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A Generalization of Hardy-Hilbert's Integral Inequality and its Reverse

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Abstract. New generalizations of Hardy-Hilbert's integral inequality and its reverse via new methods are established.

1. Introduction

Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)\,dt < \infty \ \ and \ \ 0 < \int_0^\infty g^2(t)\,dt < \infty ,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^\infty f^2(t)\,dt \int_0^\infty g^2(t)\,dt\right)^{1/2} , \tag{1}$$

where the constant factor π is the best possible (cf. Hardy et al. [2]). Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy [1] as follows

If p>1, $\frac{1}{p}+\frac{1}{q}=1$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\,dt < \infty \ \ and \ \ \int_0^\infty g^q(t)\,dt < \infty ,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty g^q(t)\,dt\right)^{1/q} , \tag{2}$$

where the constant factor $\dfrac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [3]).

Gradually, B. Yang gave the following extensions of (2) as follows :

Theorem A[4]. *If* $\lambda > 2 - \min\{p, q\}$, $f, g \geq 0$, satisfy

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\, dt < \infty \quad and \quad \int_0^\infty t^{1-\lambda} g^q(t)\, dt < \infty,$$

then

$$(3) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx\, dy < k_\lambda(p) \left(\int_0^\infty t^{1-\lambda} f^p(t)\, dt \right)^{1/p} \left(\int_0^\infty t^{1-\lambda} g^q(t)\, dt \right)^{1/q},$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible, B is the beta function.

Theorem B[5]. *If* $n \in N - \{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > 0$, $f_i \geq 0$ satisfy

$$0 < \int_0^\infty t^{p_i - 1 - \lambda} f_i^{p_i}(t)\, dt < \infty \qquad (i = 1, 2, \ldots, n),$$

then

$$(4) \qquad \int_0^\infty \ldots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i)\, dx_1 \ldots dx_n$$

$$< \frac{1}{\Gamma\lambda} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \left(\int_0^\infty t^{p_i - 1 - \lambda} f_i^{p_i}(t)\, dt \right)^{1/p_i},$$

where the constant factor $\dfrac{1}{\Gamma\lambda} \prod_{i=1}^n \Gamma\left(\dfrac{\lambda}{p_i}\right)$ is the best possible.

2. Lemmas

The following Lemmas are needed for our aim.

Lemma 1. *If* $t_i \geq 0$, $i = 1, \ldots, n$, $\lambda > 0$, *then*

(5) $$\prod_{i=1}^{n}\frac{1}{(1+t_i)^{\lambda}} \le \frac{1}{\left(1+\sum_{i=1}^{n} t_i\right)^{\lambda}}.$$

Proof . We will use induction. For n=2 it is obvious . Let (1) true for n = m.. For n=m+1, we have

$$\prod_{i=1}^{m+1}\frac{1}{(1+t_i)^{\lambda}} = \frac{1}{(1+t_{m+1})^{\lambda}}\prod_{i=1}^{m}\frac{1}{(1+t_i)^{\lambda}}$$
$$\le \frac{1}{(1+t_{m+1})^{\lambda}}\frac{1}{\left(1+\sum_{i=1}^{m} t_i\right)^{\lambda}}$$
$$\le \frac{1}{\left(1+t_{m+1}+\sum_{i=1}^{m} t_i\right)^{\lambda}} = \frac{1}{\left(1+\sum_{i=i}^{m+1} t_i\right)^{\lambda}}.$$

Lemma 2. *Let* $f, g, h \ge 0,\ \frac{1}{p}+\frac{1}{q}=1,\ \ 0<p<1$ *or* $p<0$. *Then*

(6) $$\left(\int\left(\int fg\right)^{p} h\right)^{1/p} \ge \left(\int\int f^{p}gh\left(\int g\right)^{p-1}\right)^{1/p}.$$

Proof. We will consider the case p<0 only, as the rest is similar.

