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Lyapunov Optimizing Sliding Mode Control for Linear Systems with Bounded Disturbance

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Abstract

The purpose of this treatment is the design of feedback control algorithms that asymptotically zero the states of linear systems. This objective is accomplished by extending the theory and application of Lyapunov Optimizing Control (LOC). We consider linear systems, subject to a bounded control and a bounded, but otherwise unknown disturbance. In addition, a fundamental concern during this endeavor is the minimization of a quadratic cost functional which penalizes states not at the origin. By exploiting a connection between LOC and sliding mode control (SMC) we prove asymptotic stability of the origin for instances when traditional LOC techniques cannot. The final form of the control algorithm, denoted by Lyapunov optimizing sliding mode control (LOSMC), is specified by implementing the techniques of trajectory following optimization (TFO). LOSMC produces continuous control signals and therefore eliminates the detrimental effects of chatter on mechanical actuators. The resulting algorithm provides a convenient methodology by which the analyst may address stability, disturbance, and chatter considerations while still incorporating some measure of cost into the control law design process.

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1 Introduction

Control algorithms based upon Lyapunov's second method have proven quite effective for controlling linear and nonlinear systems subject to disturbances; excellent examples being [5] and [2]. Lyapunov optimizing control, which originated in [3], produces feedback controls by selecting a candidate Lyapunov function and choosing the control to minimize this function as much as possible along system trajectories [12, pp. 273-277], [14]. The advantage of LOC algorithms comes from a built in proof of their effectiveness. Assuming the target is the origin, if the candidate Lyapunov function is decreased everywhere outside the target, a sufficient condition for asymptotic stability is satisfied [13]. Furthermore, the LOC approach provides the analyst a means to design feedback controls where cost accumulation is explicitly considered [12, pp. 295-300] and has been applied to many interesting problems [7, pp. 79-86], [9]. The difficulty in utilizing LOC methods arises during proof of asymptotic stability of the origin. For a dynamical system subject to a disturbance, without assumptions on the stability of the state equations, it may be quite difficult to guarantee that the candidate Lyapunov function decreases everywhere outside the origin. If the function doesn't decrease everywhere apart from the origin, the necessary conditions for asymptotic stability are not satisfied.

Sliding mode control has proven to be an effective approach for the control of uncertain dynamical systems. The ability of these algorithms to reject disturbances and stabilize the origin has made SMC a subject of great interest in terms of both theoretical and practical research [11], [6], [16]. The main idea of SMC is to select a switching surface in state space that is attractive, robust to disturbance, and consists of stable dynamics. Once trajectories reach this surface, they "slide" along it until the origin is reached. A primary drawback of SMC algorithms is their tendency to induce chatter; typified by high frequency (discontinuous) commutation of the control signal across this surface [17], [10]. This commutation may produce a variety of detrimental effects on a mechanical system such as excitement of resonant modes and extreme wear on actuators. To address this issue, several authors have proposed algorithms that yield a continuous time control effort while still exhibiting the desired robustness and stability properties; an excellent example is [1].

Ultimately, we propose a continuous control law based upon the techniques of trajectory following optimization. Rather than specify the control directly, a differential equation is defined for the time derivative of the control. This differential equation is integrated numerically, yielding a continuous control that is unique from other continuous-time sliding mode controllers. That is, this differential equation is derived from the optimal control necessary conditions applied to the candidate Lyapunov function, not by explicitly requiring the state to lie on the switching surface. Typically, such sliding mode controls

are found by repeated time differentiation of the switching surface.

Our approach is unique in that we combine LOC, SMC, and TFO to produce continuous feedback controls. The reasons for combining these three techniques defines the contribution of this treatment and are as follows: 1) LOC provides a means to bring cost consideration into the control design process. SMC algorithms typically place the switching surface based only on stable dynamics. The LOC approach will allow us to provide a switching surface with stable dynamics and a cost minimization interpretation. 2) The selection of the candidate Lyapunov function, which determines the placement of the switching surface, is motivated by cost minimization. This candidate is not required to decrease everywhere outside the origin. We develop necessary conditions on the quadratic cost term and system quantities that permit us to use the cost term as the candidate Lyapunov function. We prove that asymptotic stability of the origin without requiring this function to be an "actual" Lyapunov function; i.e. its time rate of change is not required to be strictly negative apart from the origin. Therefore, asymptotic stability of the origin is achieved based upon the attractiveness and stability properties of the switching surface. Thus, the applicability of LOC based control laws has been extended. 3) Trajectory following optimization will provide the continuous control effort by not only considering elimination of chatter but, once again, by considering minimization of cost.

2 Technical Background

In this section we summarize the established control and optimization theory for use in the derivation of our methodology.

2.1 Optimal Control Theory

The developed feedback controllers are similar to those found through dynamic programming, but we explicitly consider the disturbance. This research considers the optimal control problem

$$J[u(\cdot)] = \int_0^{t_f} f_0 dt = \int_0^{t_f} \Psi(x)\, dt \tag{1}$$

where

$$\Psi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\intercal}\frac{\partial^2\Psi}{\partial\mathbf{x}^2}\mathbf{x} \tag{2}$$

which is associated with transporting the state $x \in R^n$ from some initial point to the origin. The matrix $\partial^2\Psi/\partial x^2 \in R^{n\times n}$ is constant, symmetric, and posi-

tive definite. We consider systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u, v) = \mathbf{A}\mathbf{x} + \mathbf{b}(u + v), \tag{3}$$

where the control variable is an element of the constraint set

$$\mathcal{U} = \left\{ u \in R^1 : -u_{\max} \leq u \leq u_{\max} \right\}, \tag{4}$$

where $u_{\max} > 0$ and $u_{\min} = -u_{\max}$. We assume that the system (3) is in companion form; therefore,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{5}$$

