Analysis of a Chemotaxis Model for Multi-Species Host-Parasitoid Interactions

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Abstract. This paper deals with a mathematical model of multi-species host-parasitoid interactions recently proposed by Pearce et al. The model consists of five reaction-diffusion-chemotaxis equations. By a fixed point method, together with $L^p$ estimates and Schauder estimates of parabolic equations, we prove global existence and uniqueness of a classical solution for the model. Furthermore, the chemotaxis-driven linear instability for a sub-model is analytically proven.

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1. Mathematical Model

Very recently, Pearce et al proposed a chemotaxis model describing aggregative parasitoid behavior in a multi-species host-parasitoid system. The system consists of two parasitoid species and two host species. The two parasitoids are, *Cotesia glomerata* and *Cotesia rubecula*. The two hosts are, *Pieris brassicae* and *Pieris rapae*. Both hosts are common crop pests of brassical species, and both parasitoids have been used as successful biological control agents against the host [3] (and references therein). The parasitoids aggregate in response to volatile infochemicals emitted by the crops that the hosts are feeding upon, and the infochemicals can thus be considered as chemoattractants. The model assumes that the four species move randomly in a spatial domain and that both parasitoid species are also directed by the chemoattractant. Both host

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species reproduce with logistic growth and undergo death due to parasitism, which is modelled by an Ivlev functional response [5]. Two parasitoid species reproduce next-generation parasitoids by the parasitised hosts, and they are subject to natural death. The chemoattractant is produced by the two hosts, diffuses through the domain and undergoes natural decay. The (dimensionless) model consists of five reaction-diffusion-chemotaxis equations with boundary and initial conditions [4]:

\begin{align}
\frac{\partial N}{\partial t} &= D_N \triangle N + N(1 - N) - s_1 P(1 - e^{-\rho_1 N}), \\
\frac{\partial M}{\partial t} &= D_M \triangle M + \gamma_1 M(1 - M) - s_2 P(1 - e^{-\rho_2 M}) - s_3 Q(1 - e^{-\rho_3 M}), \\
\frac{\partial P}{\partial t} &= D_P \triangle P - \chi_P \nabla \cdot (P \nabla k) + c_1 P(1 - e^{-\rho_1 N}) + c_2 P(1 - e^{-\rho_2 M}) - \eta_1 P, \\
\frac{\partial Q}{\partial t} &= D_Q \triangle Q - \chi_Q \nabla \cdot (Q \nabla k) + c_3 Q(1 - e^{-\rho_3 M}) - \eta_2 Q, \\
\frac{\partial k}{\partial t} &= D_k \triangle k + \gamma_2 (N + \gamma_3 M) - \eta_3 k,
\end{align}

(1.6) \quad \partial N \bigg|_{\partial \Omega} = \partial M \bigg|_{\partial \Omega} = \partial P \bigg|_{\partial \Omega} = \partial Q \bigg|_{\partial \Omega} = \partial k \bigg|_{\partial \Omega} = 0 \quad \text{on } \partial \Omega,

(1.7) \quad N(x, 0) = N_0(x), \quad M(x, 0) = M_0(x), \quad P(x, 0) = P_0(x), \quad Q(x, 0) = Q_0(x), \quad k(x, 0) = k_0(x).

Here \( N = N(x, t) \) and \( M = M(x, t) \) denote the local population density of hosts \( P. \ brassicae \) and \( P. \ rapae \), respectively; \( P = P(x, t) \) and \( Q = Q(x, t) \) denote the local population density of parasitoids \( C. \ glomerata \) and \( C. \ rubecula \) and \( k = k(x, t) \) denotes local chemoattractant concentration. \( D_N, D_M, D_P, D_Q, D_k, s_1, s_2, s_3, c_1, c_2, c_3, \rho_1, \rho_2, \rho_3, \gamma_1, \gamma_2, \gamma_3, \eta_1, \eta_2, \eta_3, \chi_P \) and \( \chi_Q \) are some positive constants. The system (1.1)-(1.5) is posed on a given domain \( \Omega \subset \mathbb{R}^n \) \((n = 1, 2, \text{or } 3)\) with smooth boundary \( \partial \Omega \). \( \nu \) is the outward normal to \( \partial \Omega \). (1.6) is the zero-flux boundary condition, and (1.7) is the initial condition prescribed in \( \Omega \).

