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Shannon, Lévy, and Tsallis: A Note

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Abstract

The Tsallis nonextensive entropy of the statistical physics literature exactly matches the previously defined Havrda-Charvat structural α-entropy of information theory. We offer three novel results by optimizing Shannon and Havrda-Charvat entropies under different sets of conditions. The results yield generalized t-distributions that encompass the entire family of Lévy stable distributions. The Tsallis distribution is found to be just one special case of the Lévy, and not the other way around.

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1 Introduction

The Shannon entropy [2] of a variable X with density function $f(x)$ is defined as

$$h(x) \equiv \int_{-\infty}^{\infty} f(x) \ln(x) dx \tag{1}$$

where $f(x)\ln\left(f(x)\right)=0$ if $f(x)=0$. Shannon entropy has been extended in several ways. One particular generalization is Havrda-Charvat structural α-entropy [1], i.e.

$$h_\alpha(x)=\frac{1-\int_{-\infty}^{\infty}f^{\alpha}(x)dx}{\alpha-1} \tag{2}$$

where $\alpha\neq1$. One can get distinct entropy measures by choosing different degrees α. Shannon entropy is the special case of (2) as $\alpha\to1$.

In the statistical physics literature Shannon entropy (1) is called Gibbs entropy, and is meant to be the amount of "disorder" of a system. Equation (2) is well known as Tsallis nonextensive entropy [3, 5]. One standard assumption of statistical mechanics is that quantities like energy are "extensive" variables, meaning that the total energy of a system is proportional to the system size. Similarly the entropy is also assumed to be extensive. Tsallis proposed to replace the usual Gibbs extensive entropy with his nonextensive entropy, and maximize that, subject to some constraints. He got an infinite family of Tsallis nonextensive entropies, indexed by α (actually he called it "q"), which quantifies the degree of departure from extensivity. One can get back the Gibbs entropy by making $\alpha\to1$. While Gibbs entropy maximization generates exponential (Boltzmann) distributions, Tsallis nonextensive entropy generates (he claims) "type II generalized Lévy stable distributions", with the usual properties of heavy tails that follow power laws. Many classical results of statistical mechanics can be (and have been) translated into the new Tsallis setting.

That Tsallis nonextensive entropy is exactly the same as Havrda-Charvat structural α-entropy is hugely neglected by the nonextensive mechanics community. Though Tsallis himself concedes that Tsallis entropy is a new rediscovery (not a discovery) "in the labyrinthine history of the entropies" [4], he still fails to recognize that his entropy exactly matches Havrda-Charvat's. More importantly, as we will show below, Tsallis distributions follow from Lévy distributions, and not the other way around.

2 Analysis

Now we show three types of generalized t-distributions that follow directly from the definitions of entropy. The following three results can be easily obtained using the Lagrange equation of the calculus of variations.

Result 1. Shannon entropy (1) can be optimized under

$$\int_{-\infty}^{\infty}f(x)dx=1 \tag{3}$$

and, after prescribing

$$E\left[\ln\left(1+a\,|\,x\,|^{b}\right)\right]=\text{constant} \tag{4}$$

the result is the distribution

$$f(x)=\frac{c}{\left(1+a\,|\,x\,|^{b}\right)^{d}} \tag{5}$$

where $-\infty<x<\infty$, $a,b,c,d>0$, and $bd>1$. Because these are partially related through (3) and (4), we have only two free parameters as well as one constant in (5).

Result 2. Havrda-Charvat entropy (2) can be optimized under (3) and, after prescribing the absolute moment

$$\int_{-\infty}^{\infty}|\,x\,|^{b}\ f^{\alpha}(x)dx=\text{constant} \tag{6}$$

the result is the distribution

$$f(x)=\frac{c}{\left(1+a\,|\,x\,|^{b}\right)^{1/(\alpha-1)}} \tag{7}$$

where $-\infty<x<\infty$, $a,b,c>0$, and $1<\alpha<\frac{2b+1}{b+1}$. Because these are partially related through (3) and (6), we have two free parameters along with one given constant in the density function (7). Tsallis distribution [3] is the special case of (7) for $b=2$.

Result 3. Havrda-Charvat entropy (2) can be optimized under (3), and, after prescribing the absolute moment

$$\int_{-\infty}^{\infty}|\,x\,|^{b}\ f(x)dx=\text{constant} \tag{8}$$

the result is the distribution

$$f(x)=\frac{c}{\left(1+a\,|\,x\,|^{b}\right)^{1/(1-\alpha)}} \tag{9}$$

where $-\infty<x<\infty$, $a,b,c>0$, and $\frac{1}{1+b}<\alpha<1$. These are partially related through (3) and (8), and then we have two free parameters together with one constant in the density function (9). For $b=2$, (8) defines the variance.

The density functions (5), (7), and (9) are in fact generalized *t*-distributions that have polynomial and power law tails for small and large values of x respectively. In particular, they encompass the entire family of Lévy stable distributions for small and large x. Indeed, for (5) one has

$$\begin{aligned} f(x) &\sim c\,(1 - ad\,|x|^{b}) = f_1(x), \text{ for small } |x| \\ f(x) &\sim c\,(a\,|x|^{b})^{-d} = f_2(x), \text{ for large } |x| \end{aligned} \tag{10}$$

where $bd > 1$. (Similar results can be found for (7) and (9).) Because the Lévy distributions can be encompassed by the generalized *t*-distributions (5), (7), and (9), the density functions that can be derived from Shannon entropy are even more general.

3 Conclusion

This note connected Shannon and Havrda-Charvat entropies and the statistical physics concepts of entropy. It also showed that the entire family of Lévy stable distributions can be derived from generalized *t*-distributions, which in turn follow from the concept of entropy. The Tsallis nonextensive entropy is similar to the previously defined Havrda-Charvat structural α-entropy, which is itself one known generalization of Shannon entropy. This point has been hugely neglected by the nonextensive mechanics physicists. Though Tsallis himself concedes that his entropy concept is solely a new rediscovery, he fails to recognize that Tsallis entropy exactly matches Havrda-Charvat's. More importantly, Tsallis distributions follow from Lévy distributions, and not the other way around. The entire family of Lévy stable distributions are in turn less general than the *t*-distributions that can be derived directly from Shannon entropy.

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