For convenience, define the $(n-1) \times 1$ vectors x_r and $\hat{x}$ such that

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_r \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \hat{\mathbf{x}} \end{bmatrix}. \tag{6}$$

With $a^{\intercal} = [a_{n1}, \cdots, a_{nn}]$, and

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{7}$$

we may express (3) as

$$\begin{aligned} \dot{\mathbf{x}}_r &= \tilde{\mathbf{A}}\hat{\mathbf{x}} \\ \dot{x}_n &= \mathbf{a}^{\intercal}\mathbf{x} + u + v. \end{aligned} \tag{8}$$

It is assumed that the unknown disturbance satisfies

$$-v_{\max} < v < v_{\max} \tag{9}$$

where, $v_{\max} < u_{\max}$. Typically, this class of optimal control problems results in control laws of the general form [12, pp. 365-366]

$$u = \begin{cases} u_{\max} & \text{if} \quad \sigma(\mathbf{x}) < 0 \\ u_s \in [u_{\min}, u_{\max}] & \text{if} \quad \sigma(\mathbf{x}) = 0 \\ u_{\min} & \text{if} \quad \sigma(\mathbf{x}) > 0 \end{cases} \tag{10}$$

where the control switches discontinuously across the switching surface $\sigma(\mathbf{x}) = 0$. A control $u(\mathbf{x}) = u_s(\mathbf{x})$ that results in $\sigma[\mathbf{x}(t)] \equiv 0$ for a nonzero time interval is called singular control. Otherwise, the control switches between $u(\mathbf{x}) = u_{\max}$ and $u(\mathbf{x}) = u_{\min}$ and is called bang-bang (or switching) control. Singular control over a nonzero time interval requires $\sigma(\mathbf{x}) = \dot{\sigma}(\mathbf{x}, u) \equiv 0$ [12, pp. 406-408]. The switching surface is $S \triangleq \{\mathbf{x} : \sigma(\mathbf{x}) = 0\}$.

2.2 Trajectory Following Optimization

Trajectory following optimization algorithms provide solutions by solving special sets of differential equations. These differential equations are defined so that their equilibrium solutions satisfy local necessary conditions for the optimization at hand; minimization, maximization, or min-max. For example, consider a scalar valued function $G(u)$. Numerically find the minimizer by letting $\dot{u} = -\partial G/\partial u$ which yields $\dot{G} = (\partial G/\partial u)\,\dot{u} = -\partial G/\partial u\,(\partial G/\partial u)^{\intercal}$. An excellent reference on trajectory following methods in control system design is [13].

2.3 Discontinuous Ordinary Differential Equations

Closed-loop control laws may result in discontinuous state equations across a switching surface. This discontinuity may produce a phenomenon known as chatter [12, p. 14]. Let $\dot{x}(t-) = \dot{x}^{-}(t)$ and $\dot{x}(t+) = \dot{x}^{+}(t)$ denote the velocity vector just before and just after t. Assuming that neither $\dot{x}(t-)$ nor $\dot{x}(t+)$ are tangent to S at a point where it meets $x(t)$, necessary conditions for chatter are that $\partial\sigma/\partial x\,[\dot{\mathbf{x}}(t-)]$ and $\partial\sigma/\partial x\,[\dot{\mathbf{x}}(t+)]$ have opposite signs. Sufficient conditions may be expressed as $\sigma\dot{\sigma} < 0$. A function $x(t)$ is a solution to $\dot{x}(t) = f(x)$ in the sense of Fillipov [12, p. 16-17] if $\dot{x}(t)$ is a convex combination of $\dot{x}(t-)$ and $\dot{x}(t+)$. That is, there exists an $\alpha \in [0, 1]$ such that $\dot{x}_F(t) = \alpha\dot{x}(t-) + (1-\alpha)\,\dot{x}(t+)$.

2.4 Quickest Descent Lyapunov Optimizing Control

Quickest descent control provides the structure for algorithms developed in this treatment that consider cost accumulation, control bounds, and progress to the target. This is achieved by first selecting a descent function $\bar{\Psi}(\mathbf{x})$ that exhibits the properties: 1) $\bar{\Psi}(\mathbf{0}) = 0$, 2) $\bar{\Psi}(\mathbf{x}) > 0\ \forall\ x \neq 0$, and 3) $\partial\bar{\Psi}/\partial x \neq 0$ for $x \neq 0$. The control u is chosen to decrease $\bar{\Psi}(\mathbf{x})$ as quickly as possible along trajectories $x(t)$; that is the control u is chosen by

$$\min_{\mathbf{u}} \frac{d}{dt}\left[\bar{\Psi}(\mathbf{x}, \mathbf{u}, \mathbf{v})\right]. \tag{11}$$

In [12, p. 295-300] a minimum cost descent LOC algorithm is presented that generates controls that also consider cost accumulation and descent toward the target. The augmented descent function takes the form

$$\dot{\Psi}_0 = \hat{\Psi}(\mathbf{x}, \mathbf{u}) + \frac{\partial \bar{\Psi}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}) \tag{12}$$

where $\hat{\Psi}(\mathbf{x})$ is the cost accumulation rate and $\bar{\Psi}(\mathbf{x})$ is a descent function that measures progress toward the target. For the problem considered here, since we explicitly consider control bounds, cost accumulation is not a function of the control u. Therefore, minimizing (12), with respect to u (with $\hat{\Psi}(\mathbf{x}, \mathbf{u}) = \hat{\Psi}(\mathbf{x})$) would not account for cost accumulation. For this reason, we will apply the quickest descent method to decrease a suitably chosen descent function $\bar{\Psi}(\mathbf{x})$ as quickly as possible along trajectories $x(t)$.

