Applied Mathematical Sciences, Vol. 2, 2008, no. 30, 1493 - 1506

Remarks on Self-Similar Solutions to the Compressible Navier-Stokes Equations of a 1D Viscous Polytropic Ideal Gas

Yuming Qin

Department of Applied Mathematics, College of Science
Donghua University, Shanghai 200051, P.R. China
yuming@dhu.edu.cn

Xiaoke Su and Shuxian Deng

College of Information Science and Technology
Donghua University, Shanghai 200051, P. R. China
suxiaoke@mail.dhu.edu.cn dsx@mail.dhu.edu.cn

Abstract

This paper is concerned with the self-similar solutions to the compressible Navier-Stokes equations of a 1D viscous polytropic ideal gas. Our results show that there exist neither forward nor backward self-similar solutions with finite total energy, which generalizes the results for the case of the isothermal compressible Navier-Stokes equations in Z. Guo and S. Jiang (Self-similar solutions to the isothermal compressible Navier-Stokes, IMA J. Appl. Math. (2006)71, 658-669).

Keywords: self-similar solutions, Navier-Stokes equations, viscous polytropic gas

1 Introduction

In this paper, we shall study the self-similar solutions to the compressible Navier-Stokes equations of a $1D$ viscous polytropic ideal gas. The importance of the study of such kinds of solutions has been addressed in Leray [7]. Many authors studied the self-similar solutions for some models of the Navier-Stokes (see also, [1,2,3,4,5,6,11,17]), among which we would like to mentions the work by Guo and Jiang [6] where they proved that there exists neither forward nor backward self-similar solutions with finite total energy to the isothermal compressible Navier-Stokes equations in one space dimension.

To our knowledge, there have been few studies on self-similar solutions to the compressible Navier-Stokes system partially due to the complicated nonlinearities arising from both the nonlinear convection and the pressure as well as their interactions.

In this paper, we generalize the above results of Guo and Jiang [6] to the compressible Navier-Stokes equations of a 1D viscous polytropic gas. More precisely, we consider the Navier-Stokes equations for a 1D viscous polytropic gas,

$$\rho_t + (\rho v)_x = 0, \tag{1.1}$$
$$(\rho v)_t + (\rho v^2)_x = -P_x + \mu v_{xx}, \tag{1.2}$$
$$C_v\{(\rho\theta)_t + (\rho v\theta)_x\} = k\theta_{xx} - Pv_x + \mu v_x^2, \tag{1.3}$$
$$t = 0: \quad \rho = \rho_0(x), \quad v = v_0(x), \quad \theta = \theta_0(x), \; x \in R \tag{1.4}$$

where ρ, v, θ denote the density, velocity and absolute temperature, respectively; $P = \gamma\rho\theta$ is the pressure, and μ, C_v, k and γ are positive constants.

First we shall introduce the definition of H^1-solutions to the problem (1.1)-(1.4).

Definition 1 For a fixed constant $T > 0$ and some positive constants $\bar{\rho}, \bar{\theta}$, if $\rho_0 - \bar{\rho}, v_0, \theta_0 - \bar{\theta} \in H^1(R)$ and $\rho_0(x), \theta_0(x) > 0$ on R, and there exists a unique global (large) solution $(\rho(t), v(t), \theta(t))$ with positive $\rho(x,t)$ and $\theta(x,t)$ to the Cauchy problem (1.1)-(1.4) on $R \times [0, +\infty)$ such that for any $T > 0$,

$$\rho - \bar{\rho}, v, \theta - \bar{\theta} \in L^\infty([0,T], H^1(R)), \quad u_t \in L^\infty((0,T), L^2(R)), \tag{1.5}$$
$$v_t, \rho_x, \theta_t, \rho_{xt}, v_{xx}, \theta_{xx} \in L^2((0,T), L^2(R)), \tag{1.6}$$

then $(\rho(t), v(t), \theta(t))$ is said to be an H^1-generalized global solution to the Cauchy problem (1.1)-(1.4).

