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Periodic Solutions for a Predator-Prey System with Dispersal and Semi-Ratio-Dependent Functional Response

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Abstract

By using a continuation theorem of Mawhin's coincidence degree , studied a periodic predator-prey system with dispersal and semi-ratio-functional response, sufficient conditions are obtained for the existence of positive periodic solution for the system.

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1 Introduction

Traditional Lotka-Voterra type predator-prey model with functional response is described by the following system of ordinary differential equations:

$$\begin{cases} \dot{x} = x(a - bx) - p(x)y \\ \\ \dot{y} = -dy + cp(x)y \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ stand for the population density of prey and predator at time t respectively. system (1) have been studied intensively and expansively in the literature with the functional response $p(x)$ of type I-V(see $[2, 3]$ and the references cited therein).

In addition to the predator-prey systems with above functional response, there is another interesting and important predator-prey system: semi-ratio-dependent predator-prey system[4], which takes the form of

$$\begin{cases} \dot{x} = x(a - bx) - p(x)y \\ \\ \dot{y} = y(d - f \cdot \frac{y}{x}) \end{cases} \tag{2}$$

system (2) attracted much attention from both mathematicians and biologists but is far from being well studied. system (2) incorporate many famous predator-prey interaction models as its special case[5,6]. Hsu and HWang[7] have carried out thorough qualitative analysis of (2), Wang and Fan[2,3] performed version of the system

$$\begin{cases} \dot{x}_1 = x_1[a(t) - b(t)x_1] - p(t, x_1)x_2 \\ \dot{x}_2 = x_2[d(t) - f(t)\frac{x_2}{x_1}] \end{cases} \tag{3}$$

and its discrete analogue model, sufficient conditions are derived for the existence of a positive periodic solution in the system. Motivated by these work, in this present paper we are devoted to the study of the following periodic predator-prey system with prey dispersal in patch environments

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1(t) - b_1(t)x_1(t) - c(t)y(t)] + D_1(t)[x_2(t) - x_1(t)] \\ \dot{x}_2(t) = x_2(t)[r_2(t) - b_2(t)x_2(t)] + D_2(t)[x_1(t) - x_2(t)] \\ \dot{y}(t) = y(t)[r_3(t) - b_3(t)y(t) + d(t)x_1(t)] - p(t, y(t))z(t) \\ \dot{z}(t) = z(t)[r_4(t) - f(t)\frac{z(t)}{y(t)}] \end{cases} \tag{4}$$

where $x_i(t)$ is the density of prey in patch $i(i = 1, 2), y(t)$ is the density of predator species, while species x can diffuse between two patches, $D_i(t)$ $(i = 1, 2)$ are diffusion coefficient of species x. $r_i(t)$ $(i = 1, 2, 3, 4), b_i(t)$ $(i = 1, 2, 3), c(t), d(t), f(t)$ and $D_i(t)$ are positive periodic functions with periodic $\omega > 0$ and are continuous in R.

For a bounded continuous function $g(x)$ on R we note:

$$g^L = \inf\{g(t) : t \in R^+\}, \qquad g^M = \sup\{g(t) : t \in R^+\}$$

moreover, if $g(x)$ is a periodic function with periodic ω, we let $\bar{g} = \frac{1}{\omega}\int_0^\omega g(t)dt$.

For the function $P(t, y)$,we assume

(H1) $p(t, y)$ is continuous with respect to the first variable t and $p(t, 0) = 0$ for any $t \in R$. there exist a bounded continuous function $p_0(t)$ such that $p(t, y) \leq p_0(t)y$ for any $t \in R, x > 0$.

Let $x = (x_1, x_2, y, z) \in R^4_+ = \{x \in R^4 : x_i > 0(i = 1, 2), y > 0, z > 0\}$, the norm of x is defined as $\|x\| = \sum_{i=1}^2 \max|x_i(t)| + \max y(t) + \max z(t)$. From the fundamental theorem of ordinary differential equation we know that for any $t_0 \geq 0$, the solution $x(t)$ of system (4) with initial condition $x(t_0) > 0$ will remain positive while $t \geq t_0$. Such solution of system (4) are called positive solution.

2 Main result

In this section, by using Mawhin's continuation theorem of coincidence degree theory, we show the existence of periodic solution of system (4). First we introduce the notations below:

letX, Z be normed linear vector space, $L: DomL \subset X \to Z$ be a linear mapping, $N: X \to Z$ be a continuous mapping. The mapping L is called a fredholm mapping of index zero if $dim\ kerL = codim\ ImL < +\infty$ and ImL is closed in Z.

