The Homotopy Analysis Method for
the Exact Solutions of the $K(2,2)$, Burgers
and Coupled Burgers Equations

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Abstract
By means of the homotopy analysis method (HAM), the solutions of
the $K(2,2)$, Burgers and coupled Burgers equations are exactly obtained
in this paper. This analytical method is also employed to solve the sim-
ple homogeneous advection equation to obtain the exact solution. HAM
is a powerful and easy-to-use analytic tool for nonlinear problems, and
thus the present study highlights the efficiency of the method. Comp-
parison of HAM with the variational iteration method (VIM) is made,
showing that the former is more effective than the latter.

Mathematics Subject Classification: 35Q53

Keywords: Burgers equation, coupled Burgers equations, homogeneous
advection equation, homotopy analysis method, $K(2,2)$ equation

1 Introduction

The investigation of the exact solutions to nonlinear equations plays an im-
portant role in the study of nonlinear physical phenomena. To date, various
nonlinear equations were presented, which described, for example, the mo-
tion of the isolated waves, and in many fields such as hydrodynamic, plasma
physics, nonlinear optic, etc. In most cases it is difficult to solve nonlinear
problems, especially analytically. Perturbation techniques [$1$, $2$] were among
the popular ones and are based on the existence of small or large parameters,
namely the perturbation quantities. Unfortunately, many nonlinear problems in science and engineering do not contain such kind of perturbation quantities at all. Hence, some non-perturbative techniques [3, 4] have been developed, in which these techniques are independent upon small parameters. However, both perturbative and non-perturbative techniques cannot provide a simple way to adjust or control the convergence region and the rate of given approximate series [5].

To overcome such problems, the homotopy analysis method (HAM) is developed and proposed by [6] in 1992. The method is a powerful analytic method for nonlinear problems and has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors [7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and the references therein. HAM is different from the perturbation and non-perturbation methods mentioned above as it provides the convenient way to control and adjust the convergence region of solution series. HAM has also been effectively employed to solve several well-known nonlinear equations such as the Laplace equation with Dirichlet and Neumann boundary conditions [17], the generalized Hirota-Satsuma coupled KdV equation [18] and the Benjamin-Bona-Mahony-Burgers (BBBM) equations [19]. The Laplace equation has also been solved exactly using the variational iteration method (VIM) by [20].

On the other hand, the $K(n,n)$ equation developed in [21] is the pioneering equation for compactons. In solitary waves theory, compactons are defined as solitons with finite wavelengths or solitons free of exponential tails [21]. Further, the Burgers equation generally appears in fluid mechanics. This equation incorporates both convection and diffusion in fluid dynamics, and is used to describe the structure of shock waves [22]. The system of coupled Burgers equations is derived by [23]. It is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity [24]. The $K(2,2)$ equation, the Burgers equation and the coupled Burgers equations as well as the homogeneous advection equation have been solved by the VIM [22, 25, 26]. Therefore, the aim of the present work is to effectively employ the HAM to establish exact solutions for the homogeneous advection, $K(2,2)$, Burgers and coupled Burgers equations, which have not been exactly solved before using HAM. Comparison of the present method and the VIM is also presented in this paper, particularly for the solution of the $K(2,2)$ equation. It is also aimed to confirm that the HAM is efficient in handling scientific and engineering problems. In what follows we will highlight briefly the basic idea of HAM, where details can be found in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].
2 Basic idea of HAM

In this paper, we apply the HAM to the four problems to be discussed. In order to show the basic idea of HAM, consider the following differential equation:

\[ N[u(x,t)] = 0, \]

where \( N \) is a nonlinear operator, \( x \) and \( t \) denote the independent variables and \( u \) is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the HAM, we first construct the so-called zeroth-order deformation equation

\[
(1 - q) L \left[ \phi(x,t;q) - u_0(x,t) \right] = qhH(x,t) N[\phi(x,t;q)],
\]

