Viscosity Approximative Methods for Nonexpansive Nonself-Mappings without Boundary Conditions in Banach Spaces

Rabian Wangkeeree and Pramote Markshoe

Department of Mathematics, Faculty of Science
Naresuan University, Phitsanulok 65000, Thailand
rabianw@nu.ac.th (R. Wangkeeree)
pramotem@nu.ac.th (P. Markshoe)

Abstract. Let $C$ be a nonempty closed convex subset of a uniformly smooth Banach space $E$, $T : C \to E$ be a nonexpansive mapping and $P$ be a sunny nonexpansive retraction of $E$ onto $C$. For $x_0 \in C$, the explicit iterative sequence $\{x_n\}$ is given by

$$x_{n+1} = P\left(\alpha_n f(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n)\right) \quad \text{for } n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $[0, 1)$ respectively satisfying appropriate conditions, and $f : C \to C$ is a fixed contractive mapping. We prove that $\{x_n\}$ converges strongly to a fixed point of $T$ without boundary conditions. The results presented extend and improve the corresponding ones announced by Chen et al. [2], and others.

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1. Introduction

Let $C$ be a nonempty closed convex subset of a Banach space $E$, and let $T : C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). We use $\text{Fix}(T)$ to denote the set of fixed points of $T$; that is, $\text{Fix}(T) = \{x \in C : x = Tx\}$. Recall that a selfmapping $f : C \to C$ is a contraction on $C$ if there exists a constant $\beta \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \beta \|x - y\|, \quad \forall x, y \in C. \quad (1.1.1)$$

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Xu [8] defined the following two viscosity iterations for nonexpansive mappings:

\[ x_t = tf(x_t) + (1 - t)Tx_t \]  
(1.1.2)

and

\[ x_{n+1} = \alpha_nf(x_n) + (1 - \alpha_n)Tx_n \]  
(1.1.3)

where \( \{\alpha_n\} \) is a sequence in (0, 1). Xu proved the strong convergence of \( \{x_t\} \) defined by (1.1.2) as \( t \to 0 \) and \( \{x_n\} \) defined by (1.1.3) in both Hilbert space and uniformly smooth Banach space.

Recently, Song and Chen [4] proved if \( C \) is a closed subset of a real reflexive Banach space \( E \) which admits a weakly sequentially continuous duality mapping from \( E \) to \( E \), and if \( T : C \to E \) is a nonexpansive nonself-mapping satisfying the weakly inward condition, \( F(T) \neq \emptyset \), \( f : C \to C \) is a fixed contractive mapping, and \( P \) is a sunny nonexpansive retraction of \( E \) onto \( C \), then the sequences \( \{x_t\} \) and \( \{x_n\} \) defined by

\[ x_t = P(tf(x_t) + (1 - t)Tx_t) \]  
(1.1.4)

and

\[ x_{n+1} = P(\alpha_nf(x_n) + (1 - \alpha_n)Tx_n) \]  
(1.1.5)

strongly converge to a fixed point of \( T \). Very recently, Chen and Zhu [2] established the strong convergence of both \( \{x_t\} \) and \( \{x_n\} \) defined by (1.1.4) and (1.1.5) respectively, for a nonexpansive nonself-mapping \( T \) in a uniformly smooth Banach space.

Let \( C \) be a nonempty closed convex subset of a uniformly smooth Banach space \( E \), \( T : C \to E \) be a nonexpansive nonself-mapping and \( P \) be a sunny nonexpansive retraction of \( E \) onto \( C \), the purpose of this paper is to use the following iterative process : \( x_0 \in C \),

\[ x_{n+1} = P(\alpha_nf(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n)) \quad \text{for } n = 0, 1, 2, \ldots , \]  
(1.1.6)

where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in (0, 1) and [0, 1) respectively, and \( f : C \to C \) is a fixed contractive mapping, to approximate to the fixed point of nonexpansive mapping \( T \) without boundary conditions. Our results extend and improve the corresponding ones announced by Chen et al. [2], and others.

