Applied Mathematical Sciences, Vol. 2, 2008, no. 36, 1761 - 1772

Dynamic Portfolio Choice Problems with Non-Monotone Utility Functions

Lili Pan[1]

Dept. of Applied Mathematics
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu, P. R. China

Peibiao Zhao[2]

Dept. of Applied Mathematics
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu, P. R. China
pbzhao@mail.njust.edu.cn
pb_zhao@yahoo.com.cn

Abstract

In this paper, based on a non-monotone utility function being revised to a monotone utility function, we study the multi-period and continuous-time optimal consumption-investment choice model, and give an optimal solution to the model. This result can be regarded as the generalization of the portfolio selection with monotone utility functions.

Mathematics Subject Classificatin: 60J25, 60G10, 60Q15

Keywords: Portfolio monotone, Mean-variance preferences, Continuous-Time, Consumption-investment

1 Introduction

Dynamic portfolio choices consist of the discrete time model and the continuous-time model which were studied by Merton(1973) and Lucas(1978) earlier. How to generalize a standard one-period portfolio model to the multi-period one is an important subject in the financial field, the significance of the study is determined

[1]Supported by a Grant-in-Aid for Science Research from NJUST (AB96137, KN11008) and Partly by NNSF(10771102).
[2]Supported by NNSF(10771102).

by that the modern financial decision needs to reflect a complex interaction of the investment environment. For multi-period portfolios, the investors often choose investment strategies in every stage during in several continuous investment stage, their purposes are the total maximizing utility functions at the end of investment stage. If these portfolio stages are not correlative, then the multi-period strategy can be divided by many single periods, if these portfolio stages are correlative, then many investment choices become more complex. It is well know that in the practical investment environment, the returns of asset distributions change with the time. Therefore, how to adjust investment strategies based on changes of environments are the practical problem that every investor must be facing. The requests of the practical problems provide an important background for multi-period research investment portfolio analysis.

There are many conclusions for multi-period portfolio selections, but these results are all associated with monotone utility functions, and the research conclusion corresponding to the setting of non-monotone utility functions is little.

Recently, the authors [15,16] are studying the portfolio problem under the frictional market, but just studying the problem of the fixed rate of transaction costs, in particular, for a jump rate of transaction costs according to the real market, and based on this, the authors get an optimal model of minimizing risk. For this model, they use the Calculus of Variation to prove the existence of the optimal strategy. Following [15], in this paper, based on a non-monotone utility function being revised to a monotone utility function, we study the multi-period and continuous-time optimal consumption-investment choice model, and then give an optimal solution to the model.

We now single out the arrangement of this paper as follows. The first two sections of this paper are devoted to introducing some necessary notations and terminologies. The third section studies the discrete time capital asset pricing model. In the fourth section, we study the continuous time optimal consumption-investment portfolio choice. We obtain a generalization of the corresponding conclusions associated with the classical models.

2 Preliminaries

For convenience, we first give some necessary assumption conditions of the market corresponding to our models.

H1. All securities holders are of only the limited liabilities, that is the prices of securities are non-negative.

H2. There are no transaction costs, no taxes, the securities are perfectly divisible.

H3. There are many investors in the market, anyone can buy securities according to the prices of securities in the market.

H4. The market always is in the balance setting, no balance price is not existing. Further one can assume that the market is of being no-arbitrage.

H5. Agents are price-takers, they believe that their choices do not affect the distributions of asset returns.
H6. There are no institutional restrictions, so that agents are allowed to buy, sell or short sell any desired amount of any security.
H7. Economy is to be discussed indefinitely

Additional Definitions and Notations that are of a more limited scope will be introduced when needed. In most cases no proofs are given since these are readily available in the standard literature dealing with this subject area. In particular we mention the literatures of R.C. Merton(1990), H. Markowitz(1952), J.P. Bigelow(1993). In fact, a large portion of this material can be already found in the original work of Merton(1990), and that of H. Markowitz. If a particular result is of importance for our objectives and if it is not textbook material we will give a proof.

In fact, the first six assumptions are the same with the standard assumption of CAPM. We consider the discrete time, it is reasonable that the economy is indefinite.

3 The Discrete Time Capital Asset Pricing Model

We may assume that the investors on the market are also the consumers, called consumption-investors, the portfolio choice is of the multi-period model, and one purpose of these consumption-investors is to maximize the multi-period utility function. Therefore, these consumption-investors are multi-period investors, choosing security portfolios based on considering to the current and future returns, rather than one-period investment decision-makers who do not consider the situation later.