(1)

where \( q \in [0,1] \) is the embedding parameter, \( h \neq 0 \) is an auxiliary parameter, \( L \) is an auxiliary linear operator, \( \phi(x,t;q) \) is an unknown function, \( u_0(x,t) \) is an initial guess of \( u(x,t) \) and \( H(x,t) \) denotes a non-zero auxiliary function. It is obvious that when the embedding parameter \( q = 0 \) and \( q = 1 \), equation (1) becomes

\[ \phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t), \]

respectively. Thus as \( q \) increases from 0 to 1, the solution \( \phi(x,t,q) \) varies from the initial guess \( u_0(x,t) \) to the solution \( u(x,t) \). Expanding \( \phi(x,t;q) \) in Taylor series with respect to \( q \), one has

\[
\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)q^m,
\]

(2)

where

\[
u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}.
\]

(3)

The convergence of the series (2) depends upon the auxiliary parameter \( h \). If it is convergent at \( q = 1 \), one has

\[
u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t),
\]

(4)

which must be one of the solutions of the original nonlinear equation, as proven by Liao [7]. Define the vectors

\[ \bar{u}_n = \{u_0(x,t), u_1(x,t), ..., u_n(x,t)\}. \]
Differentiating the zeroth-order deformation equation (1) \(m\)-times with respect to \(q\) and then dividing them by \(m!\) and finally setting \(q = 0\), we get the following \(m\)th-order deformation equation:

\[
L \left[u_m (x, t) - \chi_m u_{m-1} (x, t)\right] = h \Re_m (\bar{u}_{m-1}),
\]

where

\[
\Re_m (\bar{u}_{m-1}) = \frac{1}{(m - 1)!} \frac{\partial^{m-1} N \left[\phi (x, t; q)\right]}{\partial q^{m-1}} \bigg|_{q=0}.
\]

and

\[
\chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases}
\]

It should be emphasized that \(u_m (x, t)\) for \(m \geq 1\) is governed by the linear equation (5) with linear boundary conditions that come from the original problem, which can be solved by the symbolic computation softwares such as Maple, Mathematica and Matlab.

3 Applications

In this part, we will apply the HAM to the equations of the homogeneous advection, \(K(2, 2)\), Burgers and coupled Burgers, respectively, in the next four examples. For all of these equations, we choose the solution expressed by the base function of the form

\[
u(x, t) = f_1 (x) \sum_{n=0}^{\infty} a_n t^n,
\]

where \(a_n\) is the coefficient and \(f_1 (x) \neq 0\) and it is determined according to the respective equation. From the style of this solution, the linear operator should be

\[
L \left[\phi (x, t; q)\right] = \frac{\partial \phi (x, t; q)}{\partial t},
\]

with the property

\[
L [c_1] = 0,
\]

where \(c_1\) is constant. Throughout this paper, \(f_1 (x)\) is taken as the initial guess (i.e. \(u_0 (x, t) = f_1 (x)\)).
Example 1
Consider the homogeneous advection equation [26],
\[ u_t + uu_x = 0, \quad u(x, 0) = -x. \]  
(10)

According to the style of the solution and the initial condition, we take the initial guess as
\[ u_0(x, t) = -x. \]

The nonlinear operator is
\[ N_u = \frac{\partial u(x, t)}{\partial t} + u(x, t) \frac{\partial u(x, t)}{\partial x}, \]  
(11)

and thus,
\[ \mathcal{R}_m (\bar{u}_{m-1}) = \frac{\partial u_{m-1}(x, t)}{\partial t} + \sum_{i=0}^{m-1} u_i(x, t) \frac{\partial u_{m-1-i}(x, t)}{\partial x}. \]  
(12)

Using equation (5), equation (12), and under the linear operator of equation (9), along with the initial condition \( u_m(x, 0) = 0 \) where \( m = 1, 2, 3, \ldots \), we have
\[
\begin{align*}
  u_1(x, t) &= -x(-ht), \\
  u_2(x, t) &= -x((-ht)^2 - ht - h^2t), \\
  u_3(x, t) &= -x((-ht)^3 + 2h^3t^2 + 2h^2t^2 - ht - 2h^2t - h^3t), \\
  &\vdots \\
  u_n(x, t) &= -x((-ht)^n + Z_1(t, h)). 
\end{align*}
\]