### 2. Preliminaries

Let \( E \) be a real Banach space and let \( J \) denote the normalized duality mapping from \( E \) into \( 2^{E^*} \) given by

\[ J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\|\|f\| = \|x\| = \|f\| \}, \forall x \in E \]

where \( E^* \) be the dual space of \( E \) and and \( \langle \cdot, \cdot \rangle \) denotes the generalized duality pairing. In the sequence, we will denote the single-valued duality mapping by \( j \), and \( x_n \to x \) will denote strong convergence of the sequence \( \{x_n\} \) to \( x \).
In Banach space $E$, the following result (the Subdifferential Inequality) is well known (Theorem 4.2.1 of [5]): \( \forall x, y \in E, \forall j(x + y) \in J(x + y), \forall j(x) \in J(x) \)

\[
\| x \|^2 + 2\langle y, j(x) \rangle \leq \| x + y \|^2 \leq \| x \|^2 + \langle y, j(x + y) \rangle.
\] (2.2.1)

Recall that the norm of $E$ is said to be Gâteaux differentiable if the limit

\[
\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}
\] (2.2.2)

exists for each $x, y$ in its unit sphere $U = \{ x \in E : \| x \| = 1 \}$. Such a Banach space $E$ is called smooth. The norm of a Banach space $E$ is also said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit of (2.2.2) is attained uniformly for $x \in U$. Finally, the norm is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth) if the limit in (2.2.2) is attained uniformly for $(x, y) \in U \times U$. A Banach space $E$ is said to be smooth if and only if $J$ is single valued. It is also well known that if $E$ is uniformly smooth, $J$ is uniformly norm-to-norm continuous. These concepts may be found in [5].

If $C$ and $D$ are nonempty subsets of a Banach space $E$ such that $C$ is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is called a retraction from $C$ to $D$ if $P^2 = P$. It is easily known that a mapping $P : C \to D$ is retraction, then $Px = x$, for all $x \in D$. A mapping $P : C \to D$ is called sunny if

\[
P(Px + t(x + Px)) = Px, \forall x \in C,
\] (2.2.3)

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A subset $D$ of $C$ is said to be a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction from $C$ onto $D$. For more detail, see [5].

The following lemma is well known [5].

**Lemma 2.1.** Let $C$ be a nonempty convex subset of a smooth Banach space $E$, $D \subset C$, $J : E \to E^\ast$ the (normalized) duality mapping of $E$, and $P : C \to D$ a retraction. Then the following are equivalent:

(i) $\langle x - Px, j(y - Px) \rangle \leq 0$ for all $x \in C$ and $y \in D$

(ii) $P$ is both sunny and nonexpansive.

Let $C$ be a nonempty convex subset of a Banach space $E$, then for $x \in C$, the inward set is given by [6, 7]

\[
I_C(x) = \{ y \in E : y = x + \lambda(z - x), z \in C, \lambda \geq 0 \}.
\] (2.2.4)

A mapping $T : C \to E$ is said to be satisfying the inward condition if $Tx \in I_C(x)$ for all $x \in C$. $T$ is also said to be satisfying the weakly inward condition if for each $x \in C$, $Tx \in I_C(x)$ where $I_C(x)$ is the closure of $I_C(x)$. Very recently for a nonself-mapping $T$ from $C$ into $E$, Matsushita and Takahashi [3] studied the following condition:

\[
Tx \in S^C_x
\] (2.2.5)
for all $x \in C$, where $S_x = \{ y \in C : y \neq x, Py = x \}$ and $P$ is a sunny nonexpansive retraction from $E$ onto $C$. Then they proved the following three lemmas.

**Lemma 2.2.** [3, Lemma 3.1] Let $C$ be a closed convex subset of a smooth Banach space $E$ and let $T$ be a mapping form $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$. If $T$ satisfies the condition (2.2.5), then $F(T) = F(PT)$, where $P$ is a sunny nonexpansive retraction from $E$ onto $C$.

**Lemma 2.3.** [3, Lemma 3.2] Let $C$ be a closed convex subset of a smooth Banach space $E$ and let $T$ be a mapping form $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$. If $T$ satisfies the weakly inward condition, then $T$ satisfies the condition (2.2.5).

**Lemma 2.4.** [3, Lemma 3.3] Let $C$ be a closed convex subset of a strictly convex Banach space $E$ and let $T$ be a nonexpansive mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$. If $F(T) \neq \emptyset$ then $T$ satisfies the condition (2.2.5).