In this paper we first prove global existence and uniqueness of a classical solution for the model (1.1)-(1.7) by a fixed point method, together with \( L^p \) estimates and Schauder estimates of parabolic equations. Furthermore, we analytically prove the chemotaxis-driven linear instability for a sub-model.
2. LOCAL EXISTENCE AND UNIQUENESS

Throughout this paper we assume that

\[(2.1) \quad 0 \leq N_0(x), M_0(x) \leq 1, \quad P_0(x), Q_0(x), k_0(x) \geq 0, \quad \partial \Omega \in C^{2+\alpha}, \quad 0 < \alpha < 1, \]
\[\frac{\partial N_0(x)}{\partial \nu} = \frac{\partial M_0(x)}{\partial \nu} = \frac{\partial P_0(x)}{\partial \nu} = \frac{\partial Q_0(x)}{\partial \nu} = \frac{\partial k_0(x)}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega.\]

For any \(0 < T \leq \infty\) we set

\[\Omega_T = \Omega \times \{0 \leq t < T\}, \quad \partial \Omega_T = \partial \Omega \times \{0 \leq t < T\}.\]

For brevity, as done in [1], we set

\[(2.2) \quad U = (N, M, P, Q, k).\]

**Theorem 2.1.** There exists a unique solution \(U \in C^{2+\alpha, 1+\alpha/2}(\Omega_T)\) of the system (1.1)-(1.7) for some small \(T > 0\) which depends on \(\|U(\cdot, 0)\|_{C^{2+\alpha}(\Omega)}\).

**Proof.** We shall prove local existence by a fixed point argument. We introduce the Banach space \(X\) of the vector-functions \(U\) (defined in (2.2)) with norm

\[\|U\| = \|U\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega_T)} \quad (0 < T < 1)\]

and a subset

\[X_B = \{U \in X, \|U\| \leq B\}, \quad B > 0\]

where \(B\) is some constant to be chosen later on. Given any \(U \in X_B\), we define a corresponding function \(\overline{U} \equiv FU\) by

\[\overline{U} = (\overline{N}, \overline{M}, \overline{P}, \overline{Q}, \overline{k})\]
where $\mathbf{U}$ satisfies the equations

\begin{align}
\frac{\partial N}{\partial t} &- D N \nabla N = N(1 - N) - s_1 P(1 - e^{-\rho_1 N}), \\
\frac{\partial M}{\partial t} &- D M \nabla M = \gamma_1 M(1 - M) - s_2 P(1 - e^{-\rho_2 M}) - s_3 Q(1 - e^{-\rho_3 M}), \\
\frac{\partial k}{\partial t} &- D k \nabla k = \gamma_2 (N + \gamma_3 M) - \eta_3 k, \\
\frac{\partial P}{\partial t} &- D P \nabla P + \chi P \nabla \cdot (P \nabla k) = c_1 P(1 - e^{-\rho_1 N}) + c_2 P(1 - e^{-\rho_2 M}) - \eta_1 P, \\
\frac{\partial Q}{\partial t} &- D Q \nabla Q + \chi Q \nabla \cdot (Q \nabla k) = c_3 Q(1 - e^{-\rho_3 M}) - \eta_2 Q.
\end{align}

By (2.3)-(2.4), $U \in X_B$, and the parabolic Schauder theory (for example, see [2]), there exists a unique solution $\mathbf{N}$, and

\begin{align}
\| \mathbf{N} \|_{C^{2+\alpha,1+\alpha/2}(\Omega_T)} &\leq \| \mathbf{N} \|_{t=0} \| \mathbf{N} \|_{C^{2+\alpha}(\Omega)} + K_1(B) \\
&\leq K_0 + K_1(B),
\end{align}

where

\[ K_0 = \| \mathbf{U}_0(x) \|_{C^{2+\alpha}(\Omega)} \]

and $K_1(B)$ is some constant which depends only on $B$. In the sequel we shall denote various constants which depend only on $K_0$ and $B$ by $K_j \equiv K_j(K_0, B)$ ($j = 2, 3, 4, \ldots$). Similarly, we have

\begin{align}
\| \mathbf{M} \|_{C^{2+\alpha,1+\alpha/2}(\Omega_T)} &\leq K_2, \\
\| \mathbf{k} \|_{C^{2+\alpha,1+\alpha/2}(\Omega_T)} &\leq K_3.
\end{align}
We now turn to Eq. (2.9). Note that Eq. (2.9) can be rewritten as