When \( h = -1 \), the polynomial \( Z_1(t, h) \) becomes 0. By equation (4), we have
\[ u(x, t) = -x \sum_{n=0}^{\infty} t^n. \]  
(13)

Equation (13) is the Taylor series that converge to
\[ u(x, t) = \frac{x}{t - 1}, \]  
(14)

under \( |t| < 1 \), which is the exact solution.

Example 2
Consider the following \( K(2, 2) \) equation [22]:
\[ u_t + u^2_x + u^2_{xxx} = 0, \quad u(x, 0) = x. \]  
(15)
According to the style of the solution and the initial condition, we take the initial guess as

\[ u_0(x, t) = x. \]

The nonlinear part is

\[ N[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t} + \frac{\partial^2 \phi(x, t; q)}{\partial x^2} + \frac{\partial^3 \phi(x, t; q)}{\partial x^3}. \] (16)

And thus by equation (6), we have

\[ \Re_m (\vec{u}_{m-1}) = \frac{\partial u_{m-1}(x, t)}{\partial t} + \frac{\partial}{\partial x} \left( \sum_{i=0}^{m-1} u_i u_{m-1-i} \right) + \frac{\partial^3}{\partial x^3} \left( \sum_{i=0}^{m-1} u_i u_{m-1-i} \right). \] (17)

By using the \( m \)th-order deformation equation (5), equation (17), and under the linear operator of equation (9), along with the initial condition \( u_m(x, 0) = 0 \) where \( m = 1, 2, 3, \ldots \), we obtain

\[
\begin{align*}
  u_1(x, t) &= 2hxt, \\
  u_2(x, t) &= 2hxt(2ht + h + 1), \\
  u_3(x, t) &= 2hxt(2ht + h + 1)^2, \\
  &\vdots \\
  u_m(x, t) &= 2hxt(2ht + h + 1)^{m-1}.
\end{align*}
\]

When \( h = -1 \), by equation (4), we have

\[ u(x, t) = x \sum_{m=0}^{+\infty} (-2)^m t^m, \] (18)

which is the Taylor series expansion for the function

\[ u(x, t) = \frac{x}{1+2t}, \] (19)

under \(|t| < 1/2\).

Hence it converges to the exact solution (19). Throughout this paper, we assume that \( h = -1 \), but in order to show the freedom of \( h \), we will discuss other value of \( h \) when comparing HAM to VIM in part 4 of this paper, with special emphasis on this particular example.

**Example 3**

Consider the one dimensional Burgers equation [25] that has the form

\[ u_t + uu_x - \nu u_{xx} = 0, \] (20)
and the solution to be obtained is subjected to the initial condition

$$u(x, 0) = \frac{\alpha + \beta + (\beta - \alpha)e^\gamma}{1 + e^\gamma}, \quad t \geq 0,$$

(21)

where $\gamma = \alpha(x/\nu)$ and the parameters $\alpha, \beta$ and $\nu$ are arbitrary constants. As in the two previous examples, we also take the initial guess to be the same as the initial condition (i.e $u(x, 0) = u_0(x, t)$). From equation (6), we have

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}(x, t)}{\partial t} + \sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}(x, t)}{\partial x} - \nu \frac{\partial^2 u_{m-1}(x, t)}{\partial x^2}.\hspace{1cm} (22)$$