3 Lyapunov Optimizing Sliding Mode Control

In this section we derive the controls applicable to the problem as formulated in (1)-(9). We apply the quickest descent method and choose the cost accumulation rate $\Psi(\mathbf{x})$ as the descent function; that is we let $\Psi(\mathbf{x}) = \bar{\Psi}(\mathbf{x})$. This selection allows us to explicitly relate cost accumulation through $\Psi(\mathbf{x})$ to the placement of the switching surface S (derived from the optimal control necessary conditions applied to the descent function as in 11), to reachability of S, and ultimately to stability of the origin. We begin by fully describing the geometric properties of the surface S.

3.1 LOSMC Geometric Properties

The control is found by the minimization

$$\min_u \dot{\Psi}(\mathbf{x}, u, v) \tag{13}$$

where

$$\dot{\Psi}(\mathbf{x},u) = \frac{1}{2}\mathbf{x}^\intercal \left[\frac{\partial^2 \Psi}{\partial \mathbf{x}^2}\mathbf{A} + \mathbf{A}^\intercal \frac{\partial^2 \Psi}{\partial \mathbf{x}^2}\right]\mathbf{x} + (u+v)\frac{\partial \sigma}{\partial \mathbf{x}}\mathbf{x}.$$

The control laws found via (13) take the form

$$u(\mathbf{x}) = \begin{cases} u_{\max} & \text{if} \quad \sigma(\mathbf{x}) < 0 \\ u_s(\mathbf{x}) & \text{if} \quad \sigma(\mathbf{x}) = 0 \\ u_{\min} & \text{if} \quad \sigma(\mathbf{x}) > 0 \end{cases}, \tag{14}$$

where

$$\sigma(\mathbf{x}) = \frac{\partial \dot{\Psi}}{\partial u} = \mathbf{b}^{\intercal} \frac{\partial^2 \Psi}{\partial \mathbf{x}^2} \mathbf{x}. \tag{15}$$

Additional entities related to the switching surface are

$$\frac{\partial \sigma}{\partial \mathbf{x}} = \mathbf{b}^{\intercal} \frac{\partial^2 \Psi}{\partial \mathbf{x}^2}, \tag{16}$$

$$\dot{\sigma}(\mathbf{x}, u) = \mathbf{b}^{\intercal} \frac{\partial^2 \Psi}{\partial \mathbf{x}^2} [\mathbf{A}\mathbf{x} + \mathbf{b}(u + v)] \tag{17}$$

Singular control $u_s(\mathbf{x})$ requires $\sigma(\mathbf{x}) = \dot{\sigma}(\mathbf{x}, u, v) = 0$, which from (15)-(17) is

$$u_s(\mathbf{x}) = -\frac{(\partial \sigma / \partial \mathbf{x})[\mathbf{A}\mathbf{x} + \mathbf{b}v]}{(\partial \sigma / \partial \mathbf{x})\mathbf{b}}. \tag{18}$$

Considering (16) and based on the assumption that $\partial^2 \Psi / \partial x^2$ is positive definite it is required that

$$\frac{\partial \sigma}{\partial \mathbf{x}} \mathbf{b} = \mathbf{b}^{\intercal} \frac{\partial^2 \Psi}{\partial \mathbf{x}^2} \mathbf{b} > \mathbf{0} \tag{19}$$

for an undisturbed system where the Kalman controllability condition is satisfied [4, p. 54]. To simplify several expressions consider the following. Substitution of (18) into (3), yields

$$\dot{\mathbf{x}} = \left[\mathbf{A} - \mathbf{b} \frac{(\partial \sigma / \partial \mathbf{x})\mathbf{A}}{(\partial \sigma / \partial \mathbf{x})\mathbf{b}} \right] \mathbf{x}. \tag{20}$$

The disturbance v does not affect state trajectories on S. However, to implement (18), yielding (20), we need to specify the unknown v; this is not possible. Let

$$\Gamma = \mathbf{A} - \mathbf{b} \frac{(\partial \sigma / \partial \mathbf{x})\mathbf{A}}{(\partial \sigma / \partial \mathbf{x})\mathbf{b}} \tag{21}$$

so that equation (20) is

$$\dot{\mathbf{x}} = \Gamma \mathbf{x}. \tag{22}$$

Throughout the analysis, it will be convenient to analyze phenomena specific to states located on or off S. Let x_S satisfy $\sigma(\mathbf{x}_S) = 0$, that is, x_S denotes a state on S. Let δx denote a displacement from S; this results in $x = x_S + \delta x$.

3.2 LOSMC Stability Necessary Conditions

Our purpose in this section is to produce necessary conditions, involving Ψ, σ, S, and the state equations (3) that must be satisfied to yield asymptotic stability of the origin. We explicitly consider nonzero equilibria, finite time convergence to S, and stability properties of S.

3.2.1 Nonzero Equilibria

Consider the possibility of nonzero equilibria on and off S.