\[
\frac{\partial P}{\partial t} - DP \triangle P + \nabla P \cdot \nabla \mathcal{F} + (\chi P \triangle \mathcal{F}) P = h
\]

with

\[
\| \nabla \mathcal{F} \|_{C^{\alpha, \alpha/2}(\Omega_T)}, \quad \| \triangle \mathcal{F} \|_{C^{\alpha, \alpha/2}(\Omega_T)}, \quad \| h \|_{C^{\alpha, \alpha/2}(\Omega_T)} \leq K_4
\]

by \( U \in X_B \) and (2.15). Hence, (2.16) is a linear parabolic equation with \( C^{\alpha, \alpha/2}(\Omega_T) \) coefficients and the right-hand term and, by the Schauder theory, it has a unique solution \( \mathcal{F} \) satisfying

\[
\| \mathcal{F} \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)} \leq K_5.
\]

Similarly, we have

\[
\| \mathcal{G} \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)} \leq K_6.
\]

We conclude from (2.13)-(2.15), (2.17) and (2.18) that

\[
\| \mathcal{U} \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)} \leq K_7.
\]

For any function \( \mathcal{U}(x, t) \), it is easily checked that

\[
\| \mathcal{U}(x, t) - \mathcal{U}(x, 0) \|_{C^{\alpha, \alpha}(\Omega_T)} \leq C \max(T^{\frac{\alpha}{2}}, T^{1-\frac{\alpha}{2}}) \cdot \| \mathcal{U} \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)}
\]

where \( C \) is some constant depending only on the domain \( \Omega \). Combining (2.19) and (2.20), we conclude that if we take \( B = K_0 + 1 = \| \mathcal{U}_0(x) \|_{C^{2+\alpha}(\Omega)} + 1 \) and \( T \) is sufficiently small, then

\[
\| \mathcal{U}(x, t) \|_{C^{\alpha, \alpha}(\Omega_T)} \leq K_0 + 1 = B
\]

i.e., \( \mathcal{U} \in X_B \). Hence, \( F \) maps \( X_B \) into itself.

We next show that \( F \) is contractive. Take \( \mathcal{U}_1, \mathcal{U}_2 \) in \( X_B \) and set \( \mathcal{U}_1 = FU_1, \mathcal{U}_2 = FU_2 \). Set

\[
\delta = \| \mathcal{U}_1 - \mathcal{U}_2 \|
\]

Using (2.3)-(2.12) and (2.19), and performing the parabolic Schauder estimate as before, it is easily proven that

\[
\| \mathcal{U}_1 - \mathcal{U}_2 \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)} \leq A_0 \delta
\]
where $A_0$ is some constant independent of $T$. Then, as before (noting $(\overline{U}_1 - \overline{U}_2)|_{t=0} = 0$),

\begin{align}
(2.23) \\
\| \overline{U}_1 - \overline{U}_2 \|_{C^{\alpha, \frac{1}{2}}(\Omega_T)} \\
& \leq C \max(T_{\alpha}^2, T^{1 - \frac{\alpha}{2}}) \cdot \| \overline{U}_1 - \overline{U}_2 \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)} \\
& \leq C \max(T_{\alpha}^2, T^{1 - \frac{\alpha}{2}}) A_0 \delta \\
& = C \max(T_{\alpha}^2, T^{1 - \frac{\alpha}{2}}) A_0 \| \overline{U}_1 - \overline{U}_2 \|_{C^{\alpha, \frac{1}{2}}(\Omega_T)}.
\end{align}

Taking $T$ small such that $C \max(T_{\alpha}^2, T^{1 - \frac{\alpha}{2}}) A_0 < \frac{1}{2}$, we conclude from (2.23) that $F$ is contractive in $X_B$. By the contraction mapping theorem $F$ has a unique fixed point $U$, which is the unique solution of (1.1)-(1.7).

\section{Global existence}

To continue the local solution in Theorem 2.1 to all $t > 0$, we need to establish some a priori estimates.