Using equation (5), equation (22) and under the linear operator of equation (9), along with the initial condition $u_m(x, 0) = 0$ where $m = 1, 2, 3, \ldots$, we have

$$u_1(x, t) = \frac{2\alpha^2 \beta e^\gamma}{\nu[1 + e^\gamma]^2} (ht),$$

$$u_2(x, t) = \frac{\alpha^3 \beta^2 e^\gamma (e^\gamma - 1)}{\nu^2 [1 + e^\gamma]^3} (ht)^2 + M_1,$$

$$u_3(x, t) = \frac{\alpha^4 \beta^3 e^\gamma [1 - 4e^\gamma + e^{2\gamma}]}{3\nu^3 [1 + e^\gamma]^4} (ht)^3 + M_2,$$

where

$$M_1 = \frac{t\alpha^2 \beta e^\gamma (2h^2 \nu e^\gamma - 2\nu e^\gamma - 2\nu - 2\nu h)}{\nu^2 [1 + e^\gamma]^3},$$

$$M_2 = \frac{2e^\gamma \alpha^3 \beta^2 (ht)^2 \nu (e^{2\gamma} (h + 1) - 1 - h) + 2\nu^2 e^\gamma \alpha^2 \beta h (e^{2\gamma} (-1 - 2h - h^2) + e^\gamma (-2h^2 - 4h - 2) - 2h - 1 - h^2)}{3\nu^3 [1 + e^\gamma]4}.$$

When $h = -1$ and $M_i = 0$, where $i = 1, 2$, we get the same value as in VIM, which means that for this example, VIM is the special case of HAM. This implies in this example that HAM is more effective than VIM. By the same manner of the previous examples and using equation (4), the solution of $u(x, t)$ in a closed form is

$$u(x, t) = \frac{\alpha + \beta + (\beta - \alpha)e^\xi}{1 + e^\xi},\hspace{1cm} (23)$$
where $\xi = (\alpha/\nu) (x - \beta t)$. This is the exact solution of the one dimensional Burgers equation and it is exactly the same as obtained by the VIM [25].

**Example 4**
Consider the following system of equations, which is also known as the coupled Burgers equations [25]:

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0,$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0,$$  \hspace{1cm} (24)

and the solutions of which are to be obtained is subjected to the initial conditions

$$u(x, 0) = \sin x, \quad v(x, 0) = \sin x.$$  \hspace{1cm} (25)

The nonlinear operators are

$$N_u = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2u \frac{\partial u}{\partial x} + \frac{\partial (uv)}{\partial x},$$  \hspace{1cm} (26)

and

$$N_v = \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - 2v \frac{\partial v}{\partial x} + \frac{\partial (uv)}{\partial x}.$$  \hspace{1cm} (27)

When we apply equation (6) on equations (26) and (27), we obtain

$$\mathcal{R}_m (\bar{u}_{m-1}) = \frac{\partial u_{m-1}(x, t)}{\partial t} - \frac{\partial^2 u_{m-1}(x, t)}{\partial x^2} - \sum_{i=0}^{m-1} 2u_i \frac{\partial u_{m-1-i}}{\partial x} + \frac{\partial (u_i v_{m-1-i})}{\partial x},$$  \hspace{1cm} (28)

$$\mathcal{R}_m (\bar{v}_{m-1}) = \frac{\partial v_{m-1}(x, t)}{\partial t} - \frac{\partial^2 v_{m-1}(x, t)}{\partial x^2} - \sum_{i=0}^{m-1} 2v_i \frac{\partial v_{m-1-i}}{\partial x} + \frac{\partial (u_i v_{m-1-i})}{\partial x},$$  \hspace{1cm} (29)

respectively. We choose the initial guess the same as the initial condition, by applying equation (5) on equations (28) and (29), and under the linear operator of equation (9), along with the initial conditions $u_m(x, 0) = 0$ and...
\( v_m(x, 0) = 0 \) where \( m = 1, 2, 3, \ldots \), we have

