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The Extended Data Envelopment Analysis/Discriminant Analysis Approach of Models with Undetermined Restrictions

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Abstract

To represent an optimization problem as a linear programming (LP), there are several assumptions needed. Among them, there is the "Deterministic" assumption that requires all the coefficients of the LP to be known deterministically. Therefore any probabilistic or stochastic element inherent in demand, costs, prices, resources and so on is needed to be approximated through some deterministic equivalent. In the cases of the classification of observations, there comes up the time when the observations (or the technological coefficients of the classification model) are inherently probabilistic. This paper represents a model that classifies such observations into two groups by minimizing the total deviation of the misclassified observations from the discriminant function. Defining the discriminant function classifying the observations, it is also possible to determine the group membership of a newly sampled probabilistic observation.

Keywords: Extended data envelopment analysis (DEA)-discriminant analysis (DA), probability, linear programming

1 Introduction

Discriminant Analysis (DA) is a method that can predict the group membership of a newly sampled observation. In a use of DA a group of observations

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whose memberships are already known are used for the estimation of weights (or parameters) of a discriminant function by some criteria such as the minimization of misclassifications or the maximization of correct classifications. A newly sampled observation is classified into one of the groups by comparing its discriminant score with the evaluation score derived from the discriminant function. Sueyoshi (1999) and Sueyoshi and Kirihara (1998) proposed a new type of non-parametric DA approach that could provide a set of weights for a linear discriminant function and consequently yield an evaluation score for the determination of the group membership of the observations. The non-parametric DA is referred to as DEA/DA. However, this method is unable to deal with negative data. In order to overcome that Sueyoshi (2001) proposed another version of DEA/DA referred to as Extended DEA/DA that is formulated in a manner that can minimize the total deviations of misclassified observations.

As an extension of the study above, this research proposes a new type of extended DEA/DA that can deal with problems possessing probabilistic observation values. To study this, in section two, part one, concepts on extended DEA/DA models is presented and in section two part two a brief basis on probability in mathematical programming is reviewed. In section three, the formulation of the new method is presented. In section four, there is an illustrative example. And finally in section five, a concluding comment on the studied model is presented.

2 Backgrond

2.1 Concepts on extended DEA/DA models

Extended DEA/DA is a non-parametric method that represents a linear separating function (instead of a separating hyperplane) and can deal with negative/positive values of the weights related to each observation factor. Mathematically this method is formulated in two stages as follows:

Stage-One: Classification and Overlap Identification (COI)

Stage-Two: Handling Overlap (HO).

Stage-One: Classification and Overlap Identification (COI):

Suppose that there are n observations each with m independent factors denoted by $z_{ij}, i = 1, \dots, k, j = 1, \dots, n$. These observations can be classified into the two groups G_1 and G_2, each with n_1 and n_2 observations, respectively. It is also assumed that $n_1 + n_2 = n$ and $G_1 \cup G_2 = G$. At this stage we intend to find the linear function that can classify the observations most powerfully. Also it is at this stage that we clarify if there exists any observation belonging to the overlap area. Stage-one of the extended DEA/DA model is formulated as follows:

$$
\begin{aligned}
\min \quad & \textstyle\sum_{j\in G_1} s_{1j}^{+} + \sum_{j\in G_2} s_{2j}^{-} \\
S.t \quad & \textstyle\sum_{i=1}^{k} \lambda_i z_{ij} + s_{1j}^{+} - s_{1j}^{-} = d+1 \quad && j \in G_1 \\
& \textstyle\sum_{i=1}^{k} \lambda_i z_{ij} + s_{2j}^{+} - s_{2j}^{-} = d && j \in G_2 \\
& \textstyle\sum_{i=1}^{k} |\lambda_i| = 1 \\
& s_{1j}^{+}, s_{1j}^{-}, s_{2j}^{+}, s_{2j}^{-} \geq 0 && j = 1, \ldots, n
\end{aligned}
\tag{1}
$$

In the model above s_{1j}^{+}, s_{1j}^{-} represent the positive and negative deviations of the G_1 members from $\sum_{i=1}^{k}$ and s_{2j}^{+}, s_{2j}^{-} represent the positive and negative deviations of the G_2 members from $sum_{i=1}^{k}$, respectively. The discriminant score is represented by d and the weights related to each observation factor by λ_i. The model obtained above is non-linear. To transfer it to an equivalent linear model we substitute $|\lambda_i|$ and λ_i by:

$$
\begin{aligned}
|\lambda_i| = \lambda_i^{+} + \lambda_i^{-} \quad & \lambda_i = \lambda_i^{+} - \lambda_i^{-} \\
\lambda_i^{-} = \tfrac{|\lambda_i| - \lambda_i}{2} \qquad & \lambda_i^{+} = \tfrac{|\lambda_i| + \lambda_i}{2}
\end{aligned}
\tag{2}
$$

