

We assume that $\| \cdot \|_*$ is a norm on $\mathbb{R}^q$. If $\text{Ker} [(H_T)^*C^*] = \{0\}$, this is equivalent to the asymptotic remediability of the system (1)+ (2) on $[0,T]$. The corresponding inner product is given by:

$$< \theta, \sigma >_* = \int_0^T < B^*e^{A^*(T-s)}C^*\theta, B^*e^{A^*(T-s)}C^*\sigma > ds$$

and the operator $\Lambda_T(C) : \mathbb{R}^q \rightarrow \mathbb{R}^q$ defined by

$$\Lambda_T(C)\theta = CH_T(H_T)^*C^*\theta = \int_0^T Ce^{(T-s)}BB^*e^{A^*(T-s)}C^*\theta ds$$

is symmetric and positive definite and then invertible. We give hereafter the expression of the optimal control ensuring the compensation of a disturbance $f$ at the final time $T$.

**Proposition 2.10**

*For $f \in L^2(0,T;\mathbb{R}^n)$, there exists a unique $\theta_f \in \mathbb{Y}^q$ such that

$$\Lambda_T(C)\theta_f = -CG_Tf$$

and the control*
\[ u_{\theta}(\cdot) = B^* e^{A^*(T-\cdot)} C^* \theta_f \]

verify

\[ CH_T u_{\theta} + CG_T f = 0 \]

Moreover, it is optimal and

\[ \| u_{\theta} \|_{L^2(0,T;R^p)} = \| \theta_f \| \]

2.3 Another approach of the compensation problem

In this section, we present a more general approach which consists to consider the compensation problem as minimization one of a cost function defined on \( L^2(0,T;R^p) \) as follows

\[
J(u) = < P(CH_Tu + CG_Tf), CH_Tu + CG_Tf > \\
+ \int_0^T < Q(CH_tu + CG_tf), CH_tu + CG_tf > dt \\
+ \int_0^T < Ru(t), u(t) > dt
\]

where \( P, Q \) and \( R \) are symmetric matrixes with \( Q \) positive, \( P \) and \( R \) are positive definite. We have the following result.

Proposition 2.11

There exists a unique control \( u^* \in L^2(0,T;R^p) \) such that

\[ J(u^*) = \inf_{u \in L^2(0,T;R^p)} J(u) \]

with \( u^* \) given by

\[
 u^* = -[(H_T)^* C^* PCH_T + (H_t)^* C^* QCH_t + R]^{-1} \\
\times [(H_T)^* C^* PCG_T + (H_t)^* C^* QCG_t] f
\]

Proof: We have
\[ J(u) = \langle PCH_T u, CH_T u \rangle + 2 \langle PCH_T u, CG_T f \rangle \]
\[ + \langle PCG_T f, CG_T f \rangle \]
\[ + \int_0^T \langle QCH_t u, CH_t u \rangle \, dt + 2 \int_0^T \langle QCH_t u, CG_t f \rangle \, dt \]
\[ + \int_0^T \langle CG_t f, CG_t f \rangle \, dt + \int_0^T \langle Ru(t), u(t) \rangle \, dt \]

and for \( h \in L^2(0, T; \mathbb{R}^p) \), we have

\[ \frac{1}{2} dJ(u) h = \langle PCH_T u, CH_T h \rangle + \langle PCH_T h, CG_T f \rangle \]
\[ + \int_0^T \langle QCH_t u, CH_t h \rangle \, dt + \int_0^T \langle QCH_t h, CG_t f \rangle \, dt \]
\[ + \int_0^T \langle Ru(t), h(t) \rangle \, dt \]
\[ = \langle (H_T)^* C^* PCH_T u, h \rangle + \langle (H_T)^* C^* PCG_T f, h \rangle \]
\[ + \langle (H_\cdot)^* C^* QCH, u \rangle + \langle (H_\cdot)^* C^* QCG, f \rangle \]
\[ + \langle Ru, h \rangle \]

It is clear that \( J \) admits a unique minimum in \( u^* \) verifying

\[ dJ(u^*) = 0 \]

i.e.