Let L be a fredholm mapping of index zero and there exist continous projectors $P: X \to X$ and $Q: Z \to Z$ such that $ImP = KerL$, $ImL = KerQ = Im(I-Q)$, then $L|_{DomL\cap KerP} : (I-P)X \to ImL$ is invertible. Denote the inverse of $L|_{DomL\cap KerP}$ by K_p. Let Ω is an open bounded set of X, the mapping N is being called $L-$ compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I-Q)N : \bar{\Omega} \to X$ is compact. Since ImQ is isomorphic to $KerL$, there exists isomorphism $J : ImQ \to KerL$.

Lemma[1] Let L be a fredholm mapping with index zero, N is $L-$ compact on open bounded set $\bar{\Omega}$, assume

(1) for each $x \in \partial\Omega \cap DomL,\ \lambda \in (0,1), Lx \neq \lambda Nx$.

(2) for each $x \in \partial\Omega \cap KerL, QNx \neq 0$.

(3) $\deg\{JQN, \Omega \cap KerL, 0\} \neq 0$.

then the operator equation $Lx = Nx$ has at least one solution in $DomL \cap \bar{\Omega}$.

Now we state our main result

Theorem Assume condition (H1) hold, moreover,if

(H2) $r_i(t) > D_i(t)\ (i = 1,2)$;

(H3) $(r_1 - D_1)^L b_3^L - c^M r_3^M > 0$;

(H4) $\bar{r}_3 - b_3^M e^K > 0, \quad \bar{b}_3\overline{(r_1 - D_1)} - \bar{c}(\bar{r}_3 + \bar{d}\theta) > 0, K := \frac{r_3^M + d^M A_2}{b_3^L}$.

$$\theta := \inf\Big\{\theta^* :\ \bar{b}_1\theta^* \leq \overline{(r_1 - D_1)} + \sqrt{\bar{b}_1\bar{D}_1}\Big[\sqrt{\frac{\overline{(r_2 - D_2)}}{\bar{b}_2}} + \Big(\frac{\bar{D}_2\theta^*}{\bar{b}_2}\Big)^{\frac{1}{4}}\Big]\Big\}$$

$$A_2 := \inf\Big\{u^* :\ b_1^L u^* \leq (r_1 - D_1)^M + \sqrt{b_1^L D_1^M}\Big[\sqrt{\frac{(r_2 - D_2)^M}{b_2^L}} + (\frac{D_2^M}{b_2^L})^{\frac{1}{4}}(u^*)^{\frac{1}{4}}\Big]\Big\}$$

then system (4) have at least one positive periodic solution.

Proof since the solution of system (4) remain positive, we make the change of variables $x_i(t) = e^{u_i(t)}, y(t) = e^{v(t)}, z(t) = e^{w(t)}$, system (4) is reformulated as

$$\begin{cases} \dot{u}_1(t) = r_1(t) - D_1(t) - b_1(t)e^{u_1(t)} - c(t)e^{v(t)} + D_1(t)e^{u_2(t)-u_1(t)} \\ \dot{u}_2(t) = r_2(t) - D_2(t) - b_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)} \\ \dot{v}(t) = r_3(t) - b_3(t)e^{v(t)} + d(t)e^{u_1(t)} - p(t, e^{v(t)})e^{w(t)-v(t)} \\ \dot{w}(t) = r_4(t) - f(t)e^{w(t)-v(t)} \end{cases} \tag{5}$$

If system (5) has a ω periodic solution u^*, then $x^* = e^{u^*}$ is a ω periodic solution for system (4). to completed the proof we only need to show that system (5) has a ω periodic solution. Take

$$X = Z = \{u(t) = (u_1(t), u_2(t), v(t), w(t))^T \in C(R, R^4) : u(t + w) = u(t)\}$$

$$\|u\| = \sum_{i=1}^{2} \max_{t\in[0,\omega]} |u_i(t)| + \max_{t\in[0,\omega]} |y(t)| + \max_{t\in[0,\omega]} |z(t)|, u \in X(\text{ or } Z).$$

then X, Z are both Banach space with the norm $\|\cdot\|$.