Therefore $\lambda_i^{+} \geq 0$ and $\lambda_i^{-} \geq 0$ and

$$
\lambda_i^{+} = \begin{cases} \lambda_i, & \lambda_i \geq 0 \\ 0 & \lambda_i < 0 \end{cases}, \lambda_i^{-} = \begin{cases} 0 & \lambda_i \geq 0 \\ -\lambda_i & \lambda_i < 0 \end{cases}
$$

So the model is changed to the linear model of:

$$
\begin{array}{lll}
min & \sum_{j\in G_1} s_{1j}^{+} + \sum_{j\in G_2} s_{2j}^{-} & \\
S.t & \sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij} + s_{1j}^{+} - s_{1j}^{-} = d+1 & j\in G_1 \\
 & \sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij} + s_{2j}^{+} - s_{2j}^{-} = d & j\in G_2 \\
 & \sum_{i=1}^{k}(\lambda_i^{+}+\lambda_i^{-}) = 1 & \\
 & s_{1j}^{+}, s_{1j}^{-}, s_{2j}^{+}, s_{2j}^{-} \geq 0 & j=1,\ldots,n \\
 & \lambda_i^{+}, \lambda_i^{-} \geq 0 & i=1,\ldots,k
\end{array}
\tag{3}
$$

Now let $\lambda_i^{+^*} - \lambda_i^{-^*} = \lambda_i^*$ and d^* be the optimal solutions obtained from stage one. The new observation z_m is classified as follows:
1. If $\sum_{i=1}^{k} \lambda_i^* z_{im} \geq d^*+1$,then the new observation belongs to G_1.
2. If $\sum_{i=1}^{k} \lambda_i^* z_{im} \leq d^*$, then the new observation belongs to G_2.
3. If $d^* < \sum_{i=1}^{k} \lambda_i^* z_{im} < d^*+1$, then the new observation belongs to $G_1 \cap G_2$.
Define:

$$
\begin{array}{l}
R_1 = \{j \in G : \sum_{i=1}^{k} \lambda_i^* z_{ij} \geq d^*+1\} \\
R_2 = \{j \in G : \sum_{i=1}^{k} \lambda_i^* z_{ij} \leq d^*\} \\
R_0 = \{j \in G : d^* < \sum_{i=1}^{k} \lambda_i^* z_{ij} < d^*+1\} \\
C_1 = \{j \in R_1 : j \in G_1\} \\
C_2 = \{j \in R_2 : j \in G_2\}
\end{array}
\tag{4}
$$

Therefore C_1 and C_2 are the sets including those members of G_1 and G_2 that are correctly classified. Also

$$
\begin{array}{l}
G = G_1 \cup G_2 = R_1 \cup R_0 \cup R_2 \\
G_1^{'} = G_1 \cap [R_0 \cup R_2], G_2^{'} = G_2 \cap [R_0 \cup R_1]
\end{array}
\tag{5}
$$

Stage-Two: Handling Overlap (HO)

Ay the end of stage one there may remain some observations belonging to $G_1^{'}, G_2^{'}$ (the overlap area). Since the group membership of theses observations is not know, the second stage starts. At this stage the observations that are correctly classified are form the first and fourth constraints. The second and third constraints try to define a more effective discriminant function. The model is formulated as follows:

$$
\begin{array}{lll}
min & \sum_{j \in G_1'} s_{1j}^+ + \sum_{j \in G_2'} s_{2j}^- & \\
S.t & \sum_{i=1}^{k} (\lambda_i^+ - \lambda_i^-) z_{ij} \geq d+1 & j \in C_1 \\
 & \sum_{i=1}^{k} (\lambda_i^+ - \lambda_i^-) z_{ij} + s_{1j}^+ - s_{1j}^- = c & j \in G_1' \\
 & \sum_{i=1}^{k} (\lambda_i^+ - \lambda_i^-) z_{ij} + s_{2j}^+ - s_{2j}^- = c & j \in G_2' \\
 & \sum_{i=1}^{k} (\lambda_i^+ - \lambda_i^-) z_{ij} \leq d & j \in C_2 \\
 & \sum_{i=1}^{k} (\lambda_i^+ + \lambda_i^-) = 1 & \\
 & d \leq c \leq d+1 & \\
 & s_{1j}^+, s_{1j}^-, s_{2j}^+, s_{2j}^- \geq 0 & j = 1, \ldots, n \\
 & \lambda_i^+, \lambda_i^- \geq 0 & i \in G_1' \cup G_2'
\end{array}
\qquad (6)
$$

Now let $\lambda_i^* (= \lambda_i^{+^*} - \lambda_i^{-^*})$ and c^* be the optimal solutions obtained from the model above. The new observation z_m is classified as follows:

1. If $\sum_{i=1}^{k} \lambda_i^* z_{im} \geq c^*$,then the new observation belongs to G_1.
2. If $\sum_{i=1}^{k} \lambda_i^* z_{im} \leq c^*$,then the new observation belongs to G_2.
3. If $c^* < \sum_{i=1}^{k} \lambda_i^* z_{im} < c^* + 1$, then the new observation belongs to $G_1 \cap G_2$