\[ (H_T)^* C^* PCH_T u^* + (H_T)^* C^* PCG_T f + \]
\[ (H_\cdot)^* C^* QCH, u^* + (H_\cdot)^* C^* QCG, f + Ru^* = 0 \]

or equivalently

\[ [(H_T)^* C^* PCH_T + (H_\cdot)^* C^* QCH, + R]u^* = \]
\[ -[(H_T)^* C^* PCG_T + (H_\cdot)^* C^* QCG,]f \]

then
\[ u^* = -[(H_T)^* C^* PCH_T + (H_*)^* QCH + R]^{-1} \times [(H_T)^* C^* PCG_T + (H_*)^* QCG_T f] \]

Remark 2.12

i) For \( Q = 0 \), we have

\[ u^* = -[(H_T)^* C^* PCH_T + R]^{-1} (H_T)^* C^* PCG_T f \]

ii) For \( Q = 0 \), \( P = I \) and \( R = \alpha I \), we have

\[ u^* = -[(H_T)^* C^* CH_T + \alpha I]^{-1} (H_T)^* C^* CG_T f \]

and if \( \alpha \ll 1 \) (i.e. \( \alpha \) is small), one obtain practically the same results established using the Hilbert Uniqueness Method.

iii) The result can be extended to the case of a minimization on a non empty closed and convex subset \( C \) of \( L^2(0, T; R^p) \).

\[ \Box \]

3 Asymptotic remediability

3.1 Problem statement

We consider the following linear disturbed dynamical system:

\[
\begin{align*}
\dot{z}(t) &= Az(t) + f(t) + Bu(t); \quad t > 0 \\
z(0) &= z_0
\end{align*}
\]

(7)

with \( A \in M_n(R), B \in M_{n,p}(R), u \in L^2(0, +\infty; R^p) \) and \( f \in L^2(0, +\infty; R^m) \)

The system (7) is augmented by the output equation:

\[ y(t) = Cz(t); \quad t > 0 \]

(8)

with \( C \in M_{q,n}(R) \), we have

\[ z(t) = e^{At}z_0 + H_tu + G_t f \]
then
\[ y(t) = C e^{At} z_0 + CH_t u + CG_t f \]

Let
\[ z = \begin{pmatrix} \dot{z}_1 \\ z_2 \end{pmatrix} \]

where \( z_1 \) and \( z_2 \) are respectively the projections of the state \( z \) on the unstable and the stable subspaces:

\[ E_1 = \bigoplus_{R(e(\lambda)) \geq 0} \ker(A - \lambda I_n)^{m(\lambda)} \quad (9) \]

and

\[ E_2 = \bigoplus_{R(e(\lambda)) < 0} \ker(A - \lambda I_n)^{m(\lambda)} \quad (10) \]

where \( m(\lambda) \) is the multiplicity of the eigenvalue \( \lambda \). The subspaces \( E_1 \) and \( E_2 \) are invariant with respect to the operator \( A \) and

\[
\begin{align*}
(S_1) \quad \dot{z}_1(t) &= A_1 z_1(t) + P Bu(t) + P f(t) \\
(S_2) \quad \dot{z}_2(t) &= A_2 z_2(t) + (I - P) Bu(t) + (I - P) f(t)
\end{align*}
\]

\( P \) is the projection operator on the unstable part and \( A_i \) is the matrix induced by \( A \) on \( E_i \), for \( i = 1, 2 \).

In the case where we observe only the stable part, i.e.
\[ E_1 \subset \ker(C) \]

the operators \( K^\infty(C) \) and \( L^\infty(C) \) defined respectively by

\[
K^\infty(C) : L^2(0, +\infty ; R^p) \longrightarrow R^q \\
u \quad \longrightarrow \quad \int_0^\infty C e^{At} Bu(t) dt
\]

and
\[ L^\infty(C) : L^2(0, +\infty; \mathbb{R}^n) \rightarrow \mathbb{R}^q \]

\[ f \rightarrow \int_0^\infty C e^{At} f(t) dt \]

are well defined. We have the same result if the considered system (7) (or the matrix \( A \)) is exponentially stable, i.e. if \( Re(\lambda_i) < 0 \) for \( i = 1, n \); where \( \lambda_1, ..., \lambda_n \) are the eigenvalues of \( A \). But as it will be shown later, this is not necessary.