\[
\begin{align*}
    u_0(x, t) &= \sin x, \\
    v_0(x, t) &= \sin x, \\
    u_1(x, t) &= \sin x(ht), \\
    v_1(x, t) &= \sin x(ht), \\
    u_2(x, t) &= \sin x\left(\frac{h^2 t^2}{2} + (h + h^2)t\right), \\
    v_2(x, t) &= \sin x\left(\frac{h^2 t^2}{2} + (h + h^2)t\right), \\
    u_3(x, t) &= \sin x\left(\frac{h^3 t^3}{3!} + (h + 2h^2 + h^3)t + (h^3 + h^2)t^2\right), \\
    v_3(x, t) &= \sin x\left(\frac{h^3 t^3}{3!} + (h + 2h^2 + h^3)t + (h^3 + h^2)t^2\right), \\
    \vdots \\
    u_m(x, t) &= \sin x\left(\frac{(ht)^m}{m!} + Z_2(h, t)\right), \\
    v_m(x, t) &= \sin x\left(\frac{(ht)^m}{m!} + Z_3(h, t)\right),
\end{align*}
\]

where \( Z_2(h, t) = Z_3(h, t) = 0 \) when \( h = -1 \). So according to equation (4), the solutions should be the following:

\[
\begin{align*}
    u(x, t) &= \sin x\left(\sum_{m=0}^{\infty} \frac{t^m}{m!}\right), \\
    v(x, t) &= \sin x\left(\sum_{m=0}^{\infty} \frac{t^m}{m!}\right),
\end{align*}
\]

Proceeding as before, the rest of the components are obtained, and the two functions \( u(x, t) \) and \( v(x, t) \) in the closed form are readily found to be

\[
\begin{align*}
    u(x, t) &= e^{-t}\sin x, \\
    v(x, t) &= e^{-t}\sin x,
\end{align*}
\]

which are the exact solutions of the coupled Burgers equations. We get the same value as in VIM when \( h = -1 \), which means also for this example, VIM is the special case of HAM. This implies in this example that HAM is more effective than VIM.

4 Comparison and discussion

In this part, we present the comparison of the analytical results between the 8th iteration of VIM and the 8th order of approximation of HAM, particularly for the solutions of the \( K(2, 2) \) equation. Figure 1 shows the VIM solution with a small range of \( t \), namely \( t \) ranging from 0 to 0.8. However, when we increase slightly the range of \( t \) to \( t \) from 0 to 1, the shape of the VIM solution,
as shown in Fig. 2, is different from the exact solution as given in Fig. 3. On the other hand, the HAM solution has the same shape as the exact solution even for larger range of $t$, i.e. $t$ ranging from 0 to 7.5 as shown in Fig. 4, when we take $h = -0.1$. This particular value of $h$ is in the convergent region as shown in Figs. 5 and 6. For these two figures we fixed the values of $x$ and $t$ to be 1. Therefore, based on these present results, we can say that HAM is more effective than VIM, at least for this particular example.

5 Conclusions

In this paper, the HAM is used to obtain the exact solutions of the homogeneous advection equation, the $K(2, 2)$ equation, the Burgers equation and the coupled Burgers equations using the PC-based MAPLE package for illustrations and for generating analytical results. The comparison between the HAM and the VIM was made particularly for the $K(2, 2)$ equation and it was found that HAM is more effective than VIM. Further, for all of the discussed examples, it was also found that there was no errors in obtaining the exact solutions using HAM. Hence, it may be concluded that this method is a powerful and an efficient technique in finding the exact solutions for wide classes of problems. This paper also illustrated the validity and the great potential of the HAM for solving nonlinear problems in science and engineering. It is also worth mentioning to this end that the advantage of this method is the fast convergent of the solutions by means of the auxiliary parameter.

References


\textbf{Received: November 5, 2007}
Figure 1: The VIM solution for the $K(2, 2)$ equation from $t = 0$ to $t = 0.8$

Figure 2: The VIM solution for the $K(2, 2)$ equation from $t = 0$ to $t = 1.0$
Figure 3: The exact solution for the $K(2, 2)$ equation from $t = 0$ to $t = 7.5$

Figure 4: The HAM solution for the $K(2, 2)$ equation from $t = 0$ to $t = 7.5$
Figure 5: The $h$-curve of the 10th-order of approximation for $x = t = 1$

Figure 6: The $h$-curve of the 20th-order of approximation for $x = t = 1$