The following lemma can be founded in [1].

**Lemma 2.5.** [1] Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\gamma_n\}$ a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \gamma_n = \infty$, $\{\beta_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\{\alpha_n\}$ a sequence of real numbers with $\limsup_{n \to \infty} \alpha_n \leq 0$. Suppose that

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n \alpha_n + \beta_n$$

for all $n \in \mathbb{N}$. Then $\lim_{n \to \infty} s_n = 0$.

The following lemma can be founded in [8].

**Theorem 2.6.** [8] Let $X$ be a uniformly smooth Banach space, $C$ a closed convex subset of $X$, $T : C \rightarrow C$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$, and $f : C \rightarrow C$ a contractive mapping. Then as $t \to 0$, $\{x_t\}$ defined by

$$x_t = tf(t) + (1 - t)Tx_t$$

(2.2.6)

converges strongly to a fixed point $q$ of $T$ such that $q$ is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (f - I)q, j(q - u) \rangle \leq 0 \text{ for all } u \in F(T).$$

(2.2.7)

### 3. Main Results

**Theorem 3.1.** Let $X$ be a uniformly smooth Banach space, $C$ a closed convex subset of $X$. Suppose that $C$ is a sunny nonexpansive retract of $E$ with $P$ a nonexpansive retraction. Let $T : C \rightarrow E$ a nonexpansive nonself-mapping with $\text{Fix}(T) \neq \emptyset$, and $f : C \rightarrow C$ be a contractive mapping. Then as $t \rightarrow 0$, $\{x_t\}$ defined by

$$x_t = tf(t) + (1 - t)PTx_t$$

(3.3.1)
converges strongly to a fixed point \( q \) of \( T \) such that \( q \) is the unique solution in \( F(T) \) to the following variational inequality:

\[
\langle (f - I)q, j(q - u) \rangle \leq 0 \text{ for all } u \in F(T).
\]

**Proof.** Applying the Theorem 2.6 with the nonexpansive self-mapping \( PT \), we obtain that \( \{x_t\} \) converges strongly as \( t \to 0 \) to a fixed point of \( PT \). Since \( F(T) \neq \emptyset \), using Lemma 2.2 and 2.4, we obtain \( F(T) = F(PT) \). The proof is complete. \( \square \)

**Theorem 3.2.** Let \( E \) be a uniformly smooth Banach space, \( C \) is a nonempty closed convex subset of \( E \). Suppose that \( C \) is a sunny nonexpansive retract of \( E \). Let \( T : C \to E \) be a nonexpansive nonself-mapping with \( F(T) \neq \emptyset \), and \( f : C \to C \) a fixed contractive mapping with coefficient \( \beta \in (0, 1) \). The sequence \( \{x_n\} \) is defined by (1.1.6), where \( P \) is the sunny nonexpansive retraction of \( E \) onto \( C \), \( \{\alpha_n\} \subset (0, 1) \) and \( \{\beta_n\} \subset [0, 1) \), and satisfying the following conditions:

(i) \( \lim_{n \to \infty} \alpha_n = 0 \);  
(ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty \);  
(iii) \( \lim_{n \to \infty} \beta_n = 0 \);  
(iv) \( \sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}| < +\infty \);  
(v) either \( \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < +\infty \) or \( \lim_{n \to \infty}(\alpha_{n+1}/\alpha_n) = 1 \).

Then as \( n \to \infty \), the sequence \( \{x_n\} \) converges strongly to a fixed point \( q \) of \( T \) such that \( q \) is the unique solution in \( F(T) \) to the variational inequality (3.3.2).