2.2 A brief review of probability in mathematical programming

When solving a linear programming, all the parameters A, b and c are determined. If there happens to exist a case in which any of the parameters mentioned above give up determinacy, the linear programming case changes to a probabilistic model for which the equality/inequality of the constraints can't be fully determined. Now let's consider we face a case in which the technological coefficients are undetermined. Therefore the linear programming formulated as:

$$\begin{array}{lll} \min & z = sum_{j=1}^{n} c_j x_j & \\ S.t & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n \end{array} \tag{7}$$

changes to the probable model of:

$$\begin{array}{lll} \min & z = sum_{j=1}^{n} c_j x_j & \\ S.t & P(sum_{j=1}^{n} a_{ij} x_j \leq b_i) = p_i & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n \end{array} \tag{8}$$

where $0 \leq p_i \leq 1$ and a_{ij}, b_i, c_j are all accidental parameters.Suppose that $\bar{a}_{ij}$ and σ^2 alternatively represent the mean and the variation values of a_{ij}. Also suppose that all a_{ij} share the same Normal distribution. Defining $d_i = \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1, \ldots, n$, we obtain:

$$\begin{array}{ll} \bar{d}_i = \sum_{j=1}^{n} \bar{a}_{ij} x_j & i = 1, \ldots, n \\ var(d_i) = X^t V_i X & i = 1, \ldots, n \end{array} \tag{9}$$

where $X^t = (x_1, x_2, \ldots, x_n)$ and

$$V_i = \left\{ \begin{array}{cccc} var(a_{i1}) & cov(a_{i1}, a_{i2}) & \ldots & cov(a_{i1}, a_{in}) \\ cov(a_{i2}, a_{i1}) & var(a_{i2}) & \ldots & cov(a_{i2}, a_{in}) \\ \vdots & \vdots & \ddots & \vdots \\ cov(a_{im}, a_{i1}) & cov(a_{im}, a_{i2}) & \ldots & var(a_{im}) \end{array} \right\}$$

Also it is defined that$P(d_i \leq b_i) = p_i$. In other words, $P(\frac{d_i - \bar{d}_i}{\sqrt{var(d_i)}} \leq \frac{b_i - \bar{d}_i}{\sqrt{var(d_i)}}) = p_i$ .Since $\frac{d_i - \bar{d}_i}{\sqrt{var(d_i)}}$ is an accidental normal standard variable, $P(d_i \leq b_i) = \varphi \frac{b_i - \bar{d}_i}{\sqrt{var(d_i)}}$. On the other hand there exists e_i such that$\varphi(e_i) = p_i$. Therefore, $\varphi \frac{b_i - \bar{d}_i}{\sqrt{var(d_i)}} = \varphi(e_i)$ and hence, $\frac{b_i - \bar{d}_i}{\sqrt{var(d_i)}} = e_i$ or $e_i \sqrt{var(d_i)} - b_i + \bar{d}_i = 0$. This leads us to having the constraints of our initial model changed as:

$\sum_{j=1}^{n} \bar{a}_{ij}x_j + e_i\sqrt{var(d_i)} - b_i = 0$ or equivalently, $\sum_{j=1}^{n} \bar{a}_{ij}x_j + e_i\sqrt{X^tV_iX} - b_i = 0$

When the $a_{ij}s$ are independent, the V_i matrix changes as:

$$V_i = \begin{pmatrix} var(a_{i1}) & 0 & \dots & 0 \\ 0 & var(a_{i2}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & var(a_{im}) \end{pmatrix}$$

And therefore the final presentation of the constraints would change to:

$$\sum_{j=1}^{n} \bar{a}_{ij}x_j + e_i\sqrt{\sum_{j=1}^{n} var(a_{ij})x_j^2} - b_i = 0 \tag{10}$$

3 The proposed algorithm

Consider G_1 and G_2 as two groups of observations each with n_1 and n_2 independent members respectively. The number of the observations in both groups reaches n altogether. Each observation is denoted by z_j for $j = 1, \dots, n$ and is characterized by m independent factors denoted by $z_{ij}$for$i = 1, \dots, m$. Assume that the value corresponded to each z_{ij} for $i = 1, \dots, m$ and $j = 1, \dots, n$ is a probabilistic number and each factor of the observation z_j for$j = 1, \dots, n$ has Normal distribution and all factors of each observation share the same values of Mean and Variation. Facing probabilistic values of the technological coefficients of an MP (mathematical programming) results in having constraints which hold true under a probability less than 1. Thus, in order to solve this MP we need to find an equal model which owes precise technological coefficients and consequently precise restrictions. This model is formulated in two stages as follows:
Stage-one: Classification and Overlap Identification of probabilistically-valued observations (COI)
Stage-two: Overlap Handling of probabilistically-valued observations (OH)