Definition 3.1 *If consumption utility functions satisfy* $\mu(c_1, \cdots, c_t) = \sum_{i=1}^{t} \mu(c_i)$, *then we call consumption utility functions the time additive.*

According to references [11], we know, when a utility function $\widetilde{\mu}(\cdot)$ is of non-monotone, we can find a monotone, minimal, concave function $\mu(\cdot)$ which is approximating to $\widetilde{\mu}(\cdot)$. In this paper we will use $\mu(\cdot)$ as a monotone utility function in the process of calculation. Here, a multi-period discrete utility function can be expressed as the sum of single monotone utility functions $\mu(\cdot)$. From these statements, one can obtain the following

Theorem 3.1 *Let the initial wealth be W_0, $W(t)$ be the gross wealth at t period. Then, the optimization problem*

$$\begin{cases} \max\limits_{c(t),\omega(t)} E_t[\sum_{t=0}^{\infty} \mu(c(t))] \\ s.t. W(t+1) = (W(t) - c(t))[\sum_{i=1}^{N} \omega_{i,t}(r_{i,t+1} - r_{t+1}) + (1 + r_{t+1})] \end{cases} \tag{3.1}$$

gives the discrete time multi-period monotone capital asset pricing model by

$$E_t[r_{i,t+1}] = r_{t+1} + \beta_{i,t+1} E_t[r_{m,t+1} - r_{t+1}]$$

where $\beta_{i,t+1} = \frac{Cov\left(\nabla V(W(t+1)), r_{i,t+1}\right)}{Cov\left(\nabla V(W(t+1)), r_{m,t+1}\right)}$, $\omega_{i,t}$ *are the investment proportion to invest risk asset* i *at* t*-period,* $r_{i,t+1}$ *are the expect returns of risk asset* i *at* $t+1$ *period,* r_{t+1} *is the risk-free asset return, and* $c(t)$ *are the* t*-period consumptions.*

Proof By virtue of the idea given by Lucas (1978), we can arrive at the solution of this model below.

In fact, we first definite the value function by

$$V_t(W(t)) = \max_{c(t),\omega(t)} E_t[\sum_{t=0}^{\infty} \mu_1(c(t))] \tag{3.2}$$

Let the value function be a Gateaux differentiable function and by the solving approach to the dynamic programming, one gets

$$V_t(W(t)) = \max_{c(t),\omega(t)} E_t\{\mu(c(t)) + V_{t+1}(W(t+1))\} \tag{3.3}$$

$\mu(c(t))$ is to be known at t, so (3.3) is to be as

$$V_t(W(t)) = \max_{c(t),\omega(t)} E_t\{\mu(c(t)) + E_t[V_{t+1}(W(t+1))]\} \tag{3.4}$$

The first order conditions for an optimum are

$$\begin{cases} \frac{\partial V_t(W(t))}{\partial c} = \nabla\mu(c(t)) - E_t[\nabla V_t(W(t+1)) \sum_{i=1}^{N} \omega_{i,t}(r_{i,t+1} - r_{t+1}) + r_{t+1}] \\ \frac{\partial V_t(W(t))}{\partial \omega} = W(t)E_t[\nabla V_t(W(t+1))(r_{i,t+1} - r_{t+1})] \end{cases} \tag{3.5}$$

According to the assumption condition, we know $c(t)$ is constant at t moment, let $\nabla\mu(c(t)) = \nabla\mu$, for the value function, integration by parts w.r.t. w, we have