Now let us recall some related results for equations (1.1)-(1.3) in the literature. For the one-dimensional Cauchy problem (1.1)-(1.4), Kanel [8] obtained the global existence and large-time behavior (only for v, θ) of H^1-solutions with small initial data. Kazhikhov and Shelukhin [9] proved the existence of H^1-generalized global solutions to the Cauchy problem (1.1)-(1.4). It is noteworthy that there is no any result on asymptotic behavior given in [9]. In this case, Okada and Kawashima [12] proved the global existence and large-time

behavior of classical (or H^1) solution with weighted small initial data. Jiang [7] proved the large-time behavior of H^1-solution with weighted small initial data. Recently, Qin et al [15] established the global existence and continuous dependence on initial data of $H^i(R),(i=1,2,4)$ (global) solutions for large initial data and showed the large-time behavior of these $H^i(R),(i=2,4)$ solutions for "small initial data".
The aim of this paper is to show that there exist neither forward nor backward self-similar H^1-generalized (global) solutions with finite total energy to the Cauchy problem (1.1)-(1.4) (for the definitions of $H^i(R)-$solutions $(i=2,4)$, we refer to Qin et al [15]).

It is worthy to point out here that since the domain is unbounded, the Poincaré inequality can not be applied to this domain, and further the large-time behavior of large initial data and the decay rate can not be anticipated. This is why we only established the large-time behavior of solutions only with "small initial data" and no decay rate was given in the above results (see, e.g., Qin et al [15], Jiang [7], Kanel [8], Okada and Kawashima [12]).

Leray's forward self-similar solutions are of the following form:

$$\rho(x,t)=\frac{1}{t}Q\left(\frac{x}{t}\right),v(x,t)=V\left(\frac{x}{t}\right),\theta(x,t)=\Theta\left(\frac{x}{t}\right),\quad x\in R,\quad t>0 \tag{1.7}$$

and Leray's backward self-similar solutions are of the following form:

$$\begin{aligned}&\rho(x,t)=\frac{1}{T-t}Q\left(\frac{x}{T-t}\right),v(x,t)=V\left(\frac{x}{T-t}\right),\\&\theta(x,t)=\Theta\left(\frac{x}{T-t}\right),\quad x\in R,\quad 0<t<T\end{aligned} \tag{1.8}$$

where $Q(y),v(y)$ and $\Theta(y)$ are defined in R. Any self-similar solution of the equation (1.1)-(1.4) must be either forward self-similar or backward self-similar solutions. Then for any constant C, the equations (1.1)-(1.3) for (ρ,v,θ) give the systems

$$Q(y)(V(y)-y)=C, \tag{1.9}$$
$$[Q(y)V(y)(V(y)-y)]'=[\mu V'(y)-\gamma Q(y)\Theta(y)]', \tag{1.10}$$
$$C_v[Q(y)\Theta(y)(V(y)-y)]'=k\Theta''(y)-V'(y)[\mu V'(y)-\gamma Q(y)\Theta(y)] \tag{1.11}$$

and

$$Q(y)(V(y)+y)=C, \tag{1.12}$$
$$[Q(y)V(y)(V(y)+y)]'=[\mu V'(y)-\gamma Q(y)\Theta(y)]', \tag{1.13}$$
$$C_v[Q(y)\Theta(y)(V(y)+y)]'=k\Theta''(y)-V'(y)[\mu V'(y)-\gamma Q(y)\Theta(y)]. \tag{1.14}$$

This paper is organized as follows. In Section 2, we give some main result. In Section 3, we shall complete the proof of main results.

2 Main Results

In this section, we shall give the notation. As pointed in Section 1, Kazhikhov and Shelukhin [9] (see also Qin [13]) proved the existence of H^1-generalized global solutions to the Cauchy problem (1.1)-(1.4). More precisely,

Lemma 2.1 *For some positive constants $\bar{\rho}, \bar{\theta}$, if $\rho_0 - \bar{\rho}, v_0, \theta_0 - \bar{\theta} \in H^1(R)$ with $\rho_0(x), \theta_0(x) > 0$ on R, then there exists a unique global (large) solution $(\rho(t), v(t), \theta(t))$ with positive $\rho(x,t)$ and $\theta(x,t)$ to the Cauchy problem (1.1)-(1.4) on $R\times[0,+\infty)$ such that for any $T>0$, (1.5)-(1.6) hold and there exists a constant $C_1(T) > 0$, such that for any $(x,t) \in R \times [0,T]$,*

$$C_1^{-1}(T) \leq \rho(x,t), \;\; \theta(x,t) \leq C_1(T). \tag{2.1}$$

Lemma 2.2 *Let (ρ, v, θ) be the H^1-generalized global solutions to (1.1)-(1.4). Then for any $T > 0$,*

$$\sup_{0\leq t\leq T}\left[E(t) + \mu\int_0^t\int_R\left(\frac{k\theta_x^2}{\theta^2} + \frac{\mu v_x^2}{\theta}\right)dxds\right] \leq 2E(0) \tag{2.2}$$

where

$$E(t) = \int_R\left[\frac{1}{2}\rho v^2 + C_v\rho(\theta - \log\theta - 1) + \gamma\rho\left(\frac{1}{\rho} - \log\frac{1}{\rho} - 1\right)\right](x,t)dx. \tag{2.3}$$