Let

$$Nu = \begin{pmatrix} r_1(t) - D_1(t) - b_1(t)e^{u_1(t)} - c(t)e^{v(t)} + D_1(t)e^{u_2(t)-u_1(t)} \\ r_2(t) - D_2(t) - b_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)} \\ r_3(t) - b_3(t)e^{v(t)} + d(t)e^{u_1(t)} - p(t, e^{v(t)})e^{w(t)-v(t)} \\ r_4(t) - f(t)e^{w(t)-v(t)} \end{pmatrix}$$

$$Lu = \dot{u} = \frac{du(t)}{dt},\ u \in X; \qquad Pu = \frac{1}{\omega}\int_0^\omega u(t)dt, u \in X;$$

$$Qz = \frac{1}{\omega}\int_0^\omega z(t)dt,\ z \in Z.$$

Then $kerL = R^4$ and

$$ImL = \{(u_1, u_2, v, w) \in Z : \int_0^\omega u_i(t)dt = \int_0^\omega v(t)dt = \int_0^\omega w(t)dt = 0\}$$

and $dimkerL = 4 = codimImL$, since ImL is closed in Z, L is a fredholm mapping of index zero.

It is easy to see that P and Q are continuous projectors such that $ImP = kerL, ImL = kerQ = Im(I - Q)$ Thus the generalized inverse $K_p : ImL \to kerP \cap DomL$ to L exist and is given by

$$K_p(z) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)dsdt$$

Then $QN : X \to Y$ and $K_P(I-Q)N : X \to X$ read

$$QNu = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega [r_1(t) - D_1(t) - b_1(t)e^{u_1(t)} - c(t)e^{v(t)} + D_1(t)e^{u_2(t)-u_1(t)}]dt \\ \\ \frac{1}{\omega}\int_0^\omega [r_2(t) - D_2(t) - b_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)}]dt \\ \\ \frac{1}{\omega}\int_0^\omega [r_3(t) - b_3(t)e^{v(t)} + d(t)e^{u_1(t)} - p(t, e^{v(t)})e^{w(t)-v(t)}]dt \\ \\ \frac{1}{\omega}\int_0^\omega [r_4(t) - f(t)e^{w(t)-v(t)}]dt \end{pmatrix}$$

$$K_P(I-Q)Nu = \int_0^t N(u(s))ds - \frac{1}{\omega}\int_0^\omega \int_0^t N(u(s))dsdt - (\frac{t}{\omega} - \frac{1}{2})\int_0^\omega N(u(s))ds$$

obviously QNu and $K_P(I-Q)Nu$ are continuous, by using the Arzela-Ascoli theorem it is easy to verify that $\overline{K_p(I-Q)N(\Omega)}$ is compact for any open bounded set $\Omega \in X$. Moreover $QN(\bar{\Omega})$ is bounded, therefore, N is $L-$ compact on open bounded set $\bar{\Omega}$.

Corresponding to the operator equation $Lu = \lambda Nu, \lambda \in (0,1)$, we have

$$\begin{cases} \dot{u}_1(t) = \lambda[r_1(t) - D_1(t) - b_1(t)e^{u_1(t)} - c(t)e^{v(t)} + D_1(t)e^{u_2(t)-u_1(t)}] \\ \\ \dot{u}_2(t) = \lambda[r_2(t) - D_2(t) - b_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)}] \\ \\ \dot{v}(t) = \lambda[r_3(t) - b_3(t)e^{v(t)} + d(t)e^{u_1(t)} - p(t, e^{v(t)})e^{w(t)-v(t)}] \\ \\ \dot{w}(t) = \lambda[r_4(t) - f(t)e^{w(t)-v(t)}] \end{cases} \tag{6}$$

Suppose $u(t) = (u_1(t), u_2(t), v(t), w(t))$ is a solution of system (6) for some $\lambda \in (0,1)$, integrate both sides of system (6) from 0 to ω, we have

$$\int_0^\omega b_1(t)e^{u_1(t)}dt + \int_0^\omega c(t)e^{v(t)}dt = \overline{(r_1 - D_1)}\omega + \int_0^\omega D_1(t)e^{u_2(t)-u_1(t)}dt \tag{7}$$

$$\int_0^\omega b_2(t)e^{u_2(t)}dt = \overline{(r_2 - D_2)}\omega + \int_0^\omega D_2(t)e^{u_1(t)-u_2(t)}dt \tag{8}$$

$$\int_0^\omega b_3(t)e^{v(t)}dt + \int_0^\omega p(t, e^{v(t)})e^{w(t)-v(t)}dt = \bar{r}_3\omega + \int_0^\omega d(t)e^{u_1(t)}dt \tag{9}$$