Theorem 1 *Assuming A is nonsingular, with $u = 0$ interior to the control constraint set U, if $\Psi(\mathbf{x})$ satisfies $(\partial\sigma/\partial\mathbf{x})[\mathbf{A}^{-1}\mathbf{b}] < 0$ trajectories generated by the control law (10) do not converge to nonzero equilibrium points in regions of bang-bang control.*

Proof. Consider the region $\sigma(\mathbf{x}) > 0$; let x_{eq}^{+} denote equilibrium in this region of state space and let δx_{eq}^{+} locate this point relative to S. With $x_{eq}^{+} = x_S + \delta x_{eq}^{+}$, $x_S + \delta x_{eq}^{+} = -A^{-1}b(u_{\min} + v)$. Now

$$(\partial\sigma/\partial\mathbf{x})\left[\mathbf{x}_S + \delta\mathbf{x}_{eq}^{+}\right] = (\partial\sigma/\partial\mathbf{x})\left[-\mathbf{A}^{-1}\mathbf{b}(u_{\min} + v)\right]$$

which implies that

$$\frac{\partial\sigma}{\partial\mathbf{x}}\delta\mathbf{x}_{eq}^{+} = \frac{\partial\sigma}{\partial\mathbf{x}}\left[-\mathbf{A}^{-1}\mathbf{b}(u_{\min} + v)\right] \tag{23}$$

since $(\partial\sigma/\partial\mathbf{x})x_S \equiv 0$. With $\sigma(\mathbf{x})$ a linear function $\sigma\left(\mathbf{x}_{eq}^{+}\right) = (\partial\sigma/\partial\mathbf{x})\delta x_{eq}^{+}$. Since $\sigma\left(\mathbf{x}_{eq}^{+}\right) > 0$, $(\partial\sigma/\partial\mathbf{x})\delta x_{eq}^{+} > 0$; this suggests that for a nonzero equilibrium point to exist in this region the right side of (23) must be greater than zero. With $(u_{\min} + v) < 0$ this requires $(\partial\sigma/\partial\mathbf{x})[\mathbf{A}^{-1}\mathbf{b}] > 0$; we have, however, specified

$$(\partial\sigma/\partial\mathbf{x})\left[\mathbf{A}^{-1}\mathbf{b}\right] < 0. \tag{24}$$

Analogous results hold for the region $\sigma(\mathbf{x}) < 0$. ■

Theorem 2 *If the system (3) is controllable (with $v = 0$), then singularity of the system matrix A does not produce equilibrium manifolds in regions of bang-bang control.*

Proof. The singularity assumption implies that there exists a nonzero vector x such that $Ax = 0$ [15, pg. 8]. A nonzero equilibrium must then satisfy

$$\begin{aligned} \mathbf{0} &= \mathbf{A}\mathbf{x} + \mathbf{b}(u + v) \\ \mathbf{0} &= \mathbf{b}(u + v) \end{aligned} \tag{25}$$

With the system (3) controllable, equation (25) can only be satisfied if $u + v = 0$. With $|u_{\max}| > |v|$, and applying (14), we conclude that the singularity of A will not induce equilibrium manifolds in regions of bang-bang control. ■

Theorem 3 *If the system (3) is controllable (with $v = 0$), LOSMC produces a one-dimensional, nonzero, equilibrium manifold.*

Proof. By expanding (21) (also proven in [7, pp. 50-51]), it can be shown that $\Gamma = [\gamma_{ij}]$ is singular, i.e. $\gamma_{n1} = 0$. The state equations

$$\begin{aligned} \dot{\mathbf{x}} &= \Gamma \mathbf{x} \\ &= \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & \gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \end{aligned} \tag{26}$$

on the singular manifold are zero for all x_1 with $x_2 = x_3 = \cdots = x_n = 0$. ■

Theorem 4 *If A is nonsingular, with $u = 0$ interior to the control constraint set U, choosing $\Psi(\mathbf{x})$ such that $(\partial\sigma/\partial\mathbf{x}) A^{-1} b \neq 0$ guarantees that the only point on the equilibrium manifold contained within S is the origin.*

Proof. With the state equations in companion form, $(\partial\sigma/\partial\mathbf{x}) A^{-1} b = (\partial\sigma/\partial x_1) a_{n1}$. To satisfy $(\partial\sigma/\partial\mathbf{x}) A^{-1} b \neq 0$ we must have $\partial\sigma/\partial x_1$ and a_{n1} nonzero. Existence of a one-dimensional equilibrium manifold for all x_1, with $x_2 = x_3 = \cdots = x_n = 0$, was proven in Theorem 3; any plane that contains nonzero points of this equilibrium manifold would have a normal vector with the first entry $\partial\sigma/\partial x_1$ is zero. The requirement $(\partial\sigma/\partial x_1) a_{n1} \neq 0$ prevents this scenario. The only point on the equilibrium manifold contained by the switching surface S with $(\partial\sigma/\partial x_1) a_{n1} \neq 0$ is the origin. ■

3.2.2 Stability of S

In this Section, we consider two scenarios. First we develop necessary conditions involving the state equations, Ψ, and σ, that provide asymptotic stability of the origin for trajectories on S. Then we propose a sub-optimal algorithm (in terms of quickest descent) that may be used if these necessary conditions are not met.

Necessary Conditions for Stability of the Sliding Surface S Necessary conditions for stability of a sliding surface are available from SMC theory. Typically, a sliding surface is selected that satisfies the conditions that follow. The necessary conditions presented here explicitly connect cost minimization to placement of the surface S. If the conditions hold then (10) is a control strategy that minimizes cost in a quickest descent fashion.