Proof By (2.3) and using (1.1)-(1.4), we obtain after a straightforward calculation that

$$\begin{aligned}\frac{dE(t)}{dt} &= \int_R[\frac{1}{2}\rho_t v^2 + \rho v v_t + C_v\rho_t(\theta - \log\theta - 1) + C_v\rho\left(\theta_t - \frac{\theta_t}{\theta}\right) \\ &\quad +\gamma\rho_t\left(\frac{1}{\rho} - \log\frac{1}{\rho} - 1\right) + \gamma\rho\left(-\frac{\rho_t}{\rho^2} + \frac{\rho_t}{\rho}\right)]dx \\ &= -\int_R\left(\frac{k\theta_x^2}{\theta^2} + \frac{\mu v_x^2}{\theta}\right)dx\end{aligned}$$

which, by integrating with respect to t, gives

$$E(t) + \int_0^t\int_R\left[\frac{k\theta_x^2}{\theta^2} + \frac{\mu v_x^2}{\theta}\right](x,s)dxds = E(0).$$

Hence for any $T > 0$,

$$\begin{aligned}&\sup_{0\leq t\leq T}\left[E(t) + \int_0^t\int_R\left[\frac{k\theta_x^2}{\theta^2} + \frac{\mu v_x^2}{\theta}\right](x,s)dxds\right] \\ \leq\; & 2\max\left\{\sup_{0\leq t\leq T}E(t), \int_0^t\int_R\left[\frac{k\theta_x^2}{\theta^2} + \frac{\mu v_x^2}{\theta}\right](x,s)dxds\right\} \leq 2E(0)\end{aligned}$$

which proves (2.2).
It follows from (2.1) that $Q(y) > 0$ for any $y \in R$ if the self-similar solutions in (1.7) (or (1.8)) satisfy the global-energy estimate (2.2).
Similarly, we can define the local-energy estimate, which is bounded from above, by

$$\sup_{t_1 \le t \le T} \int_{-R}^{R} \left[\frac{1}{2}\rho v^2 + C_v \rho(\theta - \log\theta - 1) + \gamma\rho\left(\frac{1}{\rho} - \log\frac{1}{\rho} - 1\right)\right] dx \quad (2.4)$$
$$+ \int_{t_1}^{T}\int_{-R}^{R} \left[\frac{k\theta_x^2}{\theta^2} + \frac{\mu v_x^2}{\theta}\right](x,s)dxds \le C(T), 0 < t_1 < T, R > 0.$$

□

We are now in a position to state our main theorem.

Theorem 2.1 *1. There is no forward (backward) self-similar solution to (1.1)-(1.4) that satisfies the global-energy estimate (2.2) with* $T = +\infty$.
2. If there is a forward (backward) self-similar solution that satisfies the local-energy estimate (2.4), then as $t \downarrow 0^+ (t \uparrow T^-)$, *the total energy blows up.*

3 Self-similar Solutions

In this section, we study the blow-up of the total energy by means of self-similar solutions.

Lemma 3.1 *If* (Q, V, Θ) *is a solution of (1.9)-(1.11) with* $C = 0$, *then* (ρ, v, θ) *defined by (1.7) does not satisfy the global-energy estimate (2.2) with* $T = +\infty$ *and* $\int_0^\infty \int_R \frac{\theta_x^2}{\theta^2} dxdt = +\infty$ *or* $\int_0^\infty \int_R \frac{v_x^2}{\theta} dxdt = +\infty$.

Proof Inserting (1.9) into (1.10)-(1.11), we obtain

$$CV'(y) + \gamma(Q(y)\Theta(y))' = \mu V''(y), \quad (3.1)$$
$$CC_v\Theta'(y) = k\Theta''(y) + V'(y)[\gamma Q(y)\Theta(y) - \mu V'(y)]. \quad (3.2)$$