Stage-one: Classification and Overlap Identification of probabilistically-valued observations (COI)
The MP modeled at this stage introduces two separating hyperplane that classify the observations into the groups G_1, G_2 and $G_1 \cap G_2$. Here $G_1 \cap G_2$ actually demonstrates the overlapped region between the two groups. This stage is formulated as follows:

$$
\begin{array}{lll}
\min & \sum_{j\in G_1} s_{1j}^{+} + \sum_{j\in G_2} s_{2j}^{-} & \\
S.t & P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{1j}^{+} - s_{1j}^{-} = d+1) = \alpha_j & j \in G_1 \\
 & P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{2j}^{+} - s_{2j}^{-} = d) = \alpha_j & j \in G_2 \\
 & \sum_{i=1}^{k}(\lambda_i^{+}+\lambda_i^{-}) = 1 & \\
 & s_{1j}^{+}, s_{1j}^{-}, s_{2j}^{+}, s_{2j}^{-} \geq 0 & j = 1 \in G_1 \cup G_2 \\
 & \lambda_i^{+}, \lambda_i^{-} \geq 0 & i = 1, \dots, k
\end{array}
\tag{11}
$$

where $\tilde{z}_{ij}$ determines the probabilistic value of the $i-th$ factor in the $j-th$ observation,s_{1j}^{+}, s_{1j}^{-} represent the positive and negative deviations of G_1 members from $\sum_{i=1}^{k}\lambda_i\tilde{z}_{ij}$ and s_{2j}^{+}, s_{2j}^{-} represent the positive and negative deviations of G_2 members from $\sum_{i=1}^{k}\lambda_i\tilde{z}_{ij}$ and s_{2j}^{+}, s_{2j}^{-}, respectively. The discriminant score is represented by d and the weights related to each observation factor by λ_i. On the other hand, the model model above is equal to:

$$
\begin{array}{lll}
\min & sum_{j\in G_1} s_{1j}^{+} + \sum_{j\in G_2} s_{2j}^{-} & \\
S.t & P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{1j}^{+} - s_{1j}^{-} \geq d+1) = \alpha_j & j \in G_1 \\
 & P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{1j}^{+} - s_{1j}^{-} \leq d+1) = \alpha_j & j \in G_1 \\
 & P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{2j}^{+} - s_{2j}^{-} \geq d) = \alpha_j & j \in G_2 \\
 & P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{2j}^{+} - s_{2j}^{-} \leq d) = \alpha_j & j \in G_2 \\
 & \sum_{i=1}^{k}(\lambda_i^{+}+\lambda_i^{-}) = 1 & \\
 & s_{1j}^{+}, s_{1j}^{-}, s_{2j}^{+}, s_{2j}^{-} \geq 0 & j = 1, \dots, n \\
 & \lambda_i^{+}, \lambda_i^{-} \geq 0 & i = 1, \dots, k
\end{array}
\tag{12}
$$

The first four constraints of the model above can be changed to the form of:

$$
\begin{aligned}
&P(\textstyle\sum_{i=1}^{k}(\lambda_i^+ - \lambda_i^-)\tilde{z}_{ij} \geq d + 1 - s_{1j}^+ + s_{1j}^-) = \alpha_j \quad j \in G_1\\
&P(\textstyle\sum_{i=1}^{k}(\lambda_i^+ - \lambda_i^-)\tilde{z}_{ij} \leq d + 1 - s_{1j}^+ + s_{1j}^-) = \alpha_j \quad j \in G_1\\
&P(\textstyle\sum_{i=1}^{k}(\lambda_i^+ - \lambda_i^-)\tilde{z}_{ij} \geq d - s_{2j}^+ + s_{2j}^-) = \alpha_j \qquad j \in G_2\\
&P(\textstyle\sum_{i=1}^{k}(\lambda_i^+ - \lambda_i^-)\tilde{z}_{ij} \leq d - s_{2j}^+ + s_{2j}^-) = \alpha_j \qquad j \in G_2
\end{aligned} \tag{13}
$$

In order to normalize the left hand side statements of the inequalities above, we first subtract the mean value of $\sum_{i=1}^{k}(\lambda_i^+ - \lambda_i^-)\tilde{z}_{ij}$ from the both side numerators, and next divide them by the standard deviation value of $\sum_{i=1}^{k}(\lambda_i^+ - \lambda_i^-)\tilde{z}_{ij}$. What we derive for the first four constraints will be demonstrated as:

$$
\begin{aligned}
&P\Big(\tfrac{\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)\tilde{z}_{ij}-\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}} \geq \tfrac{d+1+s_{1j}^- - s_{1j}^+ - \sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}}\Big) = \alpha_j, \quad j \in G_1\\
&P\Big(\tfrac{\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)\tilde{z}_{ij}-\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}} \leq \tfrac{d+1+s_{1j}^- - s_{1j}^+ - \sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}}\Big) = \alpha_j, \quad j \in G_1\\
&P\Big(\tfrac{\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)\tilde{z}_{ij}-\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}} \geq \tfrac{d+s_{2j}^- - s_{2j}^+ - \sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}}\Big) = \alpha_j, \quad j \in G_2\\
&P\Big(\tfrac{\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)\tilde{z}_{ij}-\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}} \leq \tfrac{d+s_{2j}^- - s_{2j}^+ - \sum_{i=1}^{k}[\lambda_i^+-\lambda_i^-]z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}}\Big) = \alpha_j, \quad j \in G_2
\end{aligned} \tag{14}
$$