$$\int_0^\omega f(t)e^{w(t)-v(t)}dt = \bar{r}_4\omega \tag{10}$$

It follows from (7),(8),(9),(10) that

$$\begin{aligned} & \int_0^\omega |\dot{u}_1(t)|dt \\ \leq\ & \int_0^\omega [b_1(t)e^{u_1(t)} + c(t)e^{v(t)}] + \int_0^\omega (r_1(t) - D_1(t))dt + \int_0^\omega D_1(t)e^{u_2(t)-u_1(t)}dt \\ =\ & 2\int_0^\omega [b_1(t)e^{u_1(t)} + c(t)e^{v(t)}]dt \leq 2b_1^M \int_0^\omega e^{u_1(t)}dt + 2c^M \int_0^\omega e^{v(t)}dt \end{aligned} \tag{11}$$

$$\begin{aligned}\int_0^\omega |\dot{u}_2(t)|dt &\le \int_0^\omega [b_2(t)e^{u_2(t)} + (r_2(t) - D_2(t))]dt + \int_0^\omega D_2(t)e^{u_1(t)-u_2(t)}dt \\ &= 2\int_0^\omega b_2(t)e^{u_2(t)}dt \le 2b_2^M \int_0^\omega e^{u_2(t)}dt \end{aligned} \quad (12)$$

$$\begin{aligned}\int_0^\omega |\dot{v}(t)|dt &\le \int_0^\omega [b_3(t)e^{v(t)} + p(t, e^{v(t)})e^{w(t)-v(t)} + r_3(t) + d(t)e^{u_1(t)}]dt \\ &= 2\bar{r}_3\omega + 2\int_0^\omega d(t)e^{u_1(t)}dt \le 2\bar{r}_3\omega + 2d^M \int_0^\omega e^{u_1(t)}dt \end{aligned} \quad (13)$$

$$\int_0^\omega |\dot{w}(t)|dt \le \int_0^\omega f(t)e^{w(t)-v(t)}dt + \int_0^\omega r_4(t)dt = 2\bar{r}_4\omega \quad (14)$$

Multiples both sides of the first equation of system (6) by $e^{u_1(t)}$ and integrate from 0 to ω, we have

$$\int_0^\omega [b_1(t)e^{2u_1(t)} + c(t)e^{v(t)+u_1(t)}]dt = \int_0^\omega [(r_1(t) - D_1(t))e^{u_1(t)} + D_1(t)e^{u_2(t)}]dt \quad (15)$$

which yields

$$b_1^L \int_0^\omega e^{2u_1(t)}dt \le (r_1 - D_1)^M \int_0^\omega e^{u_1(t)}dt + D_1^M \int_0^\omega e^{u_2(t)}dt \quad (16)$$

By using the inequality $(\int_a^b uvdt)^2 \le \int_a^b u^2dt \cdot \int_a^b v^2dt$, from (16) we have

$$\frac{b_1^L}{\omega}\Big(\int_0^\omega e^{u_1(t)}dt\Big)^2 \le (r_1 - D_1)^M \int_0^\omega e^{u_1(t)}dt + D_1^M \int_0^\omega e^{u_2(t)}dt$$

It follows that

$$\frac{2b_1^L}{\omega}\int_0^\omega e^{u_1(t)}dt \le (r_1 - D_1)^M + \sqrt{[(r_1 - D_1)^M]^2 + \frac{4b_1^L D_1^M}{\omega}\int_0^\omega e^{u_2(t)}dt}$$

Further using inequality $(a+b)^{\frac{1}{2}} \le a^{\frac{1}{2}} + b^{\frac{1}{2}}$ we have

$$\frac{b_1^L}{\omega}\int_0^\omega e^{u_1(t)}dt \le (r_1 - D_1)^M + \sqrt{\frac{b_1^L D_1^M}{\omega}}\Big(\int_0^\omega e^{u_2(t)}dt\Big)^{\frac{1}{2}} \quad (17)$$

