Conditioned Limit Theorems for Weighted Sums of Random Sequence

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Abstract. Some notions of conditionally dominated random variables are introduced and characterized. Under rather minimal assumptions on random variables \( \{X, X_n, n \geq 1\} \), some limit theorems of Jamison's type of weighted sums of random variables are obtained.

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1. Introduction

It is of interesting in probability theory and statistics to consider the convergence of weighted sums \( \sum_{k=1}^{n} w_{nk} [X_k - E(X_k | F_{k-1})] \), see e.g. [2],[4],[5],[6] and many results have been made in the field. Conditions of independent random variables are basic in historic results due to Bernoulli, Borel and Komogrov(cf.[5]). Recently, serious attempts have been made to relax these strong conditions(cf.[4]). Such as stochastically dominated conditions of some kind, and these have played an increasingly important role as a key condition in proving laws of large numbers. In Y. Adler and A. Rosasky, for example, the authors considered a sequence of independent random variables.

In the present paper, we are interesting in introducing a new set of conditions to be called conditionally dominated in Cesàro sense concerning the array \( \{w_{nk}\} \) for a sequence of random variables, and we will show some stability results of Jamison type weighted sums of arbitrary random variables in

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more general settings.

2. Preliminary work

Some definitions and preliminary results will be presented prior to establish
the main results.

Let \((\Omega, \mathcal{F}, \mathcal{F}_n, \mu, n \in \mathbb{N})\) be a probability space and \(\mathbb{N}\) denote the set of non-negative integers, where \(\{\mathcal{F}_n = \sigma(X_0, \cdots, X_n), n \in \mathbb{N}\}\) is an increasing sequence of sub \(\sigma\)-algebras of the basic \(\sigma\)-algebra \(\mathcal{F}\) and \(\mathcal{F}_0\) is the trivial \(\sigma\)-algebra \(\{\phi, \Omega\}\). Suppose that \(\{X, X_n, n \in \mathbb{N}\}\) be a stochastic sequence defined on this underlying probability space.

For all \(n \in \mathbb{N}\), \(A, A_0, A_1, \cdots, A_n \in \mathcal{B}\) (\(\mathcal{B}\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}\)),

\[
\mu\{X \in A\} = \int_{x \in A} p(x) \mu(dx) \tag{1}
\]

and

\[
\mu\{X_0 \in A_0, \cdots, X_n \in A_n\} = \int_{x_0 \in A_0} \cdots \int_{x_n \in A_n} p_n(x_0, \cdots, x_n) \mu(dx_0 \cdots dx_n) \tag{2}
\]

and denote the conditional pmf(pdf) by

\[
p_n(x_n|x_0, \cdots, x_{n-1}) = \frac{p_n(x_0, \cdots, x_n)}{p_{n-1}(x_0, \cdots, x_{n-1})} \tag{3}
\]

where \(p(x), p_n(\cdots)\) are probability mass functions(pdf) or probability density functions(pdf) w.r.t. \(\mu\). In nearly all cases \(\mu\) and is either the Lebesgue or counting measures.

Let \(\{a_k, k \geq 1\}\) be a sequence of positive real numbers, \(b = \sup\{a_k, k \geq 1\} < \infty, W = \{w_{nk}, 1 \leq k \leq n, n \geq 1\}\) be a triangular array of positive real numbers, where

\[
w_{nk} = \begin{cases} \frac{a_k}{\sigma_n}, & \text{for } k \leq n \\ 0, & \text{for } k > n \end{cases}
\]

satisfying \(\sum_{k=1}^n w_{nk} \leq 1\) and with \(\sigma_n \uparrow \infty\). We shall study the Jamison type weighted sums of the form

\[
\mathcal{T}_n(W) = \sum_{k=1}^n w_{nk}[X_k - \mathbb{E}(X_k|\mathcal{F}_{k-1})] \tag{4}
\]

for all \(n \in \mathbb{N}\).