If $C = 0$, then by (1.9), we have

$$V(y) = y. \quad (3.3)$$

By (3.1), we get $(Q\Theta)' = 0$, and hence

$$Q(y)\Theta(y) = C_1, \quad \forall y \in R \quad (3.4)$$

where C_1 is an arbitrary constant.
By (3.2)-(3.4), we have

$$\Theta'' = \frac{\mu - \gamma C_1}{k}$$

which implies

$$\Theta(y) = ay^2 + by + C_2, \quad \forall y \in R \tag{3.5}$$

where $a = \frac{\mu-\gamma C_1}{2k}, b, C_2 \in R$ are constants.
Thus it follows from (3.3)-(3.5) that

$$v(x,t) = \frac{x}{t}, \quad \theta(x,t) = a\frac{x^2}{t^2} + \frac{bx}{t} + C_2, \quad \forall x \in R, \quad t > 0.$$

Noting that (2.1), we conclude that $\Theta(y) > 0, \forall y \in R$.
Thus from (3.5) we only consider the following two cases:

$$(i) \;\; a > 0, \;\; 4aC_2 - b^2 > 0; \quad (ii) \;\; a = 0, b = 0, C_2 > 0. \tag{3.6}$$

We claim that

$$I \equiv \int_0^\infty \int_R \frac{v_x^2}{\theta}(x,t)dxdt = +\infty, \tag{3.7}$$

$$J \equiv \int_0^\infty \int_R \frac{\theta_x^2}{\theta^2}(x,t)dxdt = +\infty. \tag{3.8}$$

In fact, we know from (3.3) and (3.5)-(3.6) that

$$v(x,t) = \frac{x}{t}, \quad \forall x \in R, \quad t > 0$$

and

$$\theta(x,t) = \frac{ax^2 + btx + C_2t^2}{t^2} \quad or \quad \theta(x,t) = C_2, \quad \forall x \in R, t > 0.$$

For case (ii), we have $\theta(x,t) = C_2, v(x,t) = \frac{x}{t}$. Thus we easily deduce that

$$I = \int_0^\infty \int_R \frac{v_x^2}{\theta} dxdt = \frac{1}{C_2}\int_0^\infty \int_R \frac{1}{t^2} dxdt = +\infty.$$

For case (i), we have $v(x,t) = \frac{x}{t}, \theta(x,t) = \frac{ax^2+btx+C_2t^2}{t^2}$.
Let $A = \frac{\sqrt{4aC_2-b^2}}{2a}, x' = x + \frac{bt}{2a}$, then after a straightforward calculation, we infer that

$$\begin{aligned} I &\equiv \int_0^\infty \int_R \frac{v_x^2}{\theta}(x,t)dxdt = \int_0^\infty \int_R \frac{1}{ax^2 + btx + C_2t^2} dxdt \\ &= \frac{1}{a}\int_0^\infty \int_R \frac{dxdt}{\left[x + \frac{bt}{2a}\right]^2 + \frac{b^2-4aC_2}{4a^2}t^2} = \frac{1}{a}\int_0^\infty \int_R \frac{dx'dt}{x'^2 - A^2t^2} \\ &= +\infty. \end{aligned}$$

Similarly,

$$
\begin{aligned}
J &\equiv \int_0^\infty \int_R \frac{\theta_x^2}{\theta^2}(x,t)dxdt \\
&= \int_0^\infty \int_R \frac{(2ax+bt)^2}{(ax^2+btx+C_2t^2)^2}dxdt \\
&= 4\int_0^\infty \int_R \frac{1}{x^2-(At)^2}dxdt + 4A^2\int_0^\infty \int_R \frac{t^2}{[x'^2-A^2t^2]^2}dxdt \\
&= +\infty.
\end{aligned}
$$

The proof is complete. □

Lemma 3.2 *If $C \neq 0$ in (1.9), then we have the following result. If (Q, V, Θ) is a solution to (1.9)-(1.11) with $C \neq 0$, and (ρ, v, θ) defined by (1.7) satisfy the global-energy estimate (2.2), then the total energy of (1.1)-(1.4) must blow up, i.e., no forward self-similar solutions satisfy the global-energy estimate (2.2).*

Proof If (Q, V, Θ) solves (1.9)-(1.11), then it satisfies (1.9),(3.1) and (3.2). If $C \neq 0$, then

$$V(y) = y + \frac{C}{Q(y)}, \quad Q(y) > 0, \quad \forall y \in R.$$

Then for any $t > 0$,

$$
\begin{aligned}
E(t) &\geq \int_R \frac{1}{2}\rho(x,t)v^2(x,t)dx = \frac{1}{2}\int_R \frac{1}{t}Q\left(\frac{x}{t}\right)V^2\left(\frac{x}{t}\right)dx \\
&= \int_R \frac{1}{2}Q(y)V^2(y)dy = \frac{1}{2}\int_R Q(y)\left(y + \frac{C}{Q(y)}\right)^2 dy \\
&= \frac{1}{2}\int_R \left(Q(y)y^2 + 2Cy + \frac{C^2}{Q(y)}\right)dy \geq \int_R (Cy + |Cy|)dy = +\infty
\end{aligned}
$$

which completes the proof of the lemma. □

Lemma 3.3 *If (Q, V, Θ) is a solution of (1.9)-(1.11) with $C = 0$, then there exist $t_1 > 0, T > 0$ such that (ρ, v, θ) defined by (1.7) satisfies the local-energy estimate (2.4). But the total energy must blow up as proved in Lemma 3.1.*