$$\int fg = \int fg^{\frac{1}{p}}g^{\frac{1}{q}} \ge \left(\int f^{p}g\right)^{1/p}\left(\int g\right)^{1/q}$$

which implies

$$\left(\int fg\right)^{p} \le \left(\int f^{p}g\right)\left(\int g\right)^{p-1}$$

and hence

$$\int\left(\int fg\right)^{p} h \le \left(\int f^{p}gh\right)\left(\int g\right)^{p-1}.$$

Therefore

$$\left(\int\left(\int fg\right)^{p} h\right)^{1/p} \ge \left(\int f^{p}gh\left(\int g\right)^{p-1}\right)^{1/p}.$$

3. Main Results

We state and prove the following:

Theorem 1. *Let* $f_i, F_i, \phi_i \geq 0,\ F_i' > 0_i\ F_i(0)=0,\ F_i(\infty)=\infty,\ \lambda > p_i - 1 > 0,$ $i=1,\dots,n\ , a>0,\ \sum_{i=1}^{n}\frac{1}{p_i}=1.$ *Then*

(7) $$\int_0^\infty \dots \int_0^\infty \frac{\phi_1(F_1(x_1))\, f_1\,(x_1)\dots\phi_n(F_n(x_n))\, f_n\,(x_n)}{\left(\sum_{i=1}^{n}F_i(x_i)\right)^{\lambda}}\,dx_1\dots dx_n$$

$$\leq \frac{1}{\Gamma\left(\lambda+\frac{1}{a}-1\right)}\prod_{i=1}^{n}\left(\Gamma\left(\lambda+\frac{1}{a}-p_i\right)\int_0^\infty \frac{\phi_i^{p_i}\left(F_i(x_i)\right)\left(F_i(x_i)\right)^{p_i-\lambda-\frac{1}{a}} f_i^{p_i}(x_i)}{\left(F_i'(x_i)\right)^{p_i-1}}\,dx_i\right)^{1/p_i}$$

,

provided the integrals on the right − hand side do exist.

Proof. Write

$$I_i(t) = \int_0^\infty \phi_i\left(F_i(x_i)\right) f_i\ (x_i)\, t^{\frac{\lambda-1}{p_i}a}\, e^{-t^a F_i(x_i)}\,dx_i\ .$$

Then, we have

$$\int_0^\infty \left(\prod_{i=1}^{n} I_i(t)\right)dt$$

$$= \int_0^\infty \dots \int_0^\infty \phi_1\left(F_1(x_1)\right) f_1\ (x_1)\dots\phi_n\left(F_n(x_n)\right) f_n\ (x_n)\,dx_1\dots dx_n \int_0^\infty t^{(\lambda-1)a} e^{-t^a\sum_{i=1}^{n}F_i(x_i)}\,dt$$

$$= \int_0^\infty \dots \int_0^\infty \phi_1\left(F_1(x_1)\right) f_1\ (x_1)\dots\phi_n\left(F_n(x_n)\right) f_n\ (x_n)\,dx_1\dots dx_n \int_0^\infty \left(\frac{u}{\sum_{i=1}^{n}F_i(x_i)}\right)^{\lambda-1} e^{-u}$$

$$\times\frac{1}{a}\frac{1}{\left(\sum_{i=1}^{n}F_i(x_i)\right)^{\frac{1}{a}}}u^{\frac{1}{a}-1}\,du$$

$$= \frac{1}{a}\int_0^\infty \dots \int_0^\infty \frac{\phi_1\left(F_1(x_1)\right) f_1\ (x_1)\dots\phi_n\left(F_n(x_n)\right) f_n\ (x_n)}{\left(\sum_{i=1}^{n}F_i(x_i)\right)^{\lambda+\frac{1}{a}-1}}\,dx_1\dots dx_n \int_0^\infty u^{\lambda+\frac{1}{a}-2} e^{-u}\,du$$