\textbf{Lemma 3.1.} There holds

\begin{equation}
(3.1) \quad P, Q \geq 0, \quad 0 \leq N, M \leq 1, \quad 0 \leq k \leq \frac{\gamma_2(1 + \gamma_3)}{\eta_3} + \max_{\Omega} k_0(x).
\end{equation}

\textbf{Proof.} It is easily checked that $N \equiv 0$ is a sub-solution of Eq. (1.1) with $\frac{\partial N}{\partial \nu} |_{\partial \Omega} = 0$ and $N(x, 0) = N_0(x)$. Hence, by the maximum principle, $N \geq 0$. Similarly, $M \geq 0$, $P \geq 0$ and $Q \geq 0$. The inequality $k \geq 0$ follows from $N \geq 0$, $M \geq 0$ and the maximum principle. By $P \geq 0$ and $N \geq 0$ we easily find that $\overline{N} \equiv 1$ is a sup-solution of Eq. (1.1) with $\frac{\partial N}{\partial \nu} |_{\partial \Omega} = 0$ and $N(x, 0) = N_0(x)$. Hence, $N \leq 1$. Similarly, we have $M \leq 1$. Let $\overline{k}(t)$ be a solution of the following problem:

\begin{equation}
(3.2) \quad \frac{d\overline{k}}{dt} = \gamma_2(1 + \gamma_3) - \eta_3 \overline{k}(t), \quad \overline{k}(0) = \max_{\Omega} k_0(x).
\end{equation}

By $N \leq 1$ and $M \leq 1$, we find that $\overline{k}(t)$ is a sup-solution of Eq. (1.5). Therefore

\begin{align}
\overline{k}(x, t) & \leq \overline{k}(t) = \overline{k}(0) e^{-\eta_3 t} + \frac{\gamma_2(1 + \gamma_3)}{\eta_3} (1 - e^{-\eta_3 t}) \\
& \leq \overline{k}(0) + \frac{\gamma_2(1 + \gamma_3)}{\eta_3}.
\end{align}

This completes the proof of Lemma 3.1.
For convenience of notations, in the sequel we denote various constants which depend on $T$ by $A$. By Eq. (1.5), Lemma 3.1, Assumption (2.1) and the parabolic $L^p$-estimate, we have the following lemma.

**Lemma 3.2.** There holds

$$\| k \|_{W^{r,1}(\Omega_T)} \leq A$$

for any $r > 1$.

Next, we turn to the $L^p$-estimate of $P$ and $Q$. We first prove the following lemma.

**Lemma 3.3.** There holds

$$\| P \|_{L^{r+1}(\Omega_T)}, \| Q \|_{L^{r+1}(\Omega_T)} \leq A$$

for any $r > 1$.

**Proof.** Multiplying (1.3) by $P^r$, integrating over $\Omega_t$, and using Lemma 3.1, we get

$$\frac{1}{1+r} \int_\Omega P^{r+1}(x,t) \, dx - \frac{1}{1+r} \int_\Omega P_0^{r+1}(x) \, dx$$

$$+ r D_P \int_0^t \int_\Omega |\nabla P|^2 \, dx \, dt$$

$$\leq r \chi_P \int_0^t \int_\Omega P \cdot \nabla k \, dx \, dt + (c_1 + c_2) \int_0^t \int_\Omega P^{r+1} \, dx \, dt.$$  

By (3.3) and the Sobolev imbedding theorem (see [2; Lemma 3.3, P. 80]), we have (taking $r$ large)

$$\| \nabla k \|_{L^\infty(\Omega_t)} \leq A.$$  

It follows from (3.6) that

$$\int_0^t \int_\Omega P \cdot \nabla k \, dx \, dt \leq A \int_0^t \int_\Omega P \cdot \nabla k \, dx \, dt$$

$$\leq A \int_0^t \int_\Omega P^{r+1} \, dx \, dt$$

$$\leq \varepsilon \int_0^t \int_\Omega |\nabla P|^2 \, dx \, dt + \frac{1}{2\varepsilon} A \int_0^t \int_\Omega P^{r+1} \, dx \, dt.$$  

Inserting (3.7) into (3.5) and taking $\varepsilon$ small such that $D_P - \varepsilon \chi_P > 0$, we have

$$\int_\Omega P^{r+1}(x,t) \, dx \, dt \leq A + A \int_0^t \int_\Omega P^{r+1} \, dx \, dt.$$  

Gronwall’s lemma yields

\[ \| P \|_{L^{r+1}(\Omega_t)} \leq A. \]

Similarly, we have

\[ \| Q \|_{L^{r+1}(\Omega_t)} \leq A. \]

This completes the proof of Lemma 3.3.