$$\begin{aligned}
\nabla V(W_t) &= \nabla\mu\frac{\partial c}{\partial W} + E_t\big[\nabla V(W(t+1))(W(t)-c(t))'[\sum_{i=1}^{N}\omega_{i,t}(r_{i,t+1}-r_{t+1})+r_{t+1}] \\
&+ (W(t)-c(t))[\sum_{i=1}^{N}\omega_{i,t}(r_{i,t+1}-r_{t+1})+r_{t+1}]'\big] \\
&= \nabla\mu\frac{\partial c}{\partial W} + E_t\big[\nabla V(W(t+1))(1-\frac{\partial c}{\partial W})[\sum_{i=1}^{N}\omega_{i,t}(r_{i,t+1}-r_{t+1})+r_{t+1}] \\
&+ (W(t)-c(t))[\sum_{i=1}^{N}\frac{\partial \omega_{i,t}}{\partial W}(r_{i,t+1}-r_{t+1})]\big] \\
&= \big\{\nabla\mu - E_t[\nabla V(W(t+1))[\sum_{i=1}^{N}\omega_{i,t}(r_{i,t+1}-r_{t+1})+r_{t+1}]]\big\}\frac{\partial c}{\partial W} \\
&+ \sum_{i=1}^{N}\frac{\partial \omega_{i,t}}{\partial W}E_t[\nabla V(W(t+1))(W(t)-c(t))(r_{i,t+1}-r_{t+1})] \\
&+ E_t\Big[\nabla V(W(t+1))\big(\sum_{i=1}^{N}\omega_{i,t}(r_{i,t+1}-r_{t+1})+r_{t+1}\big)\Big].
\end{aligned}$$

Using the first order conditions and rearranging terms, one gets

$$\nabla V(W(t)) = \nabla\mu(c(t)). \tag{3.6}$$

This is the so-called envelope condition.

Substituting (3.6) into (3.5), we obtain

$$\nabla V_t(W(t)) = E_t\big[\nabla V_t(W(t+1))\sum_{i=1}^{N}\omega_{i,t}(r_{i,t+1}-r_{t+1})+r_{t+1}\big] \tag{3.7}$$

Substituting (3.6) into (3.7) we obtain

$$\nabla V_t(W(t)) = E_t\big[\nabla V_t(W(t+1))(1+r_{t+1})\big] \tag{3.8}$$

Since

$$\begin{aligned}
&E_t\big[\nabla V(W(t+1))(r_{i,t+1}-r_{t+1})\big] \\
&= E_t\big[\nabla V(W(t+1))\big]E_t\big[r_{i,t+1}-r_{t+1}\big] + Cov\big(\nabla V(W(t+1)), r_{i,t+1}\big)
\end{aligned} \tag{3.9}$$

Form (3.6) we know

$$E_t\big[\nabla V(W(t+1))\big]E_t\big[r_{i,t+1}-r_{t+1}\big] + Cov\big(\nabla V(W(t+1)), r_{i,t+1}\big) = 0 \tag{3.10}$$

$$\frac{Cov\big(\nabla V(W(t+1)), r_{i,t+1}\big)}{E_t\big[\nabla V(W(t+1))\big]} = E_t[r_{i,t+1} - r_{t+1}] \tag{3.11}$$

Then, for market portfolio m, we also arrive at the following

$$\frac{Cov\big(\nabla V(W(t+1)), r_{m,t+1}\big)}{E_t\big[\nabla V(W(t+1))\big]} = E_t[r_{m,t+1} - r_{t+1}] \tag{3.12}$$

By virtue of (3.11), (3.12) we obtain

$$\frac{Cov\big(\nabla V(W(t+1)), r_{i,t+1}\big)}{Cov\big(\nabla V(W(t+1)), r_{m,t+1}\big)} = \frac{E_t[r_{i,t+1} - r_{t+1}]}{E_t[r_{m,t+1} - r_{t+1}]} \tag{3.13}$$

Thus, we get

$$E_t[r_{i,t+1} - r_{t+1}] = \beta_{i,t+1} E_t[r_{m,t+1} - r_{t+1}]$$

where $\beta_{i,t+1} = \frac{Cov\big(\nabla V(W(t+1)), r_{i,t+1}\big)}{Cov\big(\nabla V(W(t+1)), r_{m,t+1}\big)}$. This ends the proof of Theorem 3.1. □

This is the discrete time multi-period monotone capital asset pricing model, the form of the model is the same with one period monotone capital asset pricing model, but their contents are different. We naturally consider continuous time portfolio after the discrete time multi-period monotone capital asset pricing model, we will study in detail continuous time portfolio based on monotone utility functions

4 The Continuous Time Optimal Consumption-investment Portfolio Choice

In this section we still consider a risk-free asset and n risk assets in a capital market, the returns of assets are random, the investors enter this market at zero moment, the initial wealth is W_0. Assume that One part of the wealth is used to be as a consumption, the other part of the wealth left is over used to invest the risk assets. We also assume that the investment is continuous. In fact, if there are no transaction costs or taxes and the security prices being perfectly divisible, then the investors may invest in any scale, and adjust their security portfolios with gladly at any time, so the investment is continuous with respect to time.