Theorem 5 *Assume the system (3) is controllable with $v = 0$. Let $\hat{\Gamma}$ denote an $n-1 \times n-1$ matrix obtained by deleting the first row and column of Γ. We may specify $\Psi(x)$ (equivalently place S) such that the $n-1$ eigenvalues of $\hat{\Gamma}$ have strictly negative real parts and such that there exists trajectories on S that asymptotically approach the origin.*

Proof. The proof of this theorem yields necessary conditions that must be satisfied for asymptotic stability of the origin and serves as a guide for selection of an alternate descent function if they are not. Consider the reduced system

$$\frac{d}{dt}(\hat{\mathbf{x}}) = \hat{\Gamma}\hat{\mathbf{x}} = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\dfrac{\dfrac{\partial\sigma}{\partial x_1}}{\dfrac{\partial\sigma}{\partial x_n}} & -\dfrac{\dfrac{\partial\sigma}{\partial x_2}}{\dfrac{\partial\sigma}{\partial x_n}} & \cdots & -\dfrac{\dfrac{\partial\sigma}{\partial x_{n-1}}}{\dfrac{\partial\sigma}{\partial x_n}} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}. \tag{27}$$

A necessary condition for the eigenvalues of (27) to have strictly negative real parts is that the coefficients $\gamma_{n2}, \gamma_{n3}, \ldots, \gamma_{nn}$ are negative. These are the first necessary conditions that must be satisfied by $\Psi(x)$, via $\sigma(\mathbf{x})$ to provide asymptotic stability. If these conditions are not satisfied we may alter the descent function. To see that this is possible, observe that no conditions have been imposed on $\partial\sigma/\partial x_i$, $i = 2, \ldots, n-1$; we may specify $\gamma_{n3}, \gamma_{n4}, \ldots, \gamma_{nn}$ less than zero. The problem now is to determine if we can satisfy $\gamma_{n2} < 0$ with $(\partial\sigma/\partial\mathbf{x})\, b > 0$ and $(\partial\sigma/\partial\mathbf{x})\, A^{-1}b = (\partial\sigma/\partial x_1)\, a_{n1} < 0$. Of greatest concern is the case where the eigenvalues of the matrix A have strictly negative real parts, yielding stable nonzero equilibrium points. If this is the case, we have $a_{n1} < 0$ which implies $\partial\sigma/\partial x_1 > 0$. This, along with the condition $(\partial\sigma/\partial\mathbf{x})\, b = \partial\sigma/\partial x_n > 0$ implies that $\gamma_{n2} < 0$. The necessary condition on the coefficients $\gamma_{ni}, i = 2, \ldots, n$, for eigenvalues of A to have strictly real parts is satisfied. In addition to controlling the sign of the coefficients $\gamma_{n2}, \gamma_{n3}, \ldots, \gamma_{nn}$, we have complete freedom to specify their magnitude. We may choose the coefficients such that the Hurwitz determinants of $\hat{\Gamma}$ are greater than zero; yielding $n-1$ eigenvalues with real parts that are strictly negative. For one way to do this, see [7, pp. 68-71]. Note that (27) is decoupled from x_1.

With $\partial\sigma/\partial x_1 > 0$ and S passing through the origin, the proof of Theorem 4 guarantees that the only equilibrium point on S is the origin. With $\hat{\Gamma}$ stable, $x_r \to 0$, and $\sigma(\mathbf{x}) = 0$ trajectories asymptotically approach the origin. Note that for unstable systems, we cannot satisfy $\gamma_{n2} < 0$ with $(\partial\sigma/\partial\mathbf{x})\, b > 0$ and $(\partial\sigma/\partial\mathbf{x})\, A^{-1}b = (\partial\sigma/\partial x_1)\, a_{n1} < 0$. However, any equilibrium points for this case are unstable; the condition $(\partial\sigma/\partial\mathbf{x})\, A^{-1}b = (\partial\sigma/\partial x_1)\, a_{n1} < 0$ is not critical. ■

3.3 Finite Time Convergence to S

In this section, it is shown that there exist regions of state space where S is reached in finite time even with $\dot{\Psi}(x) > 0$ for some x.

Theorem 6 *The state equations satisfy*

$$\|\mathbf{f}(\mathbf{x}, u, v)\| \leq C_1 + C_2 \|\mathbf{x}(t)\| \tag{28}$$

where $C_1 = \|\mathbf{b}(u_{\max} + v_{\max})\|$ *and* $C_2 = \|\mathbf{A}\|$.

Proof. The proof follows directly since the state equations $f(\mathbf{x}, u, v)$ are linear with a bounded control and disturbance. We have

$$\begin{aligned} \|\mathbf{f}(\mathbf{x}, u, v)\| &= \|\mathbf{A}\mathbf{x} + \mathbf{b}(u+v)\| \\ &\leq \|\mathbf{A}\mathbf{x}\| + \|\mathbf{b}(u+v)\| \\ &\leq \|\mathbf{A}\|\,\|\mathbf{x}\| + \|\mathbf{b}(u_{\max} + v_{\max})\| \\ &= C_1 + C_2\|\mathbf{x}\| \end{aligned} \tag{29}$$

■

Theorem 7 *Assume the necessary conditions of Theorem 5 are satisfied by* $\Psi(x)$ *(or an appropriate replacement). Over any finite time interval beginning at* $t = t_i$*, the state may be upper bounded by a function of* $\|\mathbf{x}(t_i)\|$ *and by the maximum absolute value of* σ *attained in that time interval.*