**Proof.** First we show that \( \{x_n\} \) is bounded. Take \( u \in F(T) \), it follows that

\[
\|x_{n+1} - u\| = \|P(\alpha_n f(x_n) + (1 - \alpha_n) (\beta_n x_n + (1 - \beta_n)Tx_n)) - Pu\| \\
\leq \|\alpha_n f(x_n) + (1 - \alpha_n) (\beta_n x_n + (1 - \beta_n)Tx_n) - u\| \\
\leq \alpha_n \|f(x_n) - u\| + (1 - \alpha_n) \|\beta_n x_n - u\| + (1 - \beta_n) \|Tx_n - u\| \\
\leq \alpha_n \beta \|x_n - u\| + \alpha_n \|f(u) - u\| + (1 - \alpha_n) \beta_n \|x_n - u\| \\
+ (1 - \beta_n)(1 - \alpha_n) \|x_n - u\| \\
= (1 - (1 - \beta_n)\alpha_n) \|x_n - u\| + \alpha_n \|f(u) - u\| \\
\leq \max\{\|x_n - u\|, \frac{1}{1 - \beta} \|f(u) - u\|\}.
\]

By induction, we have

\[
\|x_n - u\| \leq \max\{\|x_0 - u\|, \frac{1}{1 - \beta} \|f(u) - u\|\}, \forall n \geq 0.
\]

Therefore \( \{x_n\} \) is bounded, so are \( \{Tx_n\} \) and \( \{f(x_n)\} \). Then we get that

\[
\|x_{n+1} - PTx_n\| = \|P(\alpha_n f(x_n) + (1 - \alpha_n) (\beta_n x_n + (1 - \beta_n)Tx_n)) - PTx_n\| \\
\leq \|\alpha_n f(x_n) + (1 - \alpha_n) (\beta_n x_n + (1 - \beta_n)Tx_n) - Tx_n\| \\
\leq \alpha_n \|f(x_n) - Tx_n\| + (1 - \alpha_n) \|\beta_n x_n + (1 - \beta_n)Tx_n - Tx_n\| \\
= \alpha_n \|f(x_n) - Tx_n\| + (1 - \alpha_n) \beta_n \|x_n - Tx_n\|.
\]
Indeed we have
\[
\sum_{n=1}^{\infty} \frac{1}{a_n s_n} = 0 \quad \text{as} \quad n \to \infty.
\]

Next we shall show that
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]

Indeed we have
\[
\|x_{n+1} - x_n\| = \|P(\alpha_n f(x_n) + (1 - \alpha_n)\beta_n x_n + (1 - \beta_n)Tx_n) - P(\alpha_n f(x_n) + (1 - \alpha_n)\beta_n x_n + (1 - \beta_n)Tx_n)\|
\leq \alpha_n\|f(x_n) - f(x_n)\| + |\alpha_n - \alpha_n - 1|\|f(x_n)\|
+ (1 - \alpha_n)\|\beta_n x_n + (1 - \beta_n)Tx_n - \beta_n x_n - (1 - \beta_n)Tx_n\|
+ |\alpha_n - \alpha_n - 1|\|\beta_n x_n + (1 - \beta_n)Tx_n\|
\leq \alpha_n\|x_n - x_n\| + |\alpha_n - \alpha_n - 1|\|f(x_n)\|
+ (1 - \alpha_n)\|\beta_n x_n - x_n\| + |\beta_n - \beta_n - 1|\|x_n - x_n\|
+ |\beta_n - \beta_n - 1|\|Tx_n - Tx_n\| + |\alpha_n - \alpha_n - 1|\|\beta_n x_n + (1 - \beta_n)Tx_n\|
+ |\beta_n - \beta_n - 1|\|x_n - x_n\| + |\alpha_n - \alpha_n - 1|\|Tx_n - Tx_n\|
= \alpha_n\|x_n - x_n\| + (1 - \alpha_n)\|x_n - x_n\| + |\alpha_n - \alpha_n - 1|\|f(x_n)\|
+ |\beta_n - \beta_n - 1|\|x_n - x_n\| + |\alpha_n - \alpha_n - 1|\|Tx_n - Tx_n\|
\leq 1 - (1 - \beta)\alpha_n\|x_n - x_n\| + K_n,
\]
where \( K_n = |\alpha_n - \alpha_n - 1|\|f(x_n)\| + |\beta_n - \beta_n - 1|\|x_n - x_n\| + (1 - \alpha_n)\|\beta_n - \beta_n - 1|\|x_n - x_n\| + |\alpha_n - \alpha_n - 1|\|Tx_n - Tx_n\| \). Since \( \{x_n\} \) is bounded, there exists a positive constant \( K \) such that
\[
K_n \leq K(|\alpha_n - \alpha_n - 1| + |\beta_n - \beta_n - 1|),
\]
thus,
\[
\|x_{n+1} - x_n\| \leq (1 - (1 - \beta)\alpha_n)\|x_n - x_n\| + K(|\alpha_n - \alpha_n - 1| + |\beta_n - \beta_n - 1|).
\]

Assume that \( \sum_{n=0}^{\infty} |\alpha_n - \alpha_n - 1| < +\infty \). By Lemma 2.5 and the conditions on \( \{\alpha_n\} \) and \( \{\beta_n\} \) we get the required result.