**Definition 1.** Let \(\{X, X_n, n \in \mathbb{N}\}\) be a sequence of random variables, the conditionally moment generating function and conditional tail probability moment generating function of \(X_k\) with respect to \(a_k\) as follows:

\[
\mathbb{M}_n(s; x_0, \cdots, x_{n-1}) = \int_{-\infty}^{\infty} e^{a_n x_s} p_n(x|x_0, \cdots, x_{n-1}) \mu(dx). \tag{5}
\]
Conditioned limit theorems

\[ \tilde{M}_n(s; x_0, \ldots, x_{n-1}) \]

\[ = \int_0^\infty e^{a_nxs} \int_x^\infty p_n(t|x_0, \ldots, x_{n-1}) dt \mu(dx) \]

\[- \int_{-\infty}^0 e^{a_nxs} \int_{-\infty}^x p_n(t|x_0, \ldots, x_{n-1}) dt \mu(dx). \]

and let

\[ \tilde{M}_n(s) = \int_0^\infty e^{bxs} \int_x^\infty p(t) dt \mu(dx) - \int_{-\infty}^0 e^{bxs} \int_{-\infty}^x p(t) dt \mu(dx). \] (7)

\[ \tilde{M}_n^+(s) = \int_0^\infty e^{bxs} \int_x^\infty p(t) dt \mu(dx), \quad \tilde{M}_n^-(s) = \int_{-\infty}^0 e^{bxs} \int_{-\infty}^x p(t) dt \mu(dx). \] (8)

provided that the integrals exit for \( s \in (-s_0, s_0) \) for some \( s_0 > 0 \).

**Definition 2.** (cf.[6]) Let \( \{ X_n, n \in \mathbb{N} \} \) be a sequence of random variables, and is said to be:

1) conditionally dominated by a random variable \( X \) in double sides (we write \( \{ X_n, n \in \mathbb{N} \} \prec X \)) if there exists a constant \( C > 0 \), for almost every \( \omega \in \Omega \), such that

\[ \sup_{n \in \mathbb{N}} \mu\{ X_n > x | \mathcal{F}_{n-1} \} \leq C \mu\{ X > x \} \quad \text{for all } x > 0. \]

and

\[ \sup_{n \in \mathbb{N}} \mu\{ X_n < x | \mathcal{F}_{n-1} \} \leq C \mu\{ X < x \} \quad \text{for all } x < 0. \]

2) conditionally dominated in Cesàro sense by a random variable \( X \), concerning the array \( \{ w_{nk} \} \), in double sides (we write \( \{ X_n, n \in \mathbb{N} \} \prec X(C) \)) if there exists a constant \( C > 0 \), for almost every \( \omega \in \Omega \), such that

\[ \sup_{n \in \mathbb{N}} \sum_{k=1}^n w_{nk} \mu\{ X_k > x | \mathcal{F}_{k-1} \} \leq C \mu\{ X > x \} \quad \text{for all } x > 0. \]

and

\[ \sup_{n \in \mathbb{N}} \sum_{k=1}^n w_{nk} \mu\{ X_k < x | \mathcal{F}_{k-1} \} \leq C \mu\{ X < x \} \quad \text{for all } x < 0. \]

**Remark.** In the particular case of array

\[ w_{nk} = \begin{cases} 1/n, & \text{for } k \leq n \\ 0, & \text{for } k > n \end{cases} \]

the condition of \( \{ w_{nk} \} \)-stochastically dominated in Cesàro sense is the dominated in Cesàro sense.

