Applied Mathematical Sciences, Vol. 2, 2008, no