Definite 4.1 *After the decision investment strategy in entire investment stage, there is no fund to be poured into and no fund to be pulled out for some portfolio, then we say the entire transaction process is of self- financial or the transaction strategy is of self-financial.*

Now we suppose that the prices of risk assets are $P_i(t), (i = 0, 1, \cdots, n)$, the price process satisfy the following stochastic differential equations

$$dP_0(t) = rP_0(t)dt,$$

$$dP_i(t) = P_i(t)\alpha_i(t)dt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t),$$

where $\alpha_i(t)$ are the instantaneous expected returns of the risk asset i at moment t; $B^T(t) = (B_1(t), \cdots, B_n(t))$ is the $n-$dimension standard Brownian motion; σ_{ij} are the instantaneous standard variances of j-th uncertainty factor which are influenced the i-th risk asset price. To simplify computation, we assume that the assets are self-financial. Let $\omega_i(t)$ are the investment proportions of risk assets and there holds

$$\sum_{i=1}^{n} \omega_i(t) = 1$$

The process of the investment wealth

$$dW(t) = \big[\sum_{i=1}^{n} \omega_i(t_0)(u_i - r)W + rW - C(t)\big]dt + \sum_{i=1}^{n} \omega_i W \sigma_{ij} dB_j(t) \tag{4.1}$$

where ω_i are the risk asset weights, $C(t)$ is the instantaneous consumption at t moment. It is easy to check that there holds the following

$$E_t[\Delta W] = -C(t)h + W(t)\sum_{i=0}^{n} \omega_i r_{i,t} h.$$

$$Var(\Delta W) = (W(t) - C(t)h)^2 \sum_{j=1}^{n}\sum_{i=1}^{n} \omega_j \omega_i \sigma_{ij}.$$

Theorem 4.1 *The optimal consumption portfolio given by*

$$\max_{\omega(t),C(t)} E\{\int_0^t exp(-\rho t)U[C(t)]dt + \min_{y\in L_+^2(p),E[y]=1} (E[W(T)y_T]) + \frac{1}{2\theta}E[y^2] - \frac{1}{2\theta}\} \tag{4.2}$$

$$subject\ to \begin{cases} \sum_{i=1}^{n} \omega_i(t) = 1 \\ C(t) \leq 0 \\ W(t) \leq 0 \end{cases} \tag{4.3}$$

admits a solution by

$$\omega(t) = -\frac{\Psi_w}{W\Psi_{ww}}\sigma^{-1}(r - \bar{1}r_0) - \frac{\frac{\partial\Psi}{\partial y}\frac{\partial y}{\partial W}}{W\Psi_{ww}}\sigma^{-1}(r - \bar{1}r_0),$$

where the consumptive utility function $U(C)$ is strictly concave, the marginal utility function is decreasing (that is $U'(C) > 0$, $U''(C) < 0$).

Proof In fact, the utility function just as above can be recognized as the cumulative consumption utility and wealth utility, which is a monotone mean-variance utility function posed in reference [14], so the wealth utility

$$\min_{y_T \in L_+^2(p), E[y_T]=1} (E[W(T)y_T]) + \frac{1}{2\theta}E[y_T^2] - \frac{1}{2\theta} \tag{4.4}$$

is monotone. It is well know that (4.4) is a concave function, and the constraint conditions are convex, it implies that (4.4) can achieve the minimum. On the other hand, the references[9],[10],[11],[12] study deeply that the variational preference utility functions of (4.4) have the smooth nature, and can do the Gateaux differential. In the finite-dimensional space and by reference [14], one knows that when (4.4) arrives at the minimal value, that is, y is a linear function of W, then (4.4) can be written as

$$\Psi(W[T], y(W[T])) = \min_{y_T \in L_+^2(p), E[y_T]=1} \big(E[W(T)y_T]) + \frac{1}{2}(E[y_T^2] - 1)\big) \tag{4.5}$$

Then, (4.2) turns into

$$\max_{\omega(t), C(t)} E\big\{ \int_0^T \exp(-\rho t) U[C(t)]dt + \Psi(W(T), y(W(T)))\big\} \tag{4.6}$$