Defining $Z = \frac{\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)\tilde{z}_{ij}-\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}}$ and changing all the inequalities to $\leq$, summarizes inequalities above as:

$$
\begin{aligned}
&P\Big(Z \leq \tfrac{d+1+s_{1j}^- - s_{1j}^+ - \sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}}\Big) = 1-\alpha_j, \quad j \in G_1\\
&P\Big(Z \leq \tfrac{d+1+s_{1j}^- - s_{1j}^+ - \sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}}\Big) = \alpha_j, \quad j \in G_1\\
&P\Big(Z \leq \tfrac{d+s_{2j}^- - s_{2j}^+ - \sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}}\Big) = 1-\alpha_j, \quad j \in G_2\\
&P\Big(Z \leq \tfrac{d+s_{2j}^- - s_{2j}^+ - \sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^+-\lambda_i^-)z_{ij}}}\Big) = \alpha_j, \quad j \in G_2
\end{aligned} \tag{15}
$$

Regarding the properties of Normal distribution, we can next be led to

$$
\begin{aligned}
\varphi^{-1}(1-\alpha(j)) &= \frac{d+1+s_{1j}^{-}-s_{1j}^{+}-\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}}} \quad j\in G_1\\
\varphi^{-1}(\alpha(j)) &= \frac{d+1+s_{1j}^{-}-s_{1j}^{+}-\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}}} \quad j\in G_1\\
\varphi^{-1}(1-\alpha(j)) &= \frac{d+s_{2j}^{-}-s_{2j}^{+}-\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}}} \quad j\in G_2\\
\varphi^{-1}(\alpha(j)) &= \frac{d+s_{2j}^{-}-s_{2j}^{+}-\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}}{\sqrt{var\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}}} \quad j\in G_2
\end{aligned}
\tag{16}
$$

Also, $\sqrt{var\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}} = [(\lambda_1^{+}-\lambda_1^{-})^2 varz_{1j}+(\lambda_2^{+}-\lambda_2^{-})^2 varz_{2j}+\ldots+(\lambda_k^{+}-\lambda_k^{-})^2 varz_{kj}+2\sum_{m=1}^{k}\sum_{n=1,n\neq m}^{k}(\lambda_m^{+}-\lambda_m^{-})(\lambda_n^{+}-\lambda_n^{-})cov(z_{mj},z_{nj})]^{\frac{1}{2}}$ Since all the observations in this experiment are assumed to be independent, $\sqrt{var\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}} = \sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^{\frac{1}{2}}varz_{ij}}$ Therefore it is possible to have the final model formulated as:

$$
\begin{aligned}
\min\quad & \textstyle\sum_{j\in G_1}s_{1j}^{+}+\sum_{j\in G_2}s_{2j}^{-}\\
S.t\quad & \textstyle\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}+s_{1j}^{+}-s_{1j}^{-}+\varphi^{-1}(1-\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2varz_{ij}}=d+1 \quad j\in G_1\\
& \textstyle\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}+s_{1j}^{+}-s_{1j}^{-}+\varphi^{-1}(\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2varz_{ij}}=d+1 \quad j\in G_1\\
& \textstyle\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}+s_{2j}^{+}-s_{2j}^{-}+\varphi^{-1}(1-\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2varz_{ij}}=d \quad j\in G_2\\
& \textstyle\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij}+s_{2j}^{+}-s_{2j}^{-}+\varphi^{-1}(\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2varz_{ij}}=d \quad j\in G_2\\
& \textstyle\sum_{i=1}^{k}(\lambda_i^{+}+\lambda_i^{-})=1\\
& s_{1j}^{+},s_{1j}^{-},s_{2j}^{+},s_{2j}^{-}\geq 0 \qquad j=1\in G_1\cup G_2\\
& \lambda_i^{+},\lambda_i^{-}\geq 0 \qquad i=1,\ldots,k
\end{aligned}
\tag{17}
$$

For the new observation $\tilde{z}_{im}$, if
$\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*}z_{im}+\varphi^{-1}(1-\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})varz_{im}}\geq d^*+1$ and
$\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*}z_{im}+\varphi^{-1}(\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})varz_{im}}\geq d^*+1$

then $\tilde{z}_{im} \in G_1$. And if,
$\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*} z_{im} + \varphi^{-1}(1-\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) varz_{im}} \leq d^*$ and
$\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*} z_{im} + \varphi^{-1}(\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) varz_{im}} \leq d^*$

then $\tilde{z}_{im} \in G_2$. And finally if,
$d^* \leq \sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) z_{im} + \varphi^{-1}(1-\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*})^2 varz_{im}} \leq d^* + 1$ and
$d^* \leq \sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) z_{im} + \varphi^{-1}(\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*})^2 varz_{im}} \leq d^* + 1$

then the new observation belongs to the overlap region and we should go through the second stage to determine its membership. Before starting the next stage define:

$R_1 = \{j \in G : \sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) z_{ij} + \varphi^{-1}(1-\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*})^2 varz_{ij}} \geq d^* + 1 \quad and \quad \sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) z_{ij} + \varphi^{-1}(\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*})^2 varz_{ij}} \geq d^* + 1, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n\}$

$R_2 = \{j \in G : \sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) z_{ij} + \varphi^{-1}(1-\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*})^2 varz_{ij}} \leq d^* \quad and \quad \sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) z_{ij} + \varphi^{-1}(\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*})^2 varz_{ij}} \leq d^*, i = 1, \ldots, k, \quad j = 1, \ldots, n\}$

$R_0 = \{j \in G : d^* \leq \sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) z_{ij} + \varphi^{-1}(1-\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*})^2 varz_{ij}} \leq d^* + 1 \quad and \quad d^* \leq \sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*}) z_{ij} + \varphi^{-1}(\alpha)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*} - \lambda_i^{-^*})^2 varz_{ij}} \leq d^* + 1, i = 1, \ldots, k, \quad j = 1, \ldots, n\}$

And:
$C_1 = \{j \in R_1 : j \in G_1\}, C_2 = \{j \in R_2 : j \in G_2\}, G_1^{'} = G_1 - C_1, G_2^{'} = G_2 - C_2$

Stage-two: Overlap Handling of probabilistically-valued observations (OH)
After determining the existence of overlap between the two groups, it is necessary to clarify the membership of the observations belonging to the overlap region since the membership of these observations is still unknown. At this stage a new discriminant score is introduced to reclassify the observations be-

longing to $G_1^{'} \cup G_2^{'}$ as follows:

$$
\begin{array}{lll}
\min & \sum_{j\in G_1^{'}} s_{1j}^{+} + \sum_{j\in G_2^{'}} s_{2j}^{-} & \\
S.t & P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} \geq d+1) = \alpha_j & j \in C_1 \\
& P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{1j}^{+} - s_{1j}^{-} \geq c) = \alpha_j & j \in G_1^{'} \\
& P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{1j}^{+} - s_{1j}^{-} \leq c) = \alpha_j & j \in G_1^{'} \\
& P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{2j}^{+} - s_{2j}^{-} \geq c) = \alpha_j & j \in G_2^{'} \\
& P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} + s_{2j}^{+} - s_{2j}^{-} \leq c) = \alpha_j & j \in G_2^{'} \\
& P(\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})\tilde{z}_{ij} \leq d) = \alpha_j & j \in C_2 \\
& \sum_{i=1}^{k}(\lambda_i^{+}+\lambda_i^{-}) = 1 & \\
& d \leq c \leq d+1 & \\
& s_{1j}^{+}, s_{1j}^{-}, s_{2j}^{+}, s_{2j}^{-} \geq 0 & j = 1 \in G_1 \cup G_2 \\
& \lambda_i^{+}, \lambda_i^{-} \geq 0 & i = 1, \ldots, k
\end{array}
\tag{18}
$$

Therefore:

$$
\begin{array}{lll}
\min & \sum_{j\in G_1'} s_{1j}^{+} + \sum_{j\in G_2'} s_{2j}^{-} & \\
S.t & \sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij} + \varphi^{-1}(1-\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2 varz_{ij}} = d+1 & j\in C_1 \\
& \sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij} + s_{1j}^{+} - s_{1j}^{-} + \varphi^{-1}(1-\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2 varz_{ij}} = c & j\in G_1' \\
& \sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij} + s_{1j}^{+} - s_{1j}^{-} + \varphi^{-1}(\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2 varz_{ij}} = c & j\in G_1' \\
& \sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij} + s_{2j}^{+} - s_{2j}^{-} + \varphi^{-1}(1-\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2 varz_{ij}} = c & j\in G_2' \\
& \sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij} + s_{2j}^{+} - s_{2j}^{-} + \varphi^{-1}(\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2 varz_{ij}} = c & j\in G_2' \\
& \sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})z_{ij} + \varphi^{-1}(\alpha_j)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+}-\lambda_i^{-})^2 varz_{ij}} = d & j\in C_2 \\
& \sum_{i=1}^{k}(\lambda_i^{+}+\lambda_i^{-}) = 1 & \\
& d \le c \le d+1 & \\
& s_{1j}^{+}, s_{1j}^{-}, s_{2j}^{+}, s_{2j}^{-} \ge 0 \qquad j = 1 \in G_1 \cup G_2 & \\
& \lambda_i^{+}, \lambda_i^{-} \ge 0 \qquad i = 1,\ldots,k &
\end{array}
\tag{19}
$$