Proof. Consider the state equations (recall 6) in an alternate form

$$\begin{aligned} \dot{\mathbf{x}}_r &= \hat{\Gamma}\mathbf{x}_r + \mathbf{b}\frac{\sigma}{\partial\sigma/\partial x_n} \\ x_n(t) &= \mathbf{a}_r\mathbf{x}_r + \frac{\sigma}{\partial\sigma/\partial x_n} \end{aligned} \tag{30}$$

where a_r corresponds to the last row of $\hat{\Gamma}$. Integrating the first of equations (30) yields

$$\begin{aligned} \mathbf{x}_r(t) &= e^{\left[\hat{\Gamma}(t-t_i)\right]}\mathbf{x}_r(t_i) + \int_{t_i}^{t} e^{\left[\hat{\Gamma}(t-\tau)\right]}\mathbf{b}\left(\frac{\sigma}{\partial\sigma/\partial x_n}\right)d\tau \\ &\leq e^{\left[\hat{\Gamma}(t-t_i)\right]}\mathbf{x}_r(t_i) \\ &\quad + \max_{t_i\leq\tau\leq t}\left|\frac{\sigma\left[\mathbf{x}(\tau)\right]}{\partial\sigma/\partial x_n}\right|\int_{t_i}^{t} e^{\left[\hat{\Gamma}(t-\tau)\right]}\mathbf{b}d\tau. \end{aligned} \tag{31}$$

From (31), $x_r(t)$ satisfies

$$\begin{aligned} \|\mathbf{x}_r(t)\| &\leq \left\|e^{\left[\hat{\Gamma}(t-t_i)\right]}\mathbf{x}_r(t_i)\right\| \\ &\quad + \max_{t_i\leq\tau\leq t}\left|\frac{\sigma\left[\mathbf{x}(\tau)\right]}{\partial\sigma/\partial x_n}\right|\int_{t_i}^{\infty}\left\|e^{\left[\hat{\Gamma}(t-\tau)\right]}\mathbf{b}\right\|d\tau. \end{aligned} \tag{32}$$

Now

$$\begin{aligned} |x_n(t)| &= \left\|\mathbf{a}_r\mathbf{x}_r + \frac{\sigma}{\partial\sigma/\partial x_n}\right\| \\ &\leq \|\mathbf{a}_r\|\left[\left\|e^{\left[\hat{\Gamma}(t-t_i)\right]}\mathbf{x}_r(t_i)\right\|\right. \\ &\quad \left. + \max_{t_i\leq\tau\leq t}\left|\frac{\sigma\left[\mathbf{x}(\tau)\right]}{\partial\sigma/\partial x_n}\right|\int_{t_i}^{\infty}\left\|e^{\left[\hat{\Gamma}(t-\tau)\right]}\mathbf{b}\right\|d\tau\right] + \max_{t_i\leq\tau\leq t}\left|\frac{\sigma\left[\mathbf{x}(\tau)\right]}{\partial\sigma/\partial x_n}\right| \end{aligned} \tag{33}$$

and therefore (noting that $\|\mathbf{x}_r\| \leq \|\mathbf{x}\|$)

$$\begin{aligned} \|\mathbf{x}(t)\| &\leq (1+\|\mathbf{a}_r\|)\left[\left\|e^{\left[\hat{\Gamma}(t-t_i)\right]}\mathbf{x}(t_i)\right\|\right. \\ &\quad \left. + \max_{t_i\leq\tau\leq t}\left|\frac{\sigma\left[\mathbf{x}(\tau)\right]}{\partial\sigma/\partial x_n}\right|\int_{t_i}^{\infty}\left\|e^{\left[\hat{\Gamma}(t-\tau)\right]}\mathbf{b}\right\|d\tau\right] + \max_{t_i\leq\tau\leq t}\left|\frac{\sigma\left[\mathbf{x}(\tau)\right]}{\partial\sigma/\partial x_n}\right| \end{aligned} \tag{34}$$

which completes the proof. ■

As we show in the next theorem, equation (34) represents necessary conditions on $\Psi(\mathbf{x})$ for finite time convergence to the switching surface, through equation (27). The integral in (34) cannot be evaluated unless the coefficients in the vector a_r are such that the matrix $\hat{\Gamma}$ has eigenvalues with strictly negative real parts. If $\hat{\Gamma}$ is stable, then from (32) we have

$$\begin{aligned} \|\mathbf{x}(t)\| &\leq (1+\|\mathbf{a}_r\|)\left[\|\mathbf{x}(t_i)\|\right. \\ &\quad \left. + \max_{t_i\leq\tau\leq t}\left|\frac{\sigma\left[\mathbf{x}(\tau)\right]}{\partial\sigma/\partial x_n}\right|\int_{t_i}^{\infty}\left\|e^{\left[\hat{\Gamma}(t-\tau)\right]}\mathbf{b}\right\|d\tau\right] + \max_{t_i\leq\tau\leq t}\left|\frac{\sigma\left[\mathbf{x}(\tau)\right]}{\partial\sigma/\partial x_n}\right|, \end{aligned} \tag{35}$$

where the integral in (35) may be evaluated. Letting

$$\begin{aligned} C_3 &= (1+\|\mathbf{a}_r\|) \\ C_4 &= 1 + C_3\int_{t_i}^{\infty}\left\|e^{\left[\hat{\Gamma}(t-\tau)\right]}\mathbf{b}\right\|d\tau \end{aligned}$$

(35) becomes

$$\|\mathbf{x}(t)\| \leq C_3 \|\mathbf{x}(t_i)\| + C_4 \max_{t_i \leq \tau \leq t} \left| \frac{\sigma[\mathbf{x}(\tau)]}{\partial\sigma/\partial x_n} \right| \tag{36}$$

where $C_3, C_4 > 0$.

Theorem 8 *Consider the region $\sigma(\mathbf{x}) < 0$; all $x(t_i)$ that satisfy*

$$\frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}u_{\max} > \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{A} \right\| \left[C_3 \|\mathbf{x}(t_i)\| + C_4 \left| \frac{\sigma[\mathbf{x}(t_i)]}{\partial\sigma/\partial x_n} \right| \right] + \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}v_{\max} \right\| \tag{37}$$

reach the surface S ($\sigma(\mathbf{x}) = 0$) in finite time.