**Lemma 3.4.** There holds

\[ (3.9) \quad \| P, Q, N, M \|_{W^{2,1}_r(\Omega_T)} \leq A \]

for any \( r > 1 \).

**Proof.** By Eqs. (1.1) and (1.2), Lemmas 3.1 and 3.3, Assumption (2.1) and the parabolic \( L^p \)-estimate \[2\], we have

\[ \| N, M \|_{W^{2,1}_r(\Omega_T)} \leq A. \]

We now turn to Eq. (1.3). It can be rewritten as in non-divergence form:

\[ (3.10) \quad \frac{\partial P}{\partial t} - D_P \triangle P + \chi_P \nabla k \cdot \nabla P + (\chi_P \triangle k)P = f \]

where

\[ (3.11) \quad \| \nabla k \|_{L^\infty(\Omega_T)} \leq A, \quad \| \triangle k \|_{L^r(\Omega_T)} \leq A, \quad \| f \|_{L^r(\Omega_T)} \leq A \]

by (3.3), (3.6), and Lemmas 3.1 and 3.3. By (3.10), (3.11), and the parabolic \( L^p \)-estimate, we have

\[ (3.12) \quad \| P \|_{W^{2,1}_r(\Omega_T)} \leq A. \]

Similarly, we have

\[ (3.13) \quad \| Q \|_{W^{2,1}_r(\Omega_T)} \leq A. \]

This completes the proof of Lemma 3.4.

**Lemma 3.5.** There holds

\[ (3.14) \quad \| U \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)} \leq A. \]

**Proof.** By Lemmas 3.2 and 3.4, and the Sobolev imbedding Theorem (see [2; Lemma 3.3, P. 80]; taking \( r \) large),

\[ (3.15) \quad \| U \|_{C^{\alpha, \alpha/2}(\Omega_T)} \leq A. \]

This, together with Assumption (2.1) and the parabolic Schauder estimate, yields

\[ (3.16) \quad \| N, M, k \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)} \leq A. \]
Noting
\[ \| \nabla k \|_{C^\alpha, \alpha/2(\Omega_T)}, \quad \| \Delta k \|_{C^\alpha, \alpha/2(\Omega_T)}, \quad \| f \|_{C^\alpha, \alpha/2(\Omega_T)} \leq A \]
by (3.15) and (3.16), and applying Schauder estimate to Eq. (3.10), we have
(3.17) \[ \| P \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)} \leq A. \]
Similarly, we have
(3.18) \[ \| Q \|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T)} \leq A. \]
This completes the proof of Lemma 3.5. \(\square\)

With the a priori estimate (3.14), we can extend the local solution in Theorem 2.1 to all \( t > 0 \). Namely, we have

**Theorem 3.6.** There exists a unique global solution \( U \in C^{2+\alpha, 1+\alpha/2}(\Omega_\infty) \) of the model (1.1)-(1.7).

4. Chemotaxis-driven instability

In this section we will analytically prove the linear instability of the coexistence state of a sub-model for large chemotaxis coefficient \( \chi_P \). For simplicity, we consider a sub-system which consists of one parasitoid species and one host species. Setting \( M \equiv 0 \) and \( Q \equiv 0 \) in the model (1.1)-(1.7), we get the following sub-model:

(4.1) \[ \frac{\partial N}{\partial t} = D_N \Delta N + N(1 - N) - s_1 P(1 - e^{-\rho_1 N}), \]

(4.2) \[ \frac{\partial P}{\partial t} = D_P \Delta P - \chi_P \nabla \cdot (P \nabla k) + c_1 P(1 - e^{-\rho_1 N}) - \eta_1 P, \]

(4.3) \[ \frac{\partial k}{\partial t} = D_k \Delta k + \gamma_2 N - \eta_3 k, \]

(4.4) \[ \frac{\partial N}{\partial \nu} = \frac{\partial P}{\partial \nu} = \frac{\partial k}{\partial \nu} = 0 \text{ on } \partial \Omega, \]

(4.5) \[ N(x, 0) = N_0(x), \quad P(x, 0) = P_0(x), \quad k(x, 0) = k_0(x). \]

The spatially homogeneous fixed points of system (4.1)-(4.3) are found by solving the following equations