Assume that \( \lim_{n \to \infty} (\alpha_{n+1}/\alpha_n) = 1 \). Then from (3.3.5), we have
\[
\|x_{n+1} - x_n\| \leq (1 - (1 - \beta)\alpha_n)\|x_n - x_n\| + \alpha_n|1 - \alpha_{n-1}/\alpha_n|K + K|\beta_n - \beta_n - 1|.
\]

By Lemma 2.5 and the conditions on \( \{\alpha_n\} \) and \( \{\beta_n\} \) we also get the required result. Using (3.3.3) and (3.3.4), we get
\[
\|x_n - PTx_n\| \leq \|x_n - x_n\| + \|x_{n+1} - PTx_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]

Let \( q = \lim_{t \to 0} x_t \), where \( \{x_t\} \) is defined in Theorem 3.1, we get that \( q \) is the unique solution in \( F(T) \) following the variational inequality:
\[
\langle (f - I)q, j(q - u) \rangle \leq 0 \quad \text{for all} \quad u \in F(T).
\]

Next we shall show that
\[
\lim_{n \to \infty} \sup_{n \to \infty} \langle f(q) - q, j(x_n - q) \rangle \leq 0.
\]
Using the inequality (2.2.1), we have
\[ x_t - x_n = t(f(x_t) - x_n) + (1 - t)(PTx_t - x_n). \] (3.3.10)

It follows from (3.3.7) that
\[ b_n(t) = \|x_n - PTx_n\|\|x_n - PTx_n\| + 2\|x_n - x_t\| \to 0 \text{ as } n \to \infty. \] (3.3.11)

Using the inequality (2.2.1), we have
\[
\begin{align*}
\|x_t - x_n\|^2 & \leq (1 - t)^2\|PTx_t - x_n\|^2 + 2t(f(x_t) - x_n, j(x_t - x_n)) \\
& \leq (1 - t)^2\|PTx_t - PTx_n + PTx_n - x_n\|^2 + 2t(f(x_t) - x_t, j(x_t - x_n)) \\
& \quad + 2t\|x_t - x_n\|^2 \\
& \leq (1 - t)^2\|x_t - x_n\|^2 + (1 - t)^2\|x_n - PTx_n\|^2 \\
& \quad + 2(1 - t)^2\|PTx_n - x_n\|\|x_t - x_n\| + 2t(f(x_t) - x_t, j(x_t - x_n)) \\
& \quad + 2t\|x_t - x_n\|^2 \\
& \leq (1 + t)^2\|x_t - x_n\|^2 + b_n(t) + 2t(f(x_t) - x_t, j(x_t - x_n)).
\end{align*}
\] (3.3.12)

The last inequality implies
\[
\langle f(x_t) - x_t, j(x_n - x_t) \rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}b_n(t).
\] (3.3.13)

It follows from (3.3.11) that
\[
\limsup_{n \to \infty} \langle f(x_t) - x_t, j(x_n - x_t) \rangle \leq M\frac{t}{2},
\] (3.3.14)

where \( M \) is a constant such that \( M \geq \|x_t - x_n\| \) for all \( t \in (0, 1) \). By letting \( t \to 0 \) in the last inequality we have
\[
\lim_{t \to 0} \limsup_{n \to \infty} \langle f(x_t) - x_t, j(x_n - x_t) \rangle \leq 0.
\] (3.3.15)

On the other hand, for all \( \varepsilon > 0 \) there exits a positive \( \delta_1 \) such that \( t \in (0, \delta_1) \),
\[
\limsup_{n \to \infty} \langle f(x_t) - x_t, j(x_n - x_t) \rangle \leq \frac{\varepsilon}{2}.
\] (3.3.16)