Assume $c^*, \lambda_i^*(i = 1,\ldots,k)$ to be the optimal solutions obtained from the model above. Now we intend to determine the group membership of the arbitrary observation $\tilde{z_{im}}, (i = 1,\ldots,k)$. Firstly, assume that $\tilde{z_{im}} \in G_1'$. Solving the inequalities,

$$\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})z_{im} + s_{1j}^{+} - s_{1j}^{-} + \varphi^{-1}(1-\alpha_m)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})^2 varz_{im}} = c^*, \qquad j\in G_1'$$

$$\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})z_{im} + s_{1j}^{+} - s_{1j}^{-} + \varphi^{-1}(\alpha_m)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})^2 varz_{im}} = c^*, \; j\in G_1'$$

we can calculate the value of α_m. Next, suppose that $\tilde{z}_{im} \in G_2'$. Solving the inequalities

$$\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})z_{im} + s_{2j}^{+} - s_{2j}^{-} + \varphi^{-1}(1-\alpha_m)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})^2 varz_{im}} = c^*, \qquad j\in G_2'$$

$\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})z_{im}+s_{2j}^{+}-s_{2j}^{-}+\varphi^{-1}(\alpha_m)\sqrt{\sum_{i=1}^{k}(\lambda_i^{+^*}-\lambda_i^{-^*})^2 varz_{im}}=c^*,\ j\in G_2^{'}$

we can calculates the value of $\alpha_m^{'}$. To conclude to which group $\tilde{z}_m$ is more probable to belong we compare the values of $\alpha_m and \quad \alpha_m^{'}$.
1. If $\alpha_m > \alpha_m^{'}$, then $\tilde{z}_m \in G_1$.
2. If $\alpha_m < \alpha_m^{'}$, then $\tilde{z}_m \in G_2$.

4 Numerical Example

The left-hand side of Table1 documents an illustrative data set. The list has 24 observations as listed at the first column and each observation is characterized by four factors listed at the second, third, fourth and fifth column respectively. The first 10 data of Table1 correspond to G_1 observations and the rest correspond to G_2 observations as follows:

Units	I1	I2	I3	I4	Units	I1	I2	I3	I4
1	50719.14	91449.14	10123.00	50994.14	13	39927.91	41042.70	1836.64	26359.83
2	6585.20	91100.60	2266.90	31874.78	14	33373.85	41792.28	2058.71	12256.14
3	49557.15	73836.55	1896.57	27699.92	15	37656.28	271906.85	9086.14	11525.14
4	33938.57	120054.92	3455.21	16547.00	16	20415.00	52357.85	2521.57	26575.14
5	132609.15	90625.25	4855.90	35177.92	17	24735.33	64140.85	3090.14	15506.47
6	57904.20	73093.32	3543.64	47539.89	18	7711.28	59210.14	2163.00	12057.85
7	21983.44	124159.83	1824.06	40655.11	19	24339.28	35188.14	1959.00	27861.71
8	16804.71	90072.00	4819.57	22764.42	20	1264.28	143349.14	471.00	622.57
9	24775.57	19302.71	5631.42	27907.00	21	28398.14	63740.71	2784.71	17276.42
10	77207.85	63405.00	13201.57	44883.42	22	36947.57	49605.23	2965.95	27275.42
11	26073.64	32848.42	2475.21	20811.07	23	25818.71	119244.14	5312.85	24268.42
12	15239.28	38782.14	6267.28	18552.57	24	20286.99	36428.09	3909.23	19263.94

Table1

Also the probabilities under which the constraints hold true are:

$$\begin{array}{lllll}
\alpha_1=0.15 & \alpha_2=0.09 & \alpha_3=0.44 & \alpha_4=0.15 & \alpha_5=0.01\\
\alpha_6=0.99 & \alpha_7=0.99 & \alpha_8=0.00 & \alpha_9=0.00 & \alpha_{10}=0.96\\
\alpha_{11}=0.93 & \alpha_{12}=0.99 & \alpha_{13}=0.79 & \alpha_{14}=0.17 & \alpha_{15}=0.00\\
\alpha_{16}=0.83 & \alpha_{17}=0.99 & \alpha_{18}=0.17 & \alpha_{19}=0.82 & \alpha_{20}=0.99\\
\alpha_{21}=0.01 & \alpha_{22}=0.75 & \alpha_{23}=0.96 & \alpha_{24}=0.02 &
\end{array}$$

Using Mathematical Programming, we separate the above observations at the first stage with the optimal solutions as:

$$\begin{array}{llll} \lambda_1^{+^*} = 0,50 & \lambda_2^{+^*} = 0.00 & \lambda_3^{+^*} = 0.00 & \lambda_4^{+^*} = 0.00 \\ \lambda_1^{-^*} = 0.50 & \lambda_2^{-^*} = 0.00 & \lambda_3^{-^*} = 0.00 & \lambda_4^{-^*} = 0.00 \end{array}$$

Also,$d^* = 0.4116, z^* = 6.5596$

The positive deviations of the G_1 members from the discriminant function are equal to:

$$\begin{array}{lllll} s_{1j}^{+^*} = 0.00 & s_{2j}^{+^*} = 0.48 & s_{3j}^{+^*} = 0.68 & s_{4j}^{+^*} = 0.69 & s_{5j}^{+^*} = 0.00 \\ s_{6j}^{+^*} = 0.54 & s_{7j}^{+^*} = 0.70 & s_{8j}^{+^*} = 0.82 & s_{9j}^{+^*} = 0.03 & s_{10j}^{+^*} = 0.46 \\ s_{11j}^{+^*} = 0.06 & s_{12j}^{+^*} = 0.12 & s_{13j}^{+^*} = 0.33 & s_{14j}^{+^*} = 0.00 & \end{array}$$