Repeat similar process, by the second equation of (6) we obtain

$$\frac{b_2^L}{\omega}\int_0^\omega e^{u_2(t)}dt \le (r_2 - D_2)^M + \sqrt{\frac{b_2^L D_2^M}{\omega}}\Big(\int_0^\omega e^{u_1(t)}dt\Big)^{\frac{1}{2}} \quad (18)$$

which together with (17) implies

$$\begin{aligned}\frac{b_1^L}{\omega}\int_0^\omega e^{u_1(t)}dt &\le (r_1 - D_1)^M \\ &+ \sqrt{\frac{b_1^L D_1^M}{\omega}}\Big[\sqrt{\frac{\omega(r_2 - D_2)^M}{b_2^L}} + \big(\frac{\omega D_2^M}{b_2^L}\big)^{\frac{1}{4}}\Big(\int_0^\omega e^{u_1(t)dt}\Big)^{\frac{1}{4}}\Big]\end{aligned}$$

thus we know there exist a positive constant ρ_1, such that

$$\int_0^\omega e^{u_1(t)}dt < \rho_1 \tag{19}$$

Substitute (17) into (18) and from what has been discussed in (19), we also have a constant ρ_2 such that

$$\int_0^\omega e^{u_2(t)}dt < \rho_2 \tag{20}$$

Let's now consider the third equation of system (6), multiplies both sides by $e^{v(t)}$ and integrate from 0 to ω, leads to

$$\int_0^\omega r_3(t)e^{v(t)}dt + \int_0^\omega d(t)e^{u_1(t)+v(t)}dt = \int_0^\omega b_3(t)e^{2v(t)}dt + \int_0^\omega p(t, e^{v(t)})e^{w(t)}dt$$

which imply

$$\int_0^\omega b_3(t)e^{2v(t)}dt < r_3^M \int_0^\omega e^{v(t)}dt + d^M \int_0^\omega e^{u_1(t)+v(t)}dt \tag{21}$$

From (15) we also have

$$c^L \int_0^\omega e^{u_1(t)+v(t)}dt < (r_1 - D_1)^M \rho_1 + D_1^M \rho_2$$

Together with (21) leads to

$$b_3^L \int_0^\omega e^{2v(t)}dt < r_3^M \int_0^\omega e^{v(t)}dt + d^M \Big[\frac{(r_1 - D_1)^M \rho_1 + D_1^M \rho_2}{c_L}\Big]$$

Denote $d^M\Big[\frac{(r_1-D_1)^M\rho_1+D_1^M\rho_2}{c_L}\Big] := A_1$, use similar analysis on the expression above as what has taken on (16),we have $\frac{b_3^L}{\omega}\int_0^\omega e^{v(t)}dt < r_3^M + \sqrt{\frac{b_3^L A_1}{\omega}}$, hence

$$\int_0^\omega e^{v(t)}dt < \frac{(r_3^M + \sqrt{\frac{b_3^L A_1}{\omega}})\omega}{b_3^L} := \rho_3 \tag{22}$$

From (11), (12), (13) and (19), (20), (22) we have

$$\int_0^\omega |\dot{u}_1(t)|dt < 2b_1^M \rho_1 + 2c^M \rho_2 := d_1 \tag{23}$$

$$\int_0^\omega |\dot{u}_2(t)|dt < 2b_2^M \rho_2 := d_2 \tag{24}$$

$$\int_0^\omega |\dot{v}(t)|dt < 2\bar{r}_3\omega + 2d^M \rho_1 := d_3 \tag{25}$$

Since $u \in X$, there exist $\xi_i \in [0, \omega]$ $(i = 1, 2, 3, 4)$, such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t), \quad v(\xi_3) = \max_{t\in[0,\omega]} v(t), \quad w(\xi_4) = \max_{t\in[0,\omega]} w(t)$$

Note that $\dot{u}_1(\xi_1) = \dot{u}_2(\xi_2) = \dot{v}(\xi_3) = \dot{w}(\xi_4) = 0$, from system (6) we have

$$r_1(\xi_1) - D_1(\xi_1) + D_1(\xi_1)e^{u_2(\xi_1)-u_1(\xi_1)} = b_1(\xi_1)e^{u_1(\xi_1)} + c(\xi_1)e^{v(\xi_1)} \tag{26}$$

$$r_2(\xi_2) - D_2(\xi_2) + D_2(\xi_2)e^{u_1(\xi_2)-u_2(\xi_2)} = b_2(\xi_2)e^{u_2(\xi_2)} \tag{27}$$

$$r_3(\xi_3) + d(\xi_3)e^{u_1(\xi_3)} = b_3(\xi_3)e^{v(\xi_3)} + p(\xi_3, e^{v(\xi_3)})e^{w(\xi_3)-v(\xi_3)} \tag{28}$$

$$r_4(\xi_4) = f(\xi_4)e^{w(\xi_4)-v(\xi_4)} \tag{29}$$

From (28) we obtain $b_3^L e^{v(\xi_3)} < r_3^M + d^M e^{u_1(\xi_1)}$, From (26) we have $b_1^M e^{u_1(\xi_1)} + c^M e^{v(\xi_3)} > (r_1 - D_1)^L$, the two above expression yields to

$$e^{u_1(\xi_1)} > \frac{(r_1 - D_1)^L b_3^L - c^M r_3^M}{b_1^M b_3^L + c^M d^M} := B_1 > 0 \tag{30}$$