Proof. First, we are assured such a region exists; for arbitrarily small $x(t_i)$ and $\sigma(t_i)$ since

$$(\partial\sigma/\partial\mathbf{x})\,\mathbf{b}u_{\max} > 0$$

and $u_{\max} > v_{\max}$. Also note that (37) implies that $\dot{\sigma}(0) > 0$. To prove finite time convergence we must show that $\dot{\sigma}(t)$ is strictly positive for all $x(t_i)$ that satisfy (37). The proof follows by supposing for some $x(t_i)$ that satisfies (37) there is a time when $\dot{\sigma}(t) = 0$. The first instant at which this is true requires

$$\frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{A}\mathbf{x} + \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}u_{\max} + \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}v = 0.$$

Equivalently, with $\frac{\partial\sigma}{\partial\mathbf{x}}bu_{\max} > 0$, this requires

$$\left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{A}\mathbf{x} + \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}v_{\max} \right\| = \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}u_{\max}. \tag{38}$$

Now we have, in part using (36),

$$\begin{aligned} \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{A}\mathbf{x} + \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}v_{\max} \right\| &\leq \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{A}\mathbf{x} \right\| + \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}v_{\max} \right\| \\ &\leq \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{A} \right\| \|\mathbf{x}\| + \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}v_{\max} \right\| \\ &\leq \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{A} \right\| \left[C_3 \|\mathbf{x}(t_i)\| + C_4 \max_{t_i \leq \tau \leq t} \left| \frac{\sigma[\mathbf{x}(\tau)]}{\partial\sigma/\partial x_n} \right| \right] + \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}v_{\max} \right\|. \end{aligned} \tag{39}$$

After a finite time interval, at the first instant when $\dot{\sigma} = 0$, the last equation of (39) becomes

$$\left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{A} \right\| \left[C_3 \|\mathbf{x}(t_i)\| + C_4 \left| \frac{\sigma[\mathbf{x}(t_i)]}{\partial\sigma/\partial x_n} \right| \right] + \left\| \frac{\partial\sigma}{\partial\mathbf{x}}\mathbf{b}v_{\max} \right\|. \tag{40}$$

From (38) and (40) we have

$$\begin{aligned} & \left\| \frac{\partial \sigma}{\partial \mathbf{x}} \mathbf{A} \right\| \left[C_3 \left\| \mathbf{x}\left(t_i\right) \right\| + C_4 \left| \frac{\sigma\left[\mathbf{x}\left(t_i\right)\right]}{\partial \sigma / \partial x_n} \right| \right] + \left\| \frac{\partial \sigma}{\partial \mathbf{x}} \mathbf{b} v_{\max} \right\| \\ \geq \; & \frac{\partial \sigma}{\partial \mathbf{x}} \mathbf{b} u_{\max} \end{aligned}$$

which is a contradiction. Therefore, $\dot{\sigma}$ is strictly positive on any finite time interval and trajectories whose initial conditions satisfy (37) converge to the switching surface in finite time. Analogous results hold for regions where $\sigma > 0$. ■

4 Continuous LOSCB Control

In this section a continuous time control law is proposed. Consider implementing

$$\dot{u} = -\left(1/\epsilon\right)\sigma \tag{41}$$

where ϵ is a positive scalar. If the differential equation (41) would cause a violation of the control constraints, we let the control saturate. The parameter ϵ may be chosen sufficiently small so that we may approximate the control law (10) in a continuous manner. Note that (41) is an online control strategy derived by applying TFO techniques to $\dot{\Psi}$. That is, the equilibrium solution to

$$\dot{u} = -\left(1/\epsilon\right)\sigma = -\left(1/\epsilon\right)\frac{\partial \dot{\Psi}}{\partial u} \tag{42}$$

satisfies a a local necessary condition for the minimization of $\dot{\Psi}$ with respect to u. The implication of (42) is that we have a control law that eliminates discontinuous chatter, derived by considering minimization of cost.

Example 9 *Consider a system described by state equations*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + v)$$

where $-1 \leq u \leq 1$ *and for illustration purposes we take the "unknown" disturbance* $v = 0.5$. *To illustrate algorithmic performance, we select* $\Psi = 2x_1^2 + x_1 x_2 + .5x_2^2$, *which yields* $\sigma = x_1 + x_2$. *Implementing (41) with* $\epsilon = 10^{-6}$ *yields trajectories in Fig. 1. Trajectories intersect* S *with* u_S *admissible and ultimately converge to the origin under the action of the smooth control* u.

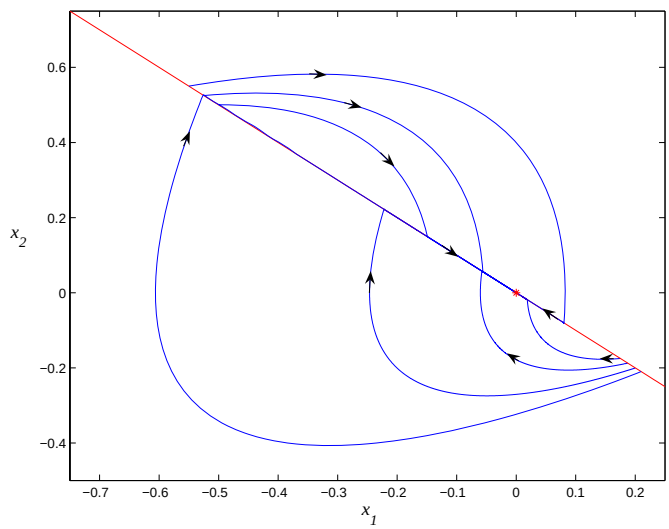

Figure 1: Typical Trajectories for Example One.