(8) $$= \frac{1}{a}\Gamma\left(\lambda+\frac{1}{a}-1\right)\int_0^\infty \dots \int_0^\infty \frac{\phi_1\left(F_1(x_1)\right) f_1\ (x_1)\dots\phi_n\left(F_n(x_n)\right) f_n\ (x)}{\left(\sum_{i=1}^{n}F_i(x_i)\right)^{\lambda+\frac{1}{a}-1}}\,dx_1\dots dx_n\ .$$

On the other hand, we have

$$\int_0^\infty\left(\prod_{i=1}^n I_i(t)\right)dt \le \prod_{i=1}^n\left(\int_0^\infty I_i^{p_i}(t)\,dt\right)^{1/p_i}$$

$$=\prod_{i=1}^n\left(\int_0^\infty\left(\int_0^\infty \phi_i(F_i(x_i))\,f_i^{p_i}(x_i)\,t^{\left(\frac{\lambda-1}{p_i}\right)a}e^{-t^aF_i(x_i)}\,dx_i\right)^{p_i}dt\right)^{1/p_i}$$

$$\le\prod_{i=1}^n\left(\int_0^\infty\int_0^\infty\frac{\phi_i^{p_i}(F_i(x_i))\,f_i^{p_i}(x_i)\,t^{(\lambda-1)a}\,e^{-t^aF_i(x_i)}}{\left(t^aF_i'(x_i)\right)^{p_i-1}}\,dx_i\left(\int_0^\infty t^{-t^aF_i(x_i)}\,t^aF_i'(x_i)\,dx_i\right)^{p_i-1}dt\right)^{1/p_i}$$

$$=\prod_{i=1}^n\left(\int_0^\infty\frac{\phi_i^{p_i}(F_i(x_i))\,f_i^{p_i}(x_i)}{(F_i'(x_i))^{p_i-1}}\,dx_i\;\int_0^\infty t^{(\lambda-p_i)a}e^{-t^aF_i(x_i)}\,dt\right)^{1/p_i}$$

$$=\prod_{i=1}^n\left(\int_0^\infty\frac{\phi_i^{p_i}(F_i(x_i))\,f_i^{p_i}(x_i)}{(F_i'(x_i))^{p_i-1}}\,dx_i\int_0^\infty\left(\frac{z}{F_i(x_i)}\right)^{\lambda-p_i}e^{-z}\,\frac{1}{a}\,\frac{1}{\left(\sum_{i=1}^n F_i(x_i)\right)^{\frac{1}{a}}}\,z^{\frac{1}{a}-1}dz\right)^{1/p_i}$$

$$=\frac{1}{a}\prod_{i=1}^n\left(\int_0^\infty\frac{\phi_i^{p_i}(F_i(x_i))(F_i(x_i))^{p_i-\lambda-\frac{1}{a_i}}\,f_i^{p_i}(x_i)}{(F_i'(x_i))^{p_i-1}}\,dx_i\int_0^\infty z^{\lambda+\frac{1}{a}-p_i-1}e^{-z}\,dz\right)^{1/p_i}$$

(9) $$=\frac{1}{a}\prod_{i=1}^n\left(\Gamma\left(\lambda+\frac{1}{a}-p_i\right)\int_0^\infty\frac{\phi_i(F_i(x_i))(F_i(x_i))^{p_i-\lambda-\frac{1}{a}}\,f_i^{p_i}(x_i)}{(F_i'(x_i))^{p_i-1}}\,dx_i\right)^{1/p_i}$$

Combining (8) and (9), the result follows.