(4.6) \[ 0 = N^*(1 - N^*) - s_1 P^*(1 - e^{-\rho_1 N^*}), \]

(4.7) \[ 0 = c_1 P^*(1 - e^{-\rho_1 N^*}) - \eta_1 P^* \]

(4.8) \[ 0 = \gamma_2 N^* - \eta_3 k^*. \]

Throughout this section we assume that
(4.9) \[ c_1(1 - e^{-\rho_1}) > \eta_1, \]
which guarantees the existence of the coexistence state. Under the assumption (4.9), it is easily checked that there are precisely three equilibrium states: the trivial state \((0, 0, 0)\), the host only state \((1, 0, \gamma_2/\eta_3)\) and coexistence state \((N^*, P^*, k^*)\)

\[
(N^*, P^*, k^*) = (-\rho_1^{-1}\ln(1 - \eta_1 c_1^{-1}), -c_1(\eta_1 s_1 \rho_1)^{-1}\ln(1 - \eta_1 c_1^{-1})[1 + \rho_1^{-1}\ln(1 - \eta_1 c_1^{-1})], -\gamma_2(\eta_3 \rho_1)^{-1}\ln(1 - \eta_1 c_1^{-1})) .
\]

In this paper we are only interested in the stability of the coexistence state \((N^*, P^*, k^*)\). By (4.7) and \(P^* \neq 0\) we find that

\[
c_1(1 - e^{-\rho_1 N^*}) = \eta_1 .
\]

**Theorem 4.1.** In addition to assumption (4.9), we further assume

\[
1 + 2\ln\left(1 - \frac{2\eta}{\rho_1}\right) + \frac{(c_1 - \eta_1)}{\eta_1} \ln\left(1 - \frac{\eta_1}{c_1}\right)
\left[1 + \frac{\ln\left(1 - \frac{2\eta}{\rho_1}\right)}{\rho_1}\right] < 0 .
\]

Then, the coexistence state \((N^*, P^*, k^*)\) is linearly unstable for large chemotaxis coefficient \(\chi_P\).

**Proof.** Introduce the following small spatio-temporal perturbations of the coexistence state \((N^*, P^*, k^*)\):

\[
N = N^* + \varepsilon N, \quad P = P^* + \varepsilon P, \quad k = k^* + \varepsilon k .
\]

Inserting (4.12) into (4.1)-(4.4), using (4.6), (4.8) and (4.10), and continuing to \(O(\varepsilon)\), we deduce that \((N, P, k)\) solves the following equations

\[
\frac{\partial N}{\partial t} = D_N \triangle N + (1 - 2N^* - s_1 \rho_1 P^* e^{-\rho_1 N^*}) N - \frac{s_1 \eta_1}{c_1} P ,
\]

\[
\frac{\partial P}{\partial t} = D_P \triangle P - \chi_P P^* \triangle k + c_1 \rho_1 P^* e^{-\rho_1 N^*} N ,
\]

\[
\frac{\partial k}{\partial t} = D_k \triangle k + \gamma_2 N - \eta_3 k
\]

with the following boundary conditions:

\[
\frac{\partial N}{\partial \nu} = \frac{\partial P}{\partial \nu} = \frac{\partial k}{\partial \nu} = 0 \quad \text{on} \ \partial \Omega .
\]

The linearized coefficient matrix \(E\) of the system (4.13)-(4.15) is as follows:

\[
E = \begin{pmatrix}
1 - 2N^* - s_1 \rho_1 P^* e^{-\rho_1 N^*} - \frac{s_1 \eta_1}{c_1} & 0 \\
c_1 \rho_1 P^* e^{-\rho_1 N^*} & 0 \\
\gamma_2 & -\eta_3
\end{pmatrix}
\]

Note that the assumption (4.11) is equivalent to the following condition:

\[
1 - 2N^* - s_1 \rho_1 P^* e^{-\rho_1 N^*} < 0 .
\]
By (4.17) we easily prove that the three eigenvalues of the matrix \(E\) have negative real parts. Therefore, with no spatial variation, the steady state \((\overline{\Omega}, \overline{\mathcal{P}}, \overline{k}) = (0, 0, 0)\) is stable for corresponding ODE system of the system (4.13)-(4.15). However, with spatial variation, in the following we will prove that the steady state \((\overline{\Omega}, \overline{\mathcal{P}}, \overline{k}) = (0, 0, 0)\) of the system (4.13)-(4.15) is unstable for large chemotaxis coefficient \(\chi_d\).