Proof If $C = 0$ in (1.9), then a straightforward calculation yields

$$I' = \int_{t_1}^T \int_{-R}^R \frac{v_x^2}{\theta}dxdt = \frac{1}{a}\int_{t_1}^T \int_{-R+\frac{bt}{2a}}^{R+\frac{bt}{2a}} \frac{1}{x'^2 - A^2t^2}dx'dt$$

$$
\begin{aligned}
&= \int_{t_1}^{T} \frac{1}{\sqrt{4aC_2-b^2}t}\left[\log\left|\frac{2aR+(b-\sqrt{4aC_2-b^2})t}{2aR+(b+\sqrt{4aC_2-b^2})t}\right| - \log\left|\frac{-2aR+(b-\sqrt{4aC_2-b^2})t}{-2aR+(b+\sqrt{4aC_2-b^2})t}\right|\right]dt\\
&= \int_{t_1}^{T} \frac{1}{\sqrt{4aC_2-b^2}t}\log\left|\frac{[2aR+(b-\sqrt{4aC_2-b^2})t][-2aR+(b+\sqrt{4aC_2-b^2})t]}{[2aR+(b+\sqrt{4aC_2-b^2})t][-2aR+(b-\sqrt{4aC_2-b^2})t]}\right|dt\\
&= \frac{1}{\sqrt{4aC_2-b^2}}\int_{t_1}^{T}\frac{1}{t}\log\left|\frac{b^2t^2-(2aR-\sqrt{4aC_2-b^2}t)^2}{b^2t^2-(2aR+\sqrt{4aC_2-b^2}t)^2}\right|dt.
\end{aligned}
$$

We discuss the following two cases:
(i) When $\pm b-\sqrt{4aC_2-b^2}\neq 0, t_0=\frac{2aR}{\pm b-\sqrt{4aC_2-b^2}}$ satisfies

$$b^2t_0^2-(2aR+\sqrt{4aC_2-b^2}t_0)^2=0,$$

thus we may choose $t_1>0$ such that $t_1>\frac{2aR}{|\pm b-\sqrt{4aC_2-b^2}|}$ and hence as $t_1<t<T$, $b^2t^2-(2aR+\sqrt{4aC_2-b^2}t)^2\neq 0$. Therefor, $I'\leq C(T)$.

(ii) When $\pm b-\sqrt{4aC_2-b^2}=0$, i.e., $2b^2=4aC_2$, hence for any $t_1>0$, as $t\in(t_1,T), b^2t^2-(2aR+\sqrt{4aC_2-b^2}t)^2\neq 0$.
Therefore $I'\leq C(T)$.
Similarly,

$$
\begin{aligned}
J' &= \int_{t_1}^{T}\int_{-R}^{R}\frac{\theta_x^2}{\theta^2}dxdt\\
&= 4I'+16A^2\int_{t_1}^{T}t^2\int_{-R+\frac{bt}{2a}}^{R+\frac{bt}{2a}}\frac{dxdt}{x^2-A^2t^2}.
\end{aligned}
\tag{3.9}
$$

By the above argument, we conclude

$$I'\leq C(T). \tag{3.10}$$

On the other hand, letting $A=\frac{\sqrt{4aC_2-b^2}}{2a}, y=\frac{x}{At}, y_1=\frac{-R+\frac{bt}{2a}}{At}, y_2=\frac{R+\frac{bt}{2a}}{At}$, we have

$$
\begin{aligned}
J_1 &= 16A^2\int_{t_1}^{T}t^2\int_{-R+\frac{bt}{2a}}^{R+\frac{bt}{2a}}\frac{dxdt}{x^2-A^2t^2}\\
&= 16A^3\int_{t_1}^{T}t^3\int_{y_1}^{y_2}\frac{dydt}{[y^2-1]^2}\\
&= 4A^3\int_{t_1}^{T}t^3\int_{y_1}^{y_2}\left[\frac{1}{(y-1)^2}+\frac{1}{(y+1)^2}-\left(\frac{1}{y-1}-\frac{1}{y+1}\right)\right]dydt\\
&= 4A^3\int_{t_1}^{T}t^3\left\{-\left[\frac{1}{y-1}+\frac{1}{y+1}\right]-\log\left|\frac{y-1}{y+1}\right|\right\}\Bigg|_{y_1}^{y_2}dt
\end{aligned}
$$