From (19), (20), there exist constant $t_1, t_2 \in [0, \omega]$ such that

$$e^{u_i(t_i)} < \frac{\rho_i}{\omega}, \ i = 1, 2 \tag{31}$$

By (27) we derive

$$e^{u_2(\xi_2)} > \frac{(r_2 - D_2)^L}{b_2^M} := B_2 \tag{32}$$

Together with (23), (24), (25) and (30), (31), (32) we obtain

$$u_i(t) \le u_i(t_i) + \int_0^\omega |\dot{u}_i(t)|dt < \ln \frac{\rho_i}{\omega} + d_i, \ i = 1, 2.$$

$$u_1(t) \ge u_1(\xi_1) - \int_0^\omega |\dot{u}_1(t)|dt > \ln B_1 - d_1$$

$$u_2(t) \le u_2(\xi_2) - \int_0^\omega |\dot{u}_2(t)|dt > \ln B_2 - d_2$$

which yields

$$|u_1(t)| \le \max\Big\{|\ln \frac{\rho_1}{\omega} + d_1|, \ |\ln B_1 - d_1|\Big\} := H_1 \tag{33}$$

$$|u_2(t)| \leq \max\Big\{|\ln\frac{\rho_2}{\omega} + d_2|,\ |\ln B_2 - d_2|\Big\} := H_2 \tag{34}$$

By (29) we notice that

$$r_4^M e^{v(\xi_3)} \geq f^L e^{w(\xi_4)} \tag{35}$$

By (28) we derive $b_3^M e^{v(\xi_3)} + p_0(\xi_3)e^{w(\xi_4)} \geq r_3^L$, thus $e^{v(\xi_3)} \geq \frac{r_3^L - p_0^M e^{w(\xi_4)}}{b_3^M}$. which together with (35) leads to

$$e^{v(\xi_3)} \geq \frac{r_3^L f^L}{b_3^M f^L + p_0^M r_4^M} := B_3 \tag{36}$$

From (22) there exist constant $\tau_1 \in [0, \omega]$ such that

$$e^{v(\tau_1)} < \frac{\rho_3}{\omega} \tag{37}$$

It follows from (36), (37) and (25) that

$$v(t) \leq v(\tau_1) + \int_0^\omega |\dot{v}(t)|dt < \ln\frac{\rho_3}{\omega} + d_3,\ v(t) \geq v(\xi_3) + \int_0^\omega |\dot{v}(t)|dt > \ln B_3 - d_3$$

Hence we have

$$|v(t)| \leq \max\Big\{|\ln\frac{\rho_3}{\omega} + d_3|,\ |\ln B_3 - d_3|\Big\} := H_3 \tag{38}$$

Now we estimates $w(t)$. Consider (26) and (27), apply the same method, which we used to estimate $\int_0^\omega e^{u_1(t)}dt$ and $\int_0^\omega e^{u_2(t)}dt$ in (15)-(20), to them we have

$$b_1^L e^{u_1(\xi_1)} \leq (r_1 - D_1)^M + \sqrt{b_1^M D_1^M}\Big[\sqrt{\frac{(r_2 - D_2)^M}{b_2^L}} + \Big(\frac{D_2^M}{b_2^L}\Big)^{\frac{1}{4}}\Big(e^{u_1(\xi_1)}\Big)^{\frac{1}{4}}\Big]$$

Set

$$A_2 = \inf\Big\{u^* :\ b_1^L u^* \leq (r_1 - D_1)^M + \sqrt{b_1^M D_1^M}\Big[\sqrt{\frac{(r_2 - D_2)^M}{b_2^L}} + \Big(\frac{D_2^M}{b_2^L}\Big)^{\frac{1}{4}}\Big(u^*\Big)^{\frac{1}{4}}\Big]$$

It's obviously that $A_2 > 0$ and $e^{u_1(\xi_1)} \leq A_2$.