For simplicity, we assume that \(\Omega = (0, 1)\). Clearly, each function \(\cos lx\) \((l = m\pi, m = 1, 2, 3, \ldots)\) satisfies the zero flux boundary conditions (4.16) (where \(l\) is called the \emph{wavenumber}). Because the problem (4.13)-(4.15) is linear, we now look for solutions \((\overline{\Omega}, \overline{\mathcal{P}}, \overline{k})\) of (4.13)-(4.15) in the form

\[
(4.18) \quad (\overline{\Omega}, \overline{\mathcal{P}}, \overline{k}) = (d_1, d_2, d_3)e^\lambda \cos lx
\]

where \(d_1, d_2\) and \(d_3\) are three constants satisfying

\[
(4.19) \quad d_1^2 + d_2^2 + d_3^2 \neq 0.
\]

Substituting (4.18) into (4.13)-(4.15) and cancelling \(e^\lambda \cos lx\), we get, for each \(l\),

\[
(4.20) \quad [\lambda + l^2 D_N - (1 - 2N^* - s_1\rho_1 P^* e^{-\rho_1 N^*})]d_1 + \frac{s_1\eta_1}{c_1} d_2 = 0,
\]

\[
(4.21) \quad -c_1\rho_1 P^* e^{-\rho_1 N^*} d_1 + (\lambda + l^2 D_P) d_2 - \chi_P P^* l^2 d_3 = 0,
\]

\[
(4.22) \quad -\gamma_2 d_1 + (\lambda + l^2 D_k + \eta_3) d_3 = 0.
\]

We deduce from (4.20)-(4.22) and (4.19) that

\[
\begin{vmatrix}
\lambda + l^2 D_N - (1 - 2N^* - s_1\rho_1 P^* e^{-\rho_1 N^*}) & \frac{s_1\eta_1}{c_1} & 0 \\
-c_1\rho_1 P^* e^{-\rho_1 N^*} & \lambda + l^2 D_P & -\chi_P P^* l^2 \\
-\gamma_2 & 0 & \lambda + l^2 D_k + \eta_3
\end{vmatrix} = 0,
\]

which yields

\[
(4.23) \quad \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0
\]

where

\[
(4.24) \quad a_0 = (\gamma_2 s_1\eta_1 c_1^{-1} l^2 P^*) \chi_P
\]

\[
+ (l^2 D_k + \eta_3) \{l^2 D_P [l^2 D_N - (1 - 2N^* - s_1\rho_1 P^* e^{-\rho_1 N^*})]
\]

\[
+ s_1\eta_1\rho_1 P^* e^{-\rho_1 N^*}\},
\]

\[
(4.25) \quad a_1 = (l^2 D_k + \eta_3) [l^2 (D_N + D_P) - (1 - 2N^* - s_1\rho_1 P^* e^{-\rho_1 N^*})]
\]

\[
+ l^2 D_P [l^2 D_N - (1 - 2N^* - s_1\rho_1 P^* e^{-\rho_1 N^*})]
\]

\[
+ s_1\eta_1\rho_1 P^* e^{-\rho_1 N^*},
\]

\[
(4.26) \quad a_2 = l^2 (D_N + D_P + D_k) + \eta_3 - (1 - 2N^* - s_1\rho_1 P^* e^{-\rho_1 N^*}).
\]
By (4.17) we easily find that

\begin{equation}
 a_0 > 0, \quad a_1 > 0, \quad a_2 > 0.
\end{equation}

Let \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) be the three roots of Equation (4.23). We deduce from (4.27) that there is no real non-negative roots of Equation (4.23). Hence, we can assume that

\[ \lambda_{1,2} = \alpha \pm \beta i, \quad \lambda_3 < 0. \]

In the following we first prove that

\begin{equation}
 \beta \neq 0 \quad \text{for large } \chi_P.
\end{equation}

Setting

\begin{equation}
 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 \equiv (\lambda^2 + b_1 \lambda + b_2)(\lambda - \lambda_3),
\end{equation}

we deduce from (4.23) and (4.29) that \( \lambda_1 \) and \( \lambda_2 \) are the two roots of the following equation:

\begin{equation}
 \lambda^2 + b_1 \lambda + b_2 = \lambda^2 + (a_2 + \lambda_3) \lambda - \frac{a_0}{\lambda_3} = 0.
\end{equation}