$$= 4A^3 \int_{t_1}^{T} t^3 \left\{ \frac{b^2 - 4aC_2}{(2aR+bt)^2 + b^2 - 4aC_2} + \frac{b^2 - 4aC_2}{(-2aR+bt)^2 + b^2 - 4aC_2} \right.$$
$$\left. - \log \left| \frac{(2aR - \sqrt{4aC_2 - b^2}t)^2 - b^2t^2}{(2aR + \sqrt{4aC_2 - b^2}t)^2 - b^2t^2} \right| \right\} dt. \qquad (3.11)$$

Therefore for any $T > \max\left\{ \left| \frac{\pm 2aR \pm \sqrt{4aC_2 - b^2}}{b} \right|, \left| \frac{\pm 2aR \mp \sqrt{4aC_2 - b^2}}{b} \right| \right\} = \hat{t_1}$, choosing $\max(0, \hat{t_1})$
$< t_1 < T$, we find that for any $t \in [t_1, T]$, the integrand in (3.11) is bounded by a positive constant $C_1(T)$.
Thus it follows from (3.9)-(3.11) that

$$J' \leq C(T).$$

The proof is complete. □

If $C \neq 0$ in (1.9), then we have the following lemma on the local-energy estimate (2.4).

Lemma 3.4 *If (Q, V, Θ) is a solution of (1.9)-(1.11) with $C \neq 0$ and (ρ, v, θ) defined by (1.7) satisfies the local-energy estimate (2.4) for $t > 0$, then as $t \downarrow 0^+$, the total energy of the self-similar solution must blow up.*

Proof By the definition of $E(t)$ and (2.4), we infer from (1.9) that for any $t > 0$,

$$\begin{aligned} E(t) &\geq \int_{-R}^{R} \frac{1}{2}\rho(x,t)v^2(x,t)dx = \int_{-R/t}^{R/t} \frac{1}{2}Q(y)V^2(y)dy \\ &= \int_{-R/t}^{R/t} \frac{1}{2}Q(y)\left[y + \frac{C}{Q(y)}\right]^2 dy \\ &= \int_{-R/t}^{R/t} \frac{1}{2}\left[Q(y)y^2 + 2Cy + \frac{C^2}{Q(y)}\right] dy \\ &\geq \int_{-R/t}^{R/t} (Cy + |Cy|)dy = \frac{R^2|C|}{t^2} \to +\infty, \quad as\ t \downarrow 0^+ \end{aligned}$$

which completes the proof of the lemma. □

4 Backward self-similar solutions

For (1.1)-(1.4), Leray's backward self-similar solutions are of the form (1.8), where $Q(y), V(y)$ and $\Theta(y)$ defined in R satisfy (1.12)-(1.14).

Similarly to the case of the forward self-similar solutions, inserting (1.12) into (1.13) and (1.14), we have

$$CV'(y) = \mu V''(y) - \gamma(Q(y)\Theta(y))' \tag{4.1}$$

$$CC_v\Theta'(y) = k\Theta''(y) - V'(y)[\mu V'(y) - \gamma Q(y)\Theta(y)]. \tag{4.2}$$

Thus if $C = 0$, we infer from (1.9), (4.1)-(4.2) that

$$V(y) = -y, \quad Q(y)\Theta(y) = C_1 \tag{4.3}$$

$$\Theta''(y) = \frac{\mu + \gamma C_1}{k}, \quad \Theta(y) = ay^2 + by + C_2 \tag{4.4}$$

where $a = \frac{\mu+\gamma C_1}{2k}, b, C, C_1 \in R$ are arbitrary constants.
By (2.1) in Lemma 2.1, we conclude $\Theta(y) > 0, \quad \forall y \in R$. Thus we have two cases:

$$(i) \;\; a > 0, \;\; b^2 - 4aC_2 < 0; \tag{4.5}$$

$$(ii) \;\; a = 0, \;\; b = 0, \;\; C_2 > 0. \tag{4.6}$$

For case (ii), we have $v(x,t) = -\frac{x}{t}, \;\; \theta(x,t) = C_2$,

$$\int_0^T \int_R \frac{v_x^2}{\theta} dxdt = \frac{1}{C_2} \int_0^T \int_R \frac{1}{t^2} dxdt = +\infty. \tag{4.7}$$