**Lemma 1.** Let \( \mathcal{M}_n(s; x_0, \ldots, x_{n-1}), \tilde{M}_n(s; x_0, \ldots, x_{n-1}) \) be defined as above, then

\[ \frac{\mathcal{M}_n(s; x_0, \ldots, x_{n-1}) - 1}{s} = a_n \tilde{M}_n(s; x_0, \ldots, x_{n-1}) \] (9)
and
\[\widetilde{M}_n(0) = \mathbb{E}(X_n | x_0, \cdots, x_{n-1})\] (10)

**Proof.** Since
\[
\frac{\mathbb{M}_n(s; x_0, \cdots, x_{n-1}) - 1}{s} = \int_0^\infty \frac{e^{a_n s x} - 1}{s} p_n(x | x_0, \cdots, x_{n-1}) \mu(dx)
\]
\[+ \int_{-\infty}^0 \frac{e^{a_n s x} - 1}{s} p_n(x | x_0, \cdots, x_{n-1}) \mu(dx)\]
\[= \int_0^\infty \frac{1 - e^{a_n s x}}{s} d \int_x^\infty p_n(t | x_0, \cdots, x_{n-1}) dt\]
\[+ \int_{-\infty}^0 \frac{1 - e^{a_n s x}}{s} d \int_{-\infty}^x p_n(t | x_0, \cdots, x_{n-1}) dt\]
\[= \left[\frac{1 - e^{a_n s x}}{s} \int_x^\infty p_n(t | x_0, \cdots, x_{n-1}) dt\right]_0^\infty\]
\[+ a_n \int_0^\infty \frac{e^{a_n s x}}{s} \int_x^\infty p_n(t | x_0, \cdots, x_{n-1}) dt \mu(dx)\]
\[+ \left[\frac{1 - e^{a_n s x}}{s} \int_{-\infty}^x p_n(t | x_0, \cdots, x_{n-1}) dt\right]_0^\infty\]
\[- a_n \int_{-\infty}^0 \frac{e^{a_n s x}}{s} \int_{-\infty}^x p_n(t | x_0, \cdots, x_{n-1}) dt \mu(dx) = a_n \widetilde{M}_n(s; x_0, \cdots, x_{n-1}).\]
(9) follows. (10) can also be obtained immediately from integration by parts. □

3. Mainstream

**Theorem 1.** Let \( \{X, X_n, n \in \mathbb{N}\} \) be defined as before. If \( \{X_n, n \in \mathbb{N}\} \prec X \) with \( \mathbb{E}X < \infty \) and let \( \sigma_n \uparrow \infty \) as \( n \to \infty \). Then
\[\lim_{n} T_n(W) = 0, \quad \mu - a.s.\] (11)

**Proof.** Putting
\[p_k(s; x_k) = \frac{e^{a_k x_k s} p_k(x_k | x_0, \cdots, x_{k-1})}{\mathbb{M}_k(s)}\]
and
\[\tilde{p}_n(s; x_0, \cdots, x_n) = p_0(x_0) \prod_{k=1}^{n} p_k(s; x_k), \quad n = 1, 2, \cdots,\]
Therefore \( \tilde{p}_n(s; x_0, \cdots, x_n) \) is a pmf or pdf of \( n + 1 \) variables, let us define
\[\Lambda_n(s, \omega) = \left\{ \begin{array}{ll} \frac{\tilde{p}_n(s; x_0, \cdots, x_n)}{p_n(x_0, \cdots, x_n)} & , \text{if the denominator > 0} \\ 0, & \text{otherwise} \end{array} \right.\]
From reference [3], we have
\[
\limsup_n \sigma_n^{-1} \log \Lambda_n(s; \omega) \leq 0, \quad \mu - a.s. \tag{12}
\]

Note that
\[
\log \Lambda_n(s; \omega) = s \sum_{k=1}^{n} a_k X_k - \sum_{k=1}^{n} \log \mathbb{M}_k(s; X_0, \ldots, X_{k-1}) \tag{13}
\]

By (12) and (13), we have
\[
\limsup_n \frac{1}{\sigma_n} \left[ s \sum_{k=1}^{n} a_k X_k - \sum_{k=1}^{n} \log \mathbb{M}_k(s; X_0, \ldots, X_{k-1}) \right] \leq 0, \quad \mu - a.s. \tag{14}
\]