Example 10 *Consider a system described by state equations*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + v)$$

where $-2 \leq u \leq 2$ *and* $-1 \leq v \leq 1$. *For illustration purposes we take the "unknown" disturbance as*

$$v = \operatorname{sgn}(x_2).$$

The cost we wish to minimize, for states not at the origin, is

$$\min_u J[u(\cdot)] = \int_0^{t_f} \Psi(x)\, dt = \int_0^{t_f} \left[2.5x_1^2 + x_1 x_2 + x_2^2\right] dt$$

which yields $\sigma = x_1 + 2x_2$. *Implementing (41) with* $\epsilon = 10^{-2}$ *yields trajectories in Fig. 2. Trajectories intersect* S *with* u_S *admissible and ultimately converge to the origin (a small limit cycle of the origin) under the action of the smooth control* u. *Observe the presence of finite time interval switching control [12, p. 294] which occurs as the trajectories overshoot* S. *Implementing (41) with* $\epsilon = 10^{-5}$ *yields trajectories in Fig. 3. Trajectories intersect* S *with* u_S *admissible and ultimately converge to the origin under the action of the smooth control* u. *The overshoot which is a characteristic of finite time interval switching control and "size" of the resulting limit cycle has been greatly decreased. However, this is achieved with a price. Due to the singular perturbation* ϵ *we have a stiff set of differential equations (41).*

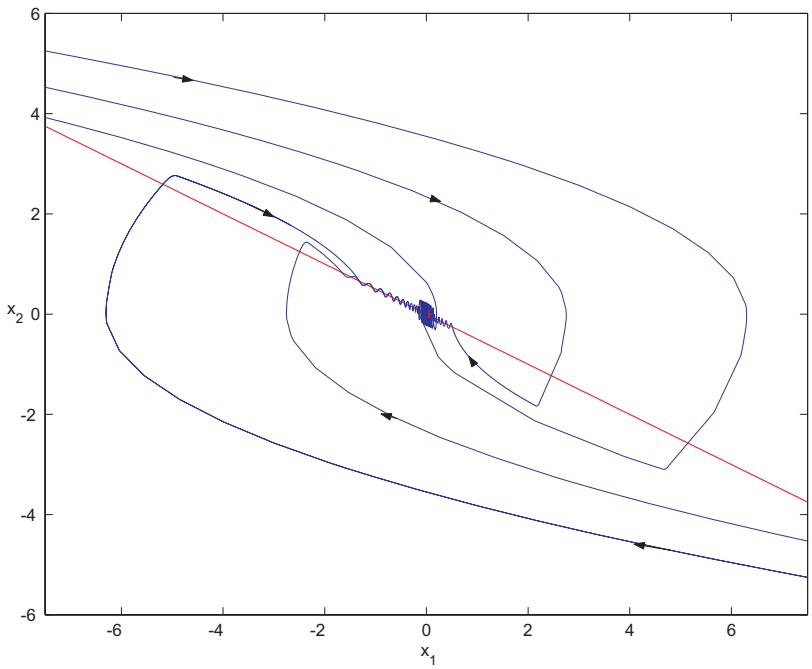

Figure 2: Typical trajectories for Example Two.

5 Conclusion

In this treatment, we have extended the applicability of Lyapunov optimizing control. This is achieved for an important class of systems by leveraging a connection to sliding mode control. Furthermore, we have used trajectory following optimization to propose a continuous time control that eliminates discontinuous chatter. This continuous control is derived not only to eliminate chatter, but to also consider cost accumulation. The form of the continuous control law will be a subject of further research. Comparison of Figure 2 and Figure 3 show that the parameter ϵ influences the manner in which system trajectories approach S and the origin. This parameter amounts to a singular perturbation, and creates a "stiff" set of differential equations when (41) is integrated along with the state equations. Investigation into strategies to eliminate the stiffness phenomena will continue.

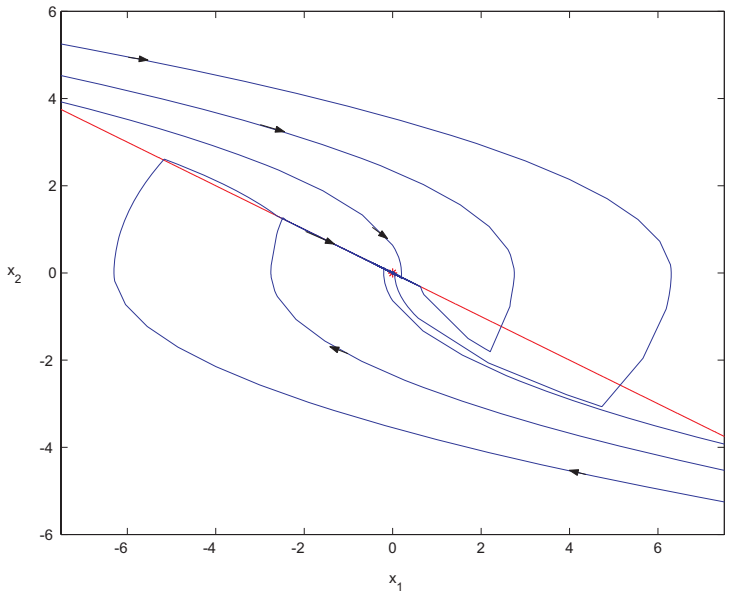

Figure 3: Typical trajectories for Example Two.

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