We now define a value function by

$$\Phi(W(t), C(t), t) = \max_{\omega(t), C(t)} E_t\big\{ \int_t^T \exp(-\rho t) U[C(t)]dt + \Psi(W(T), y(W(T)))\big\} \tag{4.7}$$

Then, we have

$$\begin{aligned} & \max_{\omega(t), C(t)} E_t\big\{ \int_t^T \exp(-\rho t) U[C(t)]dt + \Psi(W(T), y(W(T)))\big\} \\ = & \max_{\omega(t), C(t)} E_t\big\{ \int_t^{t+h} \exp(-\rho t) U[C(t)]dt + \int_{t+h}^T \exp(-\rho t) U[C(t)]dt + \Psi(W(T), y(W(T)))\big\} \\ = & \max_{\omega(t), C(t)} E_t\big\{ \int_t^{t+h} \exp(-\rho t) U[C(t)]dt + \Phi(W(t+1), C(t+1), t+1)\big\} \end{aligned} \tag{4.8}$$

Considering Taylor explosions of $\Phi(W(t+1), C(t+1), t+1)$ at t, we obtain

$$\begin{aligned} \Phi(W(t+1), C(t+1), t+1) &= \Phi(W(t), C(t), t) + \frac{\partial \Phi}{\partial t}h + \frac{\partial \Phi}{\partial W}\Delta W + \frac{\partial \Phi}{\partial C}\Delta C \\ &+ \frac{1}{2}\frac{\partial^2 \Phi}{\partial W^2}\Delta W^2 + \frac{1}{2}\frac{\partial^2 \Phi}{\partial C^2}\Delta C^2 + \frac{\partial^2 \Phi}{\partial W \partial C}\Delta W \Delta C + O(t) \end{aligned} \tag{4.9}$$

Assume that the consumption is constant within very short time, so we get

$$\frac{\partial \Phi}{\partial W}\Delta W = (\frac{\partial \Psi}{\partial W} + \frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial W})\Delta W \tag{4.10}$$

and

$$\frac{\partial^2 \Phi}{\partial W^2}\Delta W^2 = \big(\frac{\partial^2 \Psi}{\partial W^2} + \frac{\partial^2 \Psi}{\partial y \partial W} + \frac{\partial(\frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial W})}{\partial W}\big)\Delta W^2 \tag{4.11}$$

Since y is a linear function of W, therefore

$$\frac{\partial^2 \Psi}{\partial y \partial W} = 0, \frac{\partial(\frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial W})}{\partial W} = 0 \tag{4.12}$$

Substituting (4.10),(4.11) and (4.12) into (4.9), we get

$$\begin{aligned}\Phi(W(t+1), C(t+1), t+1) &= \Phi(W(t), C(t), t) + \frac{\partial \Phi}{\partial t}h + (\frac{\partial \Psi}{\partial W} + \frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial W})\Delta W \\ &+ \frac{1}{2}\frac{\partial^2 \Phi}{\partial W^2}\Delta W^2 + O(t) \end{aligned} \tag{4.13}$$

Taking the expectations of both sides of (4.13), we have

$$\begin{aligned}E_t\big[\Phi(W(t+1), C(t+1), t+1)\big] &= E_t\big[\Phi(W(t), C(t), t)\big] + \frac{\partial \Phi}{\partial t}h + (\frac{\partial \Psi}{\partial W} + \frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial W})E_t\big[\Delta W\big] \\ &+ \frac{1}{2}\frac{\partial^2 \Phi}{\partial W^2}E_t\big[\Delta W^2\big] + O(t) \end{aligned} \tag{4.14}$$

Substituting (4.14) into (4.8) we obtain

$$\begin{aligned}&\Phi(W(t), C(t), t) = \max_{\omega(t), C(t)} E_t\big\{\int_t^{t+h} \exp(-\rho t)U[C(t)]dt + \Phi(W(t+1), C(t+1), t+1)\big\} \\ = &\max_{\omega(t), C(t)} E_t\big[\exp(-\rho t)U[C(t)]h + \Phi(W(t), C(t), t) + \frac{\partial \Phi}{\partial t}h + (\frac{\partial \Psi}{\partial W} + \frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial W})\Delta W \\ + &\frac{1}{2}\frac{\partial^2 \Phi}{\partial W^2}\Delta W^2 + O(t)\big] \end{aligned} \tag{4.15}$$