Thus
\[
\limsup_n s \sum_{k=1}^{n} w_{nk} X_k \leq \limsup_n \frac{1}{\sigma_n} \sum_{k=1}^{n} \log \mathbb{M}_k(s; X_0, \ldots, X_{k-1}), \quad \mu - a.s. \tag{15}
\]

By the property of the superior limit \( \limsup_n (a_n - b_n) \leq 0 \Rightarrow \limsup_n (a_n - c_n) \leq \limsup_n (b_n - c_n) \) and note that \( \mathbb{E}(X_k | \mathcal{F}_{k-1}) < \infty, \quad a.s. \ k = 1, 2, \ldots. \)

Dividing two sides of (15) by \( s \), we have, by lemma 1, for any \( s \in (-s_0, 0) \),
\[
\liminf_n \sum_{k=1}^{n} w_{nk} [X_k - \mathbb{E}(X_k | \mathcal{F}_{k-1})] 
\geq \liminf_n \frac{1}{\sigma_n} \sum_{k=1}^{n} \left[ \frac{\log \mathbb{M}_k(s; X_0, \ldots, X_{k-1})}{s} - a_k \mathbb{E}(X_k | \mathcal{F}_{k-1}) \right], \quad \mu - a.s. \tag{16}
\]

From the inequality \( \log x \leq x - 1 (x > 0) \) and lemma 1, we have
\[
\liminf_n \mathbb{T}_n(W) \geq \liminf_n \frac{1}{\sigma_n} \sum_{k=1}^{n} \left[ \frac{\mathbb{M}_k(s; X_0, \ldots, X_{k-1}) - 1}{s} - a_k \mathbb{E}(X_k | \mathcal{F}_{k-1}) \right]
\]
\[
= \liminf_n \frac{1}{\sigma_n} \sum_{k=1}^{n} a_k [\mathbb{M}_k(s; X_0, \ldots, X_{k-1}) - \mathbb{E}(X_k | \mathcal{F}_{k-1})], \quad \mu - a.s. \tag{17}
\]

Let
\[
\varphi(s) = \liminf_n \sum_{k=1}^{n} w_{nk} [\mathbb{M}_k(s; X_0, \ldots, X_{k-1}) - \mathbb{E}(X_k | \mathcal{F}_{k-1})], \quad s \in (-s_0, 0) \tag{18}
\]

If \(-s_0 \leq s < s + \Delta s < 0\), by (18) and noticing that \( \{X_n, n \in \mathbb{N}\} \prec X \) and \( \sum_{k=1}^{n} w_{nk} \leq 1 \), we have
\[
\varphi(s + \Delta s) - \varphi(s) 
\geq \liminf_n \sum_{k=1}^{n} w_{nk} [\mathbb{M}_k(s + \Delta s; X_0, \ldots, X_{k-1}) - \mathbb{M}_k(0; X_0, \ldots, X_{k-1})]
\]
These complete the proofs of the Theorem 1.
Theorem 2. Let \( \{X, X_n, n \in \mathbb{N}\} \) be defined as above and \( \{X_n, n \in \mathbb{N}\} \prec X(C) \) with \( \mathbb{E}X < \infty \). If \( \sigma_n \uparrow \infty \) as \( n \to \infty \), then
\[
\lim_{n} \mathbb{T}_n(W) = 0, \; \mu - a.s.
\]

Corollary 1. (SLLN) Let \( \{X, X_n, n \in \mathbb{N}\} \) be a sequence of independent random variables, if
\[
\sup_{n \in \mathbb{N}} \mu\{|X_n| > x\} \leq C \mu\{|X| > x\} \quad \text{for all } x > 0.
\]
or
\[
\sup_{n \in \mathbb{N}} \sum_{k=1}^{n} w_{nk} \mu\{|X_k| > x\} \leq C \mu\{|X| > x\} \quad \text{for all } x > 0.
\]
then
\[
\lim_{n} \mathbb{T}_n(W) = 0, \; \mu - a.s.
\]

References


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