On the other hand by (28) we derive $b_3^L e^{v(\xi_3)} \leq r_3^M + d^M e^{u_1(\xi_1)}$, Thus we derive $e^{v(\xi_3)} \leq \frac{r_3^M + d^M A_2}{b_3^L} := K$. Together with (35) we have

$$e^{w(\xi_4)} \leq \frac{r_4^M(r_3^M + d^M A_2)}{f^L b_3^L} \tag{39}$$

From (9) we obtain $\int_0^\omega b_3(t)e^{v(t)}dt + \int_0^\omega p_0(t)e^{w(t)}dt \geq \bar{r}_3\omega$, thus $\int_0^\omega e^{w(t)}dt \geq \frac{\bar{r}_3\omega - b_3^M e^K \omega}{p_0^M} > 0$. It follows that there exist a constant $\tau_2 \in [0, \omega]$ such that

$$e^{\omega(\tau_2)} \geq \frac{\bar{r}_3 - b_3^M e^K}{p_0^M} \tag{40}$$

By (39), (40) and (14) we have

$$w(t) \leq w(\xi_4) + \int_0^\omega |\dot{w}(t)|dt < \ln \frac{r_4^M(r_3^M + d^M A_2)}{f^L b_3^L} + 2\bar{r}_4\omega$$

and

$$w(t) \geq w(\tau_2) - \int_0^\omega |\dot{w}(t)|dt > \ln \frac{\bar{r}_3 - b_3^M e^K}{p_0^M} - 2\bar{r}_4\omega$$

following which we derived

$$|w(t)| \leq \max\Big\{|\ln \frac{r_4^M(r_3^M + d^M A_2)}{f^L b_3^L} + 2\bar{r}_4\omega|,\ |\ln \frac{\bar{r}_3 - b_3^M e^K}{p_0^M} - 2\bar{r}_4\omega|\Big\} := H_4 \tag{41}$$

Clearly, H_1, H_2, H_3, H_4 are independent of λ.

Consider the equation $QNu = 0$, when $u \in \partial\Omega \cap kerL = R^4$, $u = (u_1, u_2, v, w)$ is constant vector. Using the integral mean valued theorem, it follows that

$$QNu = \begin{pmatrix} \bar{r}_1 - \bar{D}_1 - \bar{b}_1 e^{u_1} - \bar{c}e^v + \bar{D}_1 e^{u_2 - u_1} \\ \\ \bar{r}_2 - \bar{D}_2 - \bar{b}_2 e^{u_2} + \bar{D}_2 e^{u_1 - u_2} \\ \\ \bar{r}_3 - \bar{b}_3 e^v + \bar{d}e^{u_1} - \bar{p}(e^v)e^{w-v} \\ \\ \bar{r}_4 - \bar{f}e^{w-v} \end{pmatrix}$$

Let M_0 is a constant big enough such that each solution $u^* = (u_1^*, u_2^*, v^*, w^*)$of equation $QNu = 0$ satisfies $\|u^*\| = |u_1^*| + |u_2^*| + |v^*| + |w^*| \leq M_0$, provided that $QNU = 0$ has solution.

In order to use Mawhin's continuation theorem of coincidence degree, define homotopy $\Phi:\ DomL \times [0,1] \to X$ as

$$\Phi(u_1, u_2, v, w, \lambda) = \begin{pmatrix} \bar{r}_1 - \bar{D}_1 - \bar{b}_1 e^{u_1} \\ \\ \bar{r}_2 - \bar{D}_2 - \bar{b}_2 e^{u_2} \\ \\ \bar{r}_3 - \bar{b}_3 e^v \\ \\ \bar{r}_4 - \bar{f}e^{w-v} \end{pmatrix} + \lambda \begin{pmatrix} -\bar{c}e^v + \bar{D}_1 e^{u_2 - u_1} \\ \\ \bar{D}_2 e^{u_1 - u_2} \\ \\ \bar{d}e^{u_1} - \bar{p}(e^v)e^{w-v} \\ \\ 0 \end{pmatrix} \tag{42}$$

Note (42) as $\Phi(u_1, u_2, v, w, \lambda) := J_1 + \lambda J_2$, Of course we have $\Phi(u_1, u_2, v, w, 1) = QNu$.