On the other hand, \( \{x_t\} \) converges strongly to \( q \), as \( t \to \infty \), the set \( \{x_t - x_n\} \) is bounded, and the duality map \( J \) is norm-to-norm uniformly continuous on bounded sets of uniformly smooth space \( E \); from \( x_t \to q \) as \( t \to 0 \), we get
\[
\|f(q) - q - (f(x_t) - x_t)\| \to 0 \text{ as } t \to 0,
\]
and
\[
\begin{align*}
\|\langle f(q) - q, j(x_n - q) \rangle - \langle f(x_t) - x_t, j(x_n - x_t) \rangle\| \\
= \|\langle f(q) - q, j(x_n - q) - j(x_n - x_t) \rangle + \langle f(q) - q - (f(x_t) - x_t), j(x_n - x_t) \rangle\| \\
\leq \|f(q) - q\|\|j(x_n - q) - j(x_n - x_t)\| \\
+ \|f(q) - q - (f(x_t) - x_t)\|\|j(x_n - x_t)\| \to 0 \text{ as } t \to 0
\end{align*}
\] (3.3.17)
Hence for the above \( \varepsilon > 0 \), there exists \( \delta_2 > 0 \) such that for all \( t \in (0, \delta_2) \), for all \( n \), we have

\[
\| (f(q) - q, j(x_n - q)) - (f(x_t) - x_t, j(x_n - x_t)) \| \leq \frac{\varepsilon}{2}.
\]  

(3.3.18)

Therefore, we have

\[
\langle f(q) - q, j(x_n - q) \rangle \| \leq \langle f(x_t) - x_t, j(x_n - x_t) \rangle + \frac{\varepsilon}{2}.
\]  

(3.3.19)

Taking \( \delta = \min\{\delta_1, \delta_2\} \), for all \( t \in (0, \delta) \), we have

\[
\limsup_{n \to \infty} \langle f(q) - q, j(x_n - q) \rangle \leq \limsup_{n \to \infty} \langle f(x_t) - x_t, j(x_n - x_t) \rangle + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]  

(3.3.20)

Since \( \varepsilon \) is arbitrary, we get the required inequality (3.3.9). Finally, we shall show that \( x_n \to q \) as \( n \to \infty \). We note that

\[
x_{n+1} - (\alpha_n f(x_n) + (1 - \alpha_n)q) = (x_{n+1} - q) - \alpha_n (f(x_n - q)).
\]

Using the inequality (2.2.1), we have,

\[
\|x_{n+1} - q\|^2 = \|x_{n+1} - (\alpha_n f(x_n) + (1 - \alpha_n)q) + \alpha_n (f(x_n - q))\|^2
\]

\[
\leq \|x_{n+1} - P(\alpha_n f(x_n) + (1 - \alpha_n)q)\|^2 + 2\alpha_n (f(x_n) - q, j(x_{n+1} - q))\|
\]

\[
\leq \|\alpha_n f(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 + \beta_n)T x_n - \alpha_n f(x_n) + (1 - \alpha_n)q)\|^2
\]

\[
+ 2\alpha_n (f(x_n) - f(q), j(x_{n+1} - q))\| + 2\alpha_n (f(q) - q, j(x_{n+1} - q))\|
\]

\[
\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \|f(x_n) - f(q)\| \|x_{n+1} - q\|
\]

\[
+ 2\alpha_n (f(q) - q, j(x_{n+1} - q))\|
\]

\[
\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + \alpha_n \|f(x_n) - f(q)\|^2 + \|x_{n+1} - q\|^2
\]

\[
+ 2\alpha_n (f(q) - q, j(x_{n+1} - q))\|
\]

Therefore we have

\[
(1 - \alpha_n)\|x_{n+1} - q\|^2 \leq (1 - 2\alpha_n + \alpha_n^2) \|x_n - q\|^2 + \alpha_n \beta \|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle.
\]