Corollary 2. *Let* $\phi_i\ge 0,\ \lambda>p_i-1,\ i=1,\dots,n$, $\sum_{i=1}^n p_i=1$. *Then*

(10) $$\int_0^\infty\!\!\dots\!\int_0^\infty\frac{\phi_1(x_1)\dots\phi_n(x_n)}{\left(\sum_{i=1}^n x_i\right)^\lambda}\,dx_1\dots dx_n$$

$$\le\frac{1}{\Gamma(\lambda)}\prod_{i=1}^n\left(\Gamma(1+\lambda-p_i)\int_0^\infty\frac{x_i^{1+\lambda-p_i}\phi_i^{p_i}(x_i)}{\left(\sum_{i=1}^n x_i\right)^\lambda}\,dx_i\right)^{1/p_i},$$

Provided the integrals on the right-hand side do exist.

Proof. Follows from Theorem 1, by putting for i = 1,…,n ,

$$f_i(x)=1, \quad F_i(x)=x, \quad a=1.$$

Remark. It may be mentioned that the parameter a play a role to control the kind of the gamma function as well as the power of the variable we need. Although our method for proving Theorem 1, and hence Corollary 2 is shorter and simpler than [5]. The constant factor is better for $p_i < \lambda < p_i + \frac{1}{2}$, $i = 1,...,n$, in the sense that function Γ is decreasing on $(0, \frac{3}{2}]$. That is if $1+\lambda - p_i \in (0, \frac{3}{2}]$ with $1+\lambda - p_i > \lambda / p_i$, we have $\Gamma(1+\lambda - p_i) < \Gamma(\lambda / p_i)$.

Theorem 3. *Let* $f_i, \phi_i \geq 0$, $f_i' > 0$, $f_i(0) = 0$, $f_i(\infty) = \infty$, $\alpha_i > 0$, $\lambda > 1 + (\alpha - 1) p_i$, $i = 1,...,n$, $\lambda > \alpha$, $\sum_{i=1}^{n} \alpha_i = 1$, $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $0 < p_1 < 1$. *Then*

$$(9) \quad \int_0^\infty \cdots \int_0^\infty \frac{\phi_1(f_1(x_1))\ldots\phi_n(f_n(x_n))}{\left(\sum_{i=1}^{n} f_i(x_i)\right)^\lambda} dx_1 \ldots dx_n \geq \frac{1}{B(\alpha, \lambda - \alpha)} \times$$

$$\prod_{i=1}^{n} \left(B(1+(\alpha-1)p_i, \lambda - 1 - (\varepsilon - 1)p_i) \int_0^\infty \frac{\phi_i^{p_i}(f_i(x_i))}{(f_i'(x_i))^{p_i - 1}(f_i(x_i))^{1+(\alpha_i - 1)p_i}} dx_i \right)^{1/p_i},$$

provided the integrals on the right – hand side do exist.

Proof. Write

$$J_i(t) = \int_0^\infty \frac{\phi_i(f_i(x_i)) t^{\alpha_i - \frac{1}{p_i}}}{(1 + f_i(x_i))^\lambda} dx_i \ .$$

Then, we have, in view of Lemmas 1, and 2, we have

$$\int_0^\infty \left(\prod_{i=1}^{n} J_i(t) \right) dt$$

$$= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\phi_1(f_1(x_1))\ldots\phi_n(f_n(x_n)) t^{\alpha - 1}}{\prod_{i=1}^{n} (1 + t f_i(x_i))^\lambda} dx_1 \ldots dx_n \, dt$$

$$\leq \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\phi_1(f_1(x_1))\ldots\phi_n(f_n(x_n)) t^{\alpha - 1}}{\left(1 + t \sum_{i=1}^{n} f_i(x_i)\right)^\lambda} dx_1 \ldots dx_n \, dt$$

$$= \int_0^\infty \ldots \int_0^\infty \phi_1(f_1(x_1))\ldots\phi_n(f_n(x_n))dx_1\ldots dx_n \int_0^\infty \frac{t^{\alpha-1}}{\left(1+t\sum_{i=1}^n f_i(x_i)\right)^\lambda} dt$$