Since

$$\begin{aligned}\int_t^{t+h} \exp(-\rho t)U[C(t)]dt &= \int_t^{t+h} (\exp(-\rho t)U[C(t)] \\ &\quad +(\rho)\exp(-\rho t)U[C(t)] + \exp(-\rho t)U_t[C(t)]))hdt \\ &= \exp(-\rho t)U[C(t)]h + O(h)\end{aligned}$$

Substituting this formula above, $Var(\Delta W)$ and $E_t[\Delta]$ into (4.15) we know that

$$\begin{aligned}\Phi(W(t), C(t), t) &= \max_{w(t), C(t)} E_t\big\{\exp(-\rho t)U[C(t)]h + \Phi(W(t), C(t), t) + \frac{\partial \Phi}{\partial t}h \\ &\quad +(\frac{\partial \Psi}{\partial W} + \frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial W})E_t\big[\Delta W\big] + \frac{1}{2}\frac{\partial^2 \Phi}{\partial W^2}E_t\big[\Delta W^2\big] + O(t)\big\}\end{aligned}$$

$$\max_{w(t),C(t)} E_t\big\{\exp(-\rho t)U[C(t)] + +\frac{\partial\Phi}{\partial t} + (\frac{\partial\Psi}{\partial W} + \frac{\partial\Psi}{\partial y}\frac{\partial y}{\partial W})(-C(t) + W(t)\sum_{i=0}^{n}\omega_i r_{i,t})$$
$$+\frac{1}{2}\frac{\partial^2\Psi}{\partial W^2}W^2\sum_{j=1}^{n}\sum_{i=1}^{n}\omega_j\omega_i\sigma_{ji} + O(t)\big\} = 0 \quad (4.16)$$

Since $\sum\limits_{i=1}^{n}\omega_i(t) = 1$, then there holds $\sum\limits_{i=0}^{n}\omega_i(t)r_i = \omega_0 r + \sum\limits_{i=1}^{n}\omega_i(t)(r_i - r)$. Compute the partial derivative in (4.16) with respect to C, ω respectively, and get

$$\begin{cases} \frac{\partial[\cdot]}{\partial C} = \exp(-\rho t)U_c[C(t)] - \big(\frac{\partial\Psi}{\partial W} + \frac{\partial\Psi}{\partial y}\frac{\partial y}{\partial W}\big) = 0 \\ \frac{\partial[\cdot]}{\partial\omega} = \big(\frac{\partial\Psi}{\partial W} + \frac{\partial\Psi}{\partial y}\frac{\partial y}{\partial W}\big)W(t)(r_i - r) + \frac{\partial^2\Psi}{\partial W^2}W^2\sum\limits_{j=1}^{n}\omega_j\sigma_{ji} = 0 \end{cases} \quad (4.17)$$

To simplify calculation, we induct vector and matrix, and denote by $r = (r_1, \cdots, r_n)$ the n risk asset returns vector and by $\sigma = (\sigma_{ji})_{n\times n}$ the covariance matrix. By (4.17) we obtain

$$(\frac{\partial\Psi}{\partial W} + \frac{\partial\Psi}{\partial y}\frac{\partial y}{\partial W})W(t)(r - \bar{1}r_0) = -\frac{\partial^2\Psi}{\partial W^2}W^2\sigma\omega(t)$$

Then, one knows

$$\sigma\omega(t) = -(\frac{\Psi_w}{W\Psi_{ww}} + \frac{\Psi_y y_w}{W\Psi_{ww}})(r - \bar{1}r_0)$$

Thus, we get

$$\omega(t) = -\frac{\Psi_w}{W\Psi_{ww}}\sigma^{-1}(r - \bar{1}r_0) - \frac{\frac{\partial\Psi}{\partial y}\frac{\partial y}{\partial W}}{W\Psi_{ww}}\sigma^{-1}(r - \bar{1}r_0). \quad (4.18)$$

This completes the proof of Theorem 4.1. □

Although, we have calculated the solution form of the optimal investment- consumption problem, the explicit optimal solution is difficult to find, from (4.18) we know that the portfolio $\omega(t)$ is a proportion of risk assets in continuous consumption-investment choices, it is no relation to the investment. This also illustrates that the different investors hold the same risk asset portfolio in equilibrium, and the coefficient $-\frac{\Psi_w}{\Psi_{ww}}$ is an inverse of the absolute risk aversion, that is, an optimal portfolio is dependent on the degree of risk aversions.