By some similar argument on system (6) we can show that for each solution $u = (u_1, u_2, v, w)$ of (42), we have

$$u_1 < \ln\theta, \qquad u_1 > \ln\frac{\bar{b}_3\overline{(r_1 - D_1)} - \bar{c}(\bar{r}_3 + \bar{d}\theta)}{\bar{b}_1\bar{b}_3}$$

$$u_2 < \ln\frac{\overline{(r_2 - D_2)} + \sqrt{\bar{b}_2\bar{D}_2\theta}}{\bar{b}_2}, \qquad u_2 > \ln\frac{\overline{(r_2 - D_2)}}{\bar{b}_2}$$

$$v < \ln\frac{\bar{r}_3 + \bar{d}\theta}{\bar{b}_3}, \qquad v > \ln\frac{\bar{f}\bar{r}_3}{\bar{f}\bar{b}_3 + \bar{r}_4 p_0^M}$$

$$w < \ln\frac{\bar{r}_4(\bar{r}_3 + \bar{d}\theta)}{\bar{f}\bar{b}_3}, \qquad w > \ln\frac{\bar{r}_3\bar{r}_4}{\bar{f}\bar{b}_3 + \bar{r}_4 p_0^M}$$

Where

$$\theta = \inf\Big\{\theta^* : \ \bar{b}_1\theta^* \le \overline{(r_1 - D_1)} + \sqrt{\bar{b}_1\bar{D}_1}\Big[\sqrt{\frac{\overline{(r_2 - D_2)}}{\bar{b}_2}} + \Big(\frac{\bar{D}_2\theta^*}{\bar{b}_2}\Big)^{\frac{1}{4}}\Big]\Big\}$$

Let M_1 be a constant such that

$$\begin{aligned} & \max\Big\{|\ln\theta|, \ |\ln\frac{\bar{b}_3\overline{(r_1 - D_1)} - \bar{c}(\bar{r}_3 + \bar{d}\theta)}{\bar{b}_1\bar{b}_3}|\Big\} \\ + \ & \max\Big\{|\ln\frac{\overline{(r_2 - D_2)} + \sqrt{\bar{b}_2\bar{D}_2\theta}}{\bar{b}_2}|, \ |\ln\frac{\overline{(r_2 - D_2)}}{\bar{b}_2}|\Big\} \\ + \ & \max\Big\{|\ln\frac{\bar{r}_3 + \bar{d}\theta}{\bar{b}_3}|, \ |\ln\frac{\bar{f}\bar{r}_3}{\bar{f}\bar{b}_3 + \bar{r}_4 p_0^M}|\Big\} \\ + \ & \max\Big\{|\ln\frac{\bar{r}_4(\bar{r}_3 + \bar{d}\theta)}{\bar{f}\bar{b}_3}|, \ |\ln\frac{\bar{r}_3\bar{r}_4}{\bar{f}\bar{b}_3 + \bar{r}_4 p_0^M}|\Big\} < M_1 \end{aligned}$$

Now let $M = \sum_{i=1}^4 H_i + M_0 + M_1$, and let $\Omega = \{u \in X : \ \|u\| < M\}$. Its easy to see that the condition (1) in Lemma is satisfied because $\sum_{i=1}^4 H_i < M$; Moreover, when $u = (u_1, u_2, v, w) \in \partial\Omega \cap KerL = \partial\Omega \cap R^4$, u is a constant vector in R^4 with $|u_1| + |u_2| + |v| + |w| = M$, thus by $M_0 < M$ we know that the condition (2) in Lemma is satisfied; Finally we will show the condition (3) of Lemma satisfies, say, we will compute the topological degree of $\Phi(u_1, u_2, v, w, \lambda)$. By $M_1 < M$ we derive that $\Phi(u_1, u_2, v, w, \lambda) \ne 0$, when $u = (u_1, u_2, v, w) \in \partial\Omega \cap KerL$. And it is easy to show that equation$J_1 = 0$ has a unique solution (u_1^*, u_2^*, v^*, w^*). Using the property of topological degree and take $J = I : ImL \to KerL$ we have

$$\begin{aligned} & \deg\Big\{JQN(u_1, u_2, v, w)^T; \ \Omega \cap kerL; \ 0\Big\} \\ = \ & \deg\Big\{\Phi(u_1, u_2, v, w, 1); \ \Omega \cap kerL; \ 0\Big\} \\ = \ & \deg\Big\{\Phi(u_1, u_2, v, w, 0); \ \Omega \cap kerL; \ 0\Big\} \end{aligned}$$

$$= \operatorname{sgn}\Big\{\det \begin{bmatrix} -\bar{b}_1 e^{u_1} & 0 & 0 & 0 \\ 0 & -\bar{b}_2 e^{u_2} & 0 & 0 \\ 0 & 0 & -\bar{b}_3 e^{v} & 0 \\ 0 & 0 & 0 & -\bar{f} e^{w-v} \end{bmatrix}\Big\} = 1 \neq 0.$$

By now we have proved that the condition (3) in Lemma satisfied, thus competed the proof.

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