By (4.24) we find that

\begin{equation}
 a_0 \to +\infty \quad \text{as } \chi_P \to +\infty.
\end{equation}

Note that \( \lambda_3 \) solves the Eq. (4.23), namely,

\begin{equation}
 \lambda_3^2 + a_2 \lambda_3^2 + a_1 \lambda_3 + a_0 = 0.
\end{equation}

By (4.25) and (4.26) we find that, for each fixed \( l \),

\begin{equation}
 a_1 \text{ and } a_2 \text{ are two constants which are independent of } \chi_P.
\end{equation}

Combining (4.31), (4.32) and (4.33) we get

\begin{equation}
 |\lambda_3| \to +\infty \quad \text{as } \chi_P \to +\infty.
\end{equation}

Using (4.32) we find that

\begin{equation}
 b_1^2 - 4b_2
 = \frac{(a_2 + \lambda_3)^2 + 4a_0}{\lambda_3}
 = \frac{\lambda_3^2 + 2a_2 \lambda_3^2 + a_2^2 \lambda_3 + 4a_0}{\lambda_3}
 = \frac{(\lambda_3^2 + a_2 \lambda_3^2 + a_1 \lambda_3 + a_0) + (a_2 \lambda_3^2 + a_2^2 \lambda_3 - a_1 \lambda_3 + 3a_0)}{\lambda_3}
 = \frac{a_2 \lambda_3 (\lambda_3 + a_2) - a_1 \lambda_3 + 3a_0}{\lambda_3}.
\end{equation}
We deduce from (4.33), (4.34) and \( \lambda_3 < 0 \) that
\[
\lambda_3 + a_2 < 0 \quad \text{for large } \chi_P.
\]
This, together with (4.27), (4.35) and \( \lambda_3 < 0 \), yields
\[
b_1^2 - 4b_2 < 0 \quad \text{for large } \chi_P.
\]
Hence, we conclude from (4.30) and (4.37) that (4.28) holds.

In the following we will prove
\[
\alpha > 0 \quad \text{for large } \chi_P.
\]
Inserting \( \lambda_1 = \alpha + \beta i \) (\( \beta \neq 0 \)) into (4.23), comparing the real part and imaginary part of the resulting equation, and using (4.28), we have
\[
\beta^2 = 3\alpha^2 + 2a_2\alpha + a_1, \\
\alpha^3 - 3\alpha\beta^2 + (\alpha^2 - \beta^2)a_2 + \alpha a_1 + a_0 = 0.
\]
By (4.39), we get
\[
\alpha^3 - 3\alpha\beta^2 + (\alpha^2 - \beta^2)a_2 + \alpha a_1 + a_0 \\
= -8\alpha^3 - 8a_2\alpha^2 - (2a_1 + 2a_2^2)\alpha + a_0 - a_1a_2 \\
= : f(\alpha).
\]
We deduce from (4.31) that
\[
f(0) = a_0 - a_1a_2 > 0 \quad \text{for large } \chi_P.
\]
clearly,
\[
\lim_{\alpha \to +\infty} f(\alpha) < 0,
\]
\[
f'(\alpha) = -24\alpha^2 - 16a_2\alpha - (2a_1 + 2a_2^2) < 0 \quad \text{for } \alpha > 0.
\]
Using (4.41)-(4.43) and the continuity of function \( f(\alpha) \) on \([0, +\infty)\), we find that there exists a unique \( \alpha \in (0, +\infty) \) such that \( f(\alpha) = 0 \). Hence,
\[
Re\lambda_1 = Re\lambda_2 = \alpha > 0 \quad \text{for large } \chi_P.
\]
This shows that, for large \( \chi_P \), the steady state \((N, P, K) = (0, 0, 0)\) is spatio-temporally unstable for the system (4.13)-(4.15).

**Remark 4.1.** For the typical parameter values given in [4]: \( c_1 = 0.3, \rho_1 = 2.5, \eta_1 = 0.2 \), the assumptions (4.9) and (4.11) hold.

**Remark 4.2.** Theorem 4.1 implies that, for large chemotaxis coefficient \( \chi_P \), the spatially inhomogeneous patterns for the system (4.1)-(4.5) may evolve by chemotaxis-driven instability.
References


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