Remark 4.1 *Traditional continuous time optimal investment-consumption portfolios are*[2] *as follows*

$$\omega_i^*(t) = -\frac{\Psi_w}{\Psi_{ww}}\sum_{j=1}^{n}\sigma_{ji}^{-1}(r_j - r)$$

Based on the non-monotone utility function being revised to a monotone utility function, the solution of investment-consumption problem, in this paper, is given by

$$\omega(t) = -\frac{\Psi_w}{\Psi_{ww}}\sigma^{-1}(r - \bar{1}r_0) - \frac{\frac{\partial\Psi}{\partial y}\frac{\partial y}{\partial w}}{w\Psi_{ww}}\sigma^{-1}(r - \bar{1}r_0).$$

This formula can be regarded as a natural generalization of that posed by [2].

Since the monotone utility function is decreasing of marginal utilities, that is, $\Psi_w > 0, \Psi_{ww} < 0$, thus we know that the coefficient $-\frac{\Psi_w}{\Psi_{ww}} > 0$ and $-\frac{\frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial w}}{w\Psi_{ww}} > 0$. Obviously, one has $\omega(t) \geq \omega^*(t)$, this implies that a investor associated with the global monotone utility functions more grasps the investment opportunity than the tradition investor associated with the mean-variance utility, so the investors with the global monotone utility functions can take more returns. Therefore, the results of the present paper are more close to the real marketplace, and further reduce the gap between Theory and Reality wield. In other words, the results given here have had certain actual meaningful.

Acknowledgments. This work was supported by the Foundation of Nanjing University of Science and Technology and the Natural Science Foundations of Province, China.

References

[1] G. Z. Liu, "Dynamic Programming - Theory and Application," Chengdu: Chengdu Science and Technology University Press, 1991

[2] Y. Shao, "Microeconomic Finance and Mathematic," Peking: Qinghua University Press, 2003

[3] Bigelow, J. P, "Consistency of mean-variance analysis and expected utility analysis: a complete characterization," *Economic Letters*, **43**, pp, 187-192, 1993

[4] P. H. Dybivg and J. E. Ingersoll, "Mean-Variance theory in complete markets", *The Journal of Business*, **55**, pp, 233-251, 1982

[5] L. G. Epstein, "A definition of uncertainty aversion," *Review of Economic Studies*, 579-608, 1999

[6] H. Markowitz, 'Portfolio selection," *Finance*, **7**, pp, 77-91, 1952

[7] R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economy Theory*, pp, 1373-413, 1971

[8] R. C. Merton, "Continuous-time Finance," Oxford: Blackwell Publishers, 1990

[9] F. Maccheroni, M. Marinacci and A. Rustichini, "A variational formula for the relative Gini concentration index," http://web.econ.unito.it/gma/fabio/mmrr.pdf, 2006

[10] F. Maccheroni, M. Marinacci, and A. Rustichini, “Variational representation of preferences under ambiguity”, *J. of Economic Theory*, preprint, 2004

[11] F. Maccheroni, M. Marinacci, A. Rustichini and M. Taboga, “ Portfolio Selection with Monotone Mean-Variance Preferences,” *J. of Economic Theory*, pp, 973-982, 2005

[12] F. Maccheroni, M. Marinacci and A. Rustichini, “Ambiguity aversion, Robustness and the Variational Representation of Preferences”, *Econometrica*, **74**:6, pp, 1447-1498, 2006

[13] Zhongfei Li, Hailiang Yang and Xiao-tie Deng. “An exact solution to a continuous-time portfolio selection problem with EAR”, *Lecture Notes in Decision Sciences*, **2**, pp, 75-83, 2003

[14] D. Li and W. Ng, “Optimal dynamic portfolio selection: Multi-period mean variance formulation,” *Mathematical Finance*, **10**:3, pp, 387-406, 2003

[15] Chungfang Yang and Peibiao Zhao, “The Portfolio Problem Under the Variable Rate of Transaction Costs,” to appear in Bulletin of the Korean Mathematical Society.

[16] Shi Juan and Peibiao Zhao, “The geometric anslysis of portfolio selections based on Shanghai-50ETF,” *System Enginnering*, **25**:I, pp, 117-119, 2007

Received: December 21, 2007