$$= \int_0^\infty \ldots \int_0^\infty \phi_1(f_1(x_1))\ldots\phi_n(f_n(x_n))dx_1\ldots dx_n \int_0^\infty \frac{\left(\frac{u}{\sum_{i=1}^n f_i(x_i)}\right)^{\alpha-1}}{(1+u)^\lambda} \frac{du}{\sum_{i=1}^n f_i(x_i)}$$

$$= \int_0^\infty \ldots \int_0^\infty \frac{\phi_1(f_1(x_1))\ldots\phi_n(f_n(x_n))}{\left(\sum_{i=1}^n f_i(x_i)\right)^\alpha} dx_1\ldots dx_n \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^\lambda} du$$

(10) $$= B(\alpha, \lambda-\alpha) \int_0^\infty \ldots \int_0^\infty \frac{\phi_1(f_1(x_1))\ldots\phi_n(f_n(x_n))}{\left(\sum_{i=1}^n f_i(x_i)\right)^\alpha} dx_1\ldots dx_n \ .$$

Also, we have

$$\int_0^\infty \left(\prod_{i=1}^n J_i(t)\right) dt \geq \prod_{i=1}^n \left(\int_0^\infty J_i^{p_i}(t)\, dt\right)^{1/p_i}$$

$$= \prod_{i=1}^n \left(\int_0^\infty \left(\int_0^\infty \frac{\phi_i(f_i(x_i))t^{\alpha_i - \frac{1}{p_i}}}{(1+tf_i(x_i))^\lambda} dx_i\right)^{p_i} dt\right)^{1/p_i}$$

$$\geq \prod_{i=1}^n \left(\int_0^\infty \left(\int_0^\infty \frac{\phi_i^{p_i}(f_i(x_i))t^{\alpha_i p_i - 1}}{(t\, f_i'(x_i))^{p_i-1}(1+tf_i(x_i))^\lambda} dx_i\right)\left(\int_0^\infty \frac{t\, f_i'(x_i)}{(1+tf_i(x_i))^\lambda} dx_i\right)^{p_i-1} dt\right)^{1/p_i}$$

$$= \prod_{i=1}^n \left(\int_0^\infty \frac{\phi_i^{p_i}(f_i(x_i))}{(f_i'(x_i))^{p_i-1}} dx_i \int_0^\infty \frac{t^{(\alpha_i-1)p_i}}{(1+t\, f_i(x_i))^\lambda} dt\right)^{1/p_i}$$

$$= \prod_{i=1}^n \left(\int_0^\infty \frac{\phi_i^{p_i}(f_i(x_i))}{\left(f_i'(x_i)\right)^{p_i-1}} dx_i \int_0^\infty \frac{\left(\frac{z}{f_i(x_i)}\right)^{(\alpha_i-1)p_i}}{(1+z)^\lambda} \frac{dz}{f_i(x_i)}\right)^{1/p_i}$$

(11) $$= \prod_{i=1}^n \left(B(1+(\alpha_i-1)p_i\, , \lambda-1-(\alpha_i-1)p_i) \int_0^\infty \frac{\phi_i^{p_i}(f_i(x_i))}{\left(f_i'(x_i)\right)^{p_i-1}(f_i(x_i))^{1+(\alpha_i-1)p_i}} dx_i\right)^{1/p_i}$$

In view of (10) and (11), the proof is complete.

References

[1] G. H. Hardy, Note on a theorem of Hilbert concerning series of positive terms, Proc. Math. Soc. 23(2) (1925), Records of Proc. XLV-XLVI.

[2] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, Cambridge University Press, Cambridge, 1952.

[3] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, Inequalities involving functions and their integrals and derivatives, Kluwer Academic Publishers, Boston, 1991.

[4] B. Yang, On Hardy-Hilbert's integral inequality, J. Math. Anal. Appl. 261 (2001) 295-306.

[5] B. Yang, On a new multiple extension of Hilbert's integral inequality, JIPAM, Volume 6, Issue 2, Article 39, 2005.

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