For case (i), we infer

$$v(x,t) = -\frac{x}{t}, \;\; \theta(x,t) = \frac{ax^2 + btx + C_2t^2}{t^2}, \;\; \forall x \in R, \;\; t > 0.$$

Letting $A = \frac{\sqrt{4aC_2 - b^2}}{2a}$, then we get

$$\begin{aligned} &\int_0^T \int_R \frac{v_x^2}{\theta} dxdt = \int_0^T \int_R \frac{dxdt}{ax^2 + btx + C_2t^2} \\ = &\;\; \frac{1}{a} \int_0^T \int_R \frac{dxdt}{x^2 - A^2t^2} = +\infty, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\int_0^T \int_R \frac{\theta_x^2}{\theta^2}(x,t) dxdt \\ = &\;\; 4\int_0^T \int_R \frac{dxdt}{x^2 - A^2t^2} + 4A^2 \int_0^T \int_R \frac{t^2}{(x^2 - A^2t^2)^2} dxdt \\ = &\;\; +\infty. \end{aligned} \tag{4.9}$$

Then we have proved the following lemma.

Lemma 4.1 *If* (Q, V, Θ) *is a solution of (1.12)-(1.14) with* $C = 0$*, then* (ρ, v, θ) *defined by (1.8) does not satisfy the global-energy estimate (2.2) with* $T = +\infty$*.*

The above lemma means that the total energy must blow up.
Similarly to Lemma 3.2, we easily prove the following lemma.

Lemma 4.2 *If* (Q, V, Θ) *is a solution of (1.12)-(1.14) with* $C = 0$*, then there exist* $t_1 > 0, T > 0$ *such that* (ρ, v, θ) *satisfies the local-energy estimate (2.4).*

Lemma 4.3 *If* (Q, V, Θ) *is a solution of (1.12)-(1.14) with* $C \neq 0$ *and* (ρ, v, θ) *defined by (1.8) satisfies the global-energy estimate (2.2), then the total energy of (1.1)-(1.4) must blow up, i.e.,no backward self-similar solutions satisfy the global-energy estimate (2.2).*

Proof If (Q, V, Θ) solves (1.12)-(1.14), then

$$V(y) = -y + \frac{C}{Q(y)}, \quad C \neq 0, \quad Q(y) > 0 \tag{4.10}$$

whence, for any $0 < t < T$,

$$\begin{aligned} E(t) &\geq \int_R \frac{1}{2}\rho(x,t)v^2(x,t)dx \\ &= \int_R \frac{1}{2(T-t)} Q\left(\frac{1}{T-t}\right) V^2 \left(\frac{1}{T-t}\right) dx \\ &= \int_R \frac{1}{2}\left[Q(y)y^2 - 2Cy + \frac{C^2}{Q(y)}\right] dy \\ &\geq \int_R (|Cy| - Cy)dy = +\infty. \end{aligned}$$

The proof is complete. □

Lemma 4.4 *If* (Q, V, Θ) *is a solution of (1.12)-(1.14) and* (ρ, v, θ) *defined by (1.8) satisfies the global-energy estimate (2.4) for* $0 < t < T$*, then as* $t \uparrow T^-$*, the total energy of (1.12)-(1.14) must blow up.*

Proof By the definition of $E(t)$, (2.4) and (4.10), we infer that for any $0 < t < T$,

$$E(t) \geq \int_{-R}^{R} \frac{1}{2}\rho(x,t)v^2(x,t)dx$$

$$
\begin{aligned}
&= \int_{\frac{-R}{T-t}}^{\frac{R}{T-t}} \frac{1}{2}Q(y)\left[-y+\frac{C}{Q(y)}\right]^2 dy \\
&= \int_{\frac{-R}{T-t}}^{\frac{R}{T-t}} \frac{1}{2}\left[Q(y)y^2-2Cy+\frac{C^2}{Q(y)}\right] dy \\
&\geq \int_{\frac{-R}{T-t}}^{\frac{R}{T-t}} (|Cy|-Cy)dy \\
&= \frac{R^2|C|}{(T-t)^2} \to +\infty \quad as\ t\uparrow T^-.
\end{aligned}
$$