**Definition 3.1**

(7)\( + (8) \) is asymptotically remediable, if for every \( f \in L^2(0, +\infty; \mathbb{R}^n) \), there exists \( u \in L^2(0, +\infty; \mathbb{R}^p) \) such that:

\[ K^\infty(C)u + L^\infty(C)f = 0 \]

We give hereafter characterizations of the asymptotic remediability.

**Proposition 3.2** The following properties are equivalent:

i) (7)\( + (8) \) is asymptotically remediable.

ii) \( \text{Im} [L^\infty(C)] \subset \text{Im} [K^\infty(C)] \)

iii) \( \text{Ker} [K^\infty(C)] \subset \text{Ker} [L^\infty(C)] \)

iv) \( \exists \gamma > 0 \) such that \( \forall \theta \in \mathbb{R}^q \), we have

\[ \| e^{A^*} C^* \theta \|_{L^2(0, +\infty; \mathbb{R}^n)} \leq \gamma \| B^* e^{A^*} C^* \theta \|_{L^2(0, +\infty; \mathbb{R}^p)} \]

We have also the following rank characterization.

**Proposition 3.3**

If the operators \( K^\infty(C) \) and \( L^\infty(C) \) are well defined, then (7)\( + (8) \) is asymptotically remediable if and only if

\[ \text{rank} \left( \begin{array}{ccc} CB & CAB & \ldots & CA^{n-1}B \end{array} \right) = \text{rank} \left( \begin{array}{c} C \end{array} \right) \]

**Proof:** Similar to that given in the finite time case. \( \Box \)
Example 3.4

Let us consider the case of an unstable system with \( n = 2, \ p = q = 1 \) and

\[
A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & -3 \end{pmatrix}
\]

We have

\[
e^{tA} = Pe^{tD}P^{-1}
\]

where

\[
D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}; \quad P = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}
\]

\[
Ce^{tA} = \begin{pmatrix} e^{-t} & -3e^{-t} \end{pmatrix}
\]

In this case \( K^\infty(C) \) and \( L^\infty(C) \) are well defined and

\[
\begin{pmatrix} CB & CAB \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix}
\]

and \( \text{rank} \left( C \right) = 1 \), consequently, (7)\+ (8) is asymptotically remediable.

Hereafter, we examine the relationship between the asymptotic remediability and the other asymptotic notions of controllability, stability and stabilizability. But first let us recall that the system (7) or \((A, B)\) is asymptotically controllable if

\[
\text{Im}(H^\infty) = \mathbb{R}^n
\]

where \( H^\infty \) is defined by

\[
H^\infty : L^2(0, +\infty; \mathbb{R}^p) \longrightarrow \mathbb{R}^n
\]

\[
u \quad \longrightarrow \quad \int_0^{\infty} e^{At} Bu(t) dt
\]

This is equivalent to

\[
\text{Ker} [(H^\infty)^\ast] = \{0\}
\]

or (using Caylay Hamilton theorem):

\[
\text{rank} \left( B \quad AB \quad \ldots \quad A^{n-1}B \right) = n
\]
If (7) (or the matrix $A$) is exponentially stable, then $H^\infty$ is well defined.

Also in the asymptotic case, the remediability remain weaker than the controllability. Let us note equally that asymptotically, a system may be remediable without being controllable or stabilizable. This is illustrated in the following example.

**Example 3.5** We consider an unstable system with $n = 2$, $p = q = 1$

i) For

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}
\]

we have

\[
A + BF = \begin{pmatrix} 1 + a & b \\ 0 & 2 \end{pmatrix}
\]

Then for any $F = \begin{pmatrix} a & b \end{pmatrix}$, $A + BF$ is not stable, i.e. $(A,B)$ is not stabilizable, but (7)+(8) is asymptotically remediable.

ii) Let

\[
A = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}
\]

For $F = \begin{pmatrix} a & b \end{pmatrix}$, the matrix $A + BF = \begin{pmatrix} -1 & 0 \\ a + 1 & b + 2 \end{pmatrix}$ is stable for $b < -2$, then $(A,B)$ is stabilizable.