And the negative deviations of the G_2 members from the discriminant function equal to:

$$\begin{array}{lllll} s_{1j}^{-^*} = 0.00 & s_{2j}^{-^*} = 0.00 & s_{3j}^{-^*} = 0.10 & s_{4j}^{-^*} = 0.00 & s_{5j}^{-^*} = 0.00 \\ s_{6j}^{-^*} = 0.02 & s_{7j}^{-^*} = 0.04 & s_{8j}^{-^*} = 0.00 & s_{9j}^{-^*} = 0.00 & s_{10j}^{-^*} = 0.39 \\ s_{11j}^{-^*} = 0.06 & s_{12j}^{-^*} = 0.12 & s_{13j}^{-^*} = 0.33 & s_{14j}^{-^*} = 0.00 & \end{array}$$

For $j = 1, \dots, 10 (j \in G_1)$, define y_j as:

$$y_j = \begin{cases} 1, & z_j \in G_1 \\ 0, & otherwise \end{cases}$$

And for $j = 11, \dots, 24 (j \in G_2)$, define y_j as:

$$y_j = \begin{cases} 1, & z_j \in G_2 \\ 0, & otherwise \end{cases}$$

In other words y_j clarifies whether or not the observation z_j has been classified properly. The values of y_j obtained from stage one for the above example are:

$$\begin{array}{lllll} y_1 = 1.00 & y_2 = 0.00 & y_3 = 0.00 & y_4 = 0.00 & y_5 = 1.00 \\ y_6 = 0.00 & y_7 = 0.00 & y_8 = 0.00 & y_9 = 0.00 & y_{10} = 0.00 \\ y_{11} = 1.00 & y_{12} = 1.00 & y_{13} = 1.00 & y_{14} = 1.00 & y_{15} = 1.00 \\ y_{16} = 1.00 & y_{17} = 1.00 & y_{18} = 1.00 & y_{19} = 1.00 & y_{20} = 1.00 \\ y_{21} = 1.00 & y_{22} = 1.00 & y_{23} = 1.00 & y_{24} = 1.00 & \end{array}$$

The results obtained above suggest that only $\tilde{z}_1, \tilde{z}_5$ from group one have been successfully classified whereas all of group two's observations have been successfully classified. To determine the membership of $\tilde{z}_1, \tilde{z}_5$, we start stage two of our MP. The results obtained at this stage come up as:

$$\begin{array}{llll} \lambda_1^{+^*} = 0,00 & \lambda_2^{+^*} = 0.00 & \lambda_3^{+^*} = 0.49 & \lambda_4^{+^*} = 0.00 \\ \lambda_1^{-^*} = 0.00 & \lambda_2^{-^*} = 0.00 & \lambda_3^{-^*} = 0.50 & \lambda_4^{-^*} = 0.00 \end{array}$$

Also,$c^* = 1.1423, z^* = 4.0039$

5 Conclusion

When trying to classify observations, we sometimes come across observations with probabilistic observation factors in real world problems. For instance, take some banks as the observations and factors like Return on total assets, Equality to total assets, Cost-profit ratio, Return on equity as the observation factors. The problem is to classify the banks in to two groups of efficient or inefficient ones. Now suppose that the observation factors are probabilistic values that are not precisely evaluated. The method introduced in this paper is applicable in this case to classify the observations. The objective function of this model is based on minimizing the total deviation of misclassified observations. As a future study it is suggested to implement probabilistic values of technological coefficients into DEA models and compare the efficiencies of DEA probabilistic method and Extended DEA/DA MIP probabilistic approach.

References

[1] Jahanshhloo, G., Hosseinzadeh Lotfi, F., Rezai Balf, F., Zhiani Rezai, H., 2005, Discriminant Analysis on Interval Data and using Monte Carlo Method in Assessment of Overlap in DEA.

[2] Sueyoshi, T., 2004, DEA- discriminant analysis and methodological comparison among eight discriminant analysis approaches, European Journal of Operational Research.

[3] Jahanshahloo, G., HosseinzadehLotfi, F., 2003, Data Envelopement Analysis.

[4] Sueyoshi, T., 2002, Mixed integer programming approach of extended DEA- discriminant analysis, European Journal of Operational Research.

[5] Cooper, W., Seiford, L., Tone, K., 2002, Data Envelopement Analysis, Kluwer Academic publishers.

[6] Sueyoshi, T., 2001, Extended DEA-discriminant analysis, European Journal of Operational Research, 131, 324-351.

[7] Sueyoshi, T., 1997, DEA-Discriminant analysis in the view of goal programming, European Journal of Operational Research.

[8] Charnes, A., Cooper W., 1997, Goal Programming and multiple objective optimization, European Journal of Operational Research, 1, 39-54.

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