Thus,

\[
\|x_{n+1} - q\|^2 \leq \left(1 - \frac{\beta^2}{1 - \alpha_n}\right) \|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n} \|x_n - q\|^2
\]

\[
+ \frac{2\alpha_n}{1 - \alpha_n} \langle f(q) - q, j(x_{n+1} - q) \rangle
\]

\[
\leq (1 - \gamma_n) \|x_n - q\|^2 + \lambda \gamma_n \alpha_n + \frac{2}{1 - \beta_n^2} \gamma_n \langle f(q) - q, j(x_{n+1} - q) \rangle,
\]

where \( \gamma_n = \frac{1 - \beta^2}{1 - \alpha_n} \) and \( \lambda \) is a constant such that \( \lambda > \frac{1}{1 - \beta_n} \|x_n - q\|^2 \). Hence

\[
\|x_{n+1} - q\|^2 \leq (1 - \gamma_n) \|x_n - q\|^2
\]
\[ + \gamma_n (\lambda \alpha_n + \frac{2}{1 - \beta_n^2} \gamma_n \langle f(q) - q, j(x_{n+1} - q) \rangle). \quad (3.3.21) \]

It is easily seen that \( \gamma_n \to 0, \sum_{n=1}^{\infty} \gamma_n = \infty \), and noting that

\[ \lim_{n \to \infty} (\lambda \alpha_n + \frac{2}{1 - \beta_n^2} \gamma_n \langle f(q) - q, j(x_{n+1} - q) \rangle) \leq 0. \]

Applying Lemma 2.5 onto (3.3.21), we have \( \{x_n\} \) converges strongly to \( q \). The proof is complete.

If in Theorem 3.2, \( \beta_n = 0 \) for all \( n \geq 0 \), then the iteration (1.1.6) reduces to the iteration (1.1.5). Note that, the weakly inward conditions on the mapping \( T \) can be dropped. In fact, the following Corollary can be obtained from Theorem 3.2 immediately.

**Corollary 3.3.** [2, Theorem 3.4] Let \( E \) be a uniformly smooth Banach space, \( C \) is a nonempty closed convex subset of \( E \), let \( T : C \to E \) be a nonexpansive nonself-mapping satisfying the weakly inward conditions, and \( F(T) \neq \emptyset \). Let \( f : C \to C \) a fixed contractive mapping. The sequence \( \{x_n\} \) is defined by (1.1.5), where \( P \) is the sunny nonexpansive retraction of \( E \) onto \( C \), and \( \{\alpha_n\} \subset (0, 1) \), and satisfying the following conditions:

(i) \( \lim_{n \to \infty} \alpha_n = 0 \);
(ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty \);
(iii) either \( \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < +\infty \) or \( \lim_{n \to \infty} (\alpha_{n+1}/\alpha_n) = 1 \).

Then as \( n \to \infty \), the sequence \( \{x_n\} \) converges strongly to a fixed point \( q \) of \( T \) such that \( q \) is the unique solution in \( F(T) \) to the following variational inequality:

\[ \langle (f - I)q, j(q - u) \rangle \leq 0 \text{ for all } u \in F(T). \]

If in Theorem 3.2, \( T : C \to C \) is the nonexpansive mapping and \( \beta_n = 0 \) for all \( n \geq 0 \), then the iteration (1.1.6) reduces to the iteration (1.1.3). In fact, the following Corollary can be obtained from Theorem 3.2 immediately.

**Corollary 3.4.** [8, Theorem 4.2] Let \( E \) be a uniformly smooth Banach space, \( C \) is a nonempty closed convex subset of \( E \), let \( T : C \to C \) be a nonexpansive mapping with \( F(T) \neq \emptyset \). Let \( f : C \to C \) a fixed contractive mapping. The sequence \( \{x_n\} \) is defined by (1.1.3) and \( \{\alpha_n\} \subset (0, 1) \) satisfying the following conditions:

(i) \( \lim_{n \to \infty} \alpha_n = 0 \);
(ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty \);
(iii) either \( \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < +\infty \) or \( \lim_{n \to \infty} (\alpha_{n+1}/\alpha_n) = 1 \).

Then as \( n \to \infty \), the sequence \( \{x_n\} \) converges strongly to a fixed point \( q \) of \( T \) such that \( q \) is the unique solution in \( F(T) \) to the following variational inequality:

\[ \langle (f - I)q, j(q - u) \rangle \leq 0 \text{ for all } u \in F(T). \]
References


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