□

Remark 4.1 The similar conclusion holds for the following model of compressible fluids between two horizontal parallel plates in R^3 (see, e.g., [14]):

$$
\begin{aligned}
&(\rho u)_t + (\rho u u)_x = -P_x + \nu u_{xx}, \\
&\rho_t + (\rho u)_x = 0, \\
&C_v\{(\rho\theta)_t + (\rho u\theta)_x\} = k\theta_{xx} - Pu_x + \nu u_x^2 + \mu|v_x|^2, \\
&(\rho\vec{v})_t + (\rho u\vec{v})_x = \mu\vec{v}_{xx}
\end{aligned}
$$

where $\nu = \lambda + 2\mu, 3\lambda + 2\mu \geq 0, \mu \geq 0$ for the viscosities λ and μ, ρ is the mass density, $P = \gamma\rho\theta$ is the pressure, θ is the temperature, μ is the vertical velocity component, and $\vec{v} = (v_1, v_2)$ is the two-dimensional vector of horizontal velocity in which v_1 and v_2 are the components along the directions y and z, respectively. This model describes the three-dimensional Navier-Stokes equations governing the flow under the assumption that, given a Cartesian coordinates system (x, y, z), the flow is independent of the horizontal variables y and z provided that the gravity force acts in the negative x-direction.
The energy under consideration takes the following form:

$$
E(t) = \int_R \left[\frac{1}{2}\rho(u^2 + |\vec{v}|^2) + C_v\rho(\theta - \log\theta - 1) + \gamma\rho(\frac{1}{\rho} - \log\frac{1}{\rho} - 1)\right] dx.
$$

ACKNOWLEDGEMENTS This work was supported in part by the NNSF of China (contract number 10571024).

References

[1] O. A. Barraza, Self-similar solutions in weak L^p-spaces of the Navier-Stokes equations, Rev.Mat.Iberoam 12(1996), 411-439.

[2] L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularlity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure. Appl. Math. XXXV(1982), 771-831.

[3] M. Cannone and F. Planchon, Self-similar solutions for Navier-Stokes equations in R^3, Comm. PDE, 21(1996), 179-193.

[4] S. V. Ershkov and V. V. Shchennikov, Self-similar solutions to the complete system of Navier-Stokes equations for axially symmetric suirlying viscous compressible gas flow, Comput. Math. Math. Phys., 41(2001).

[5] Y. Giga and T. Miyakawa, Navier-Stokes flow in R^3 with measures as initial vorticity and Morrey spaces, Comm. PDE, 14(1989), 577-618.

[6] Z. Guo and S. Jiang, Self-similar solutions to the isothermal compressible Navier-Stokes equations, IMA. J. Appl. Math. 71(2006), 658-669.

[7] S. Jiang, Large-time behavior of solutions to the equations of a viscous polytropic ideal gas, Ann. Mat. Pura. Appl., CLXXV(1998), 253-275.

[8] Y. I. Kanel, Cauchy problem for the equations of gas dynamics with viscosity, Siberian math. J., 20(1979), 208-218.

[9] A. V. kazhikhov and V. V. Shelukhin, Unique global solution with respect to time of initial boundary value problems for one-dimensional equations of a viscous gas, J. Appl. Mech., 41(1977), 273-282.

[10] J. Leray, Sur le mouvement dun liquide visqueux emplissant lespace, Acta. Math. 63(1934), 193-248.

[11] J. Nečas, M. Ružič and v. Šverǎk, On Leray's self-similar solutions of the Navier-Stokes equations, Acta. Math., 176(1996), 283-294.

[12] M. Okada and S. Kawashima, On the equations of one-dimensional motion of compressible viscous fluids, J. Math. kyoto univ., 23(1983), 55-71.

[13] Y. Qin, Global existence and asympototic behavior of solutions to nonlinear hyperbolic-parabolic coupled systems with arbitrary initial data, Ph. D. Thesis, Fudan University (1998).

[14] Y. Qin and J. E. Muñoz Rivera, Exponential stability and universal attractors for the Navier-Stokes equations of compressible fluids between two horizontal parallel plates in R^3, Appl. Num. Math., 47(2003), 209-235.

[15] Y. Qin, Y. Wu and F. Liu, On the Cauchy problem for a one-dimensional compressible viscous polytropic ideal gas, Portugaliae Math., 64(2007), 87-126.

[16] T. P. Tasi, On Leray's self-similar solutions of the Navier-Stokes equations satisfying local energy estimates, Arch. Ration. Mech. Anal., 143(1998), 29-51.

[17] G. Tian and Z. Xin, Gradient estimation on Navier-Stokes equations, Comm. Anal. Geom., 7(1999), 221-257.

Received: December 8, 2007