On the other hand $AB = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, then $\text{rank}(B \ AB) = 1 \neq 2$ and consequently $(A,B)$ is not asymptotically controllable. But (7)+(8) is asymptotically remediable because

\[
\text{rank} \begin{pmatrix} CB & CAB \end{pmatrix} = \text{rank} \begin{pmatrix} C \end{pmatrix}
\]

Let us note also that in the general case, if $C$ is an invertible matrix (for example if $C$ is the identity matrix), then the notions of remediability and controllability are equivalent. But this hypothesis is very strong and not practice.
3.2 Asymptotic remediability with minimum energy:

For $z_0 \in \mathbb{R}^n$ and $f \in L^2(0, +\infty; R^p)$, we study the existence of an optimal control $u \in L^2(0, +\infty; R^p)$ ensuring asymptotically the compensation of the disturbance $f$, i.e. such that

$$K^\infty(C)u + L^\infty(C)f = 0$$

Here also, we use an extension of the Hilbert Uniqueness Method. For $\theta \in \mathbb{R}^q$, let:

$$\| \theta \|_* = \left( \int_0^\infty \| [K^\infty(C)]^* \theta \|_{L^2_{R^p}}^2 dt \right)^{\frac{1}{2}}$$

$\| \theta \|_*$ is a semi-norm on $\mathbb{R}^q$.

We assume that it is a norm on $\mathbb{R}^q$. If $Ker [K^\infty(C)]^* = \{0\}$, this is equivalent to the asymptotic remediability of the system (7)+(8). The corresponding inner product is defined by:

$$< \theta; \sigma >_* = \int_0^\infty < B^* e^{A^*t} C^* \theta, B^* e^{A^*t} C^* \sigma > dt$$

and the operator (matrix) $\Lambda^\infty(C) : \mathbb{R}^q \rightarrow \mathbb{R}^q$ given by

$$\Lambda^\infty(C) \theta = K^\infty(C) [K^\infty(C)]^* \theta = \int_0^\infty C e^{At} B B^* e^{A^*t} C^* \theta dt$$

is symmetric, positive definite and then invertible. In the following proposition, we give the optimal control ensuring the asymptotic compensation of a disturbance $f$. With convenient operators and spaces, the result is analogous to that obtained in the finite time case.

**Proposition 3.6**

For $f \in L^2(0, +\infty; R^p)$, there exists a unique $\theta_f \in Y^q$ such that

$$\Lambda^\infty(C) \theta_f = -L^\infty(C)f$$

and the control

$$u_{\theta_f}(.) = B^* e^{A^* \cdot} C^* \theta_f$$

is such that

$$K^\infty(C)u_{\theta_f} + K^\infty(C)f = 0$$

and is optimal with

$$\| u_{\theta_f} \|_{L^2(0, +\infty; R^p)} = \| \theta_f \|_*$$
3.3 Infinite time minimization problem

We consider the compensation problem as minimization on $L^2(0, +\infty; \mathbb{R}^p)$ of a cost function defined as follows

$$J(u) = <P(K^\infty(C)u + L^\infty(C)f), K^\infty(C)u + L^\infty(C)f> + \int_0^\infty <Q(CH_tu + CG_tf), CH_tu + CG_tf> dt + \int_0^\infty <Ru(t), u(t)> dt$$

where $P$, $Q$ and $R$ are symmetric with $Q$ positive, $P$ and $R$ are positive definite. We have the following result which is analogous to that established in the finite time case.

**Proposition 3.7**

There exists a unique control $u^* \in L^2(0, +\infty; \mathbb{R}^p)$ such that

$$J(u^*) = \inf_{u \in L^2(0, +\infty; \mathbb{R}^p)} J(u)$$

with $u^*$ given by

$$u^*(.) = -[[K^\infty(C)]^* PK^\infty(C) + (H_\cdot)^*C^*QCH_\cdot + R]^{-1} \times [[[K^\infty(C)]^* PL^\infty(C) + (H_\cdot)^*C^*QCG_\cdot]f$$

**Proof**: Similar to that given in the case of a finite time horizon.

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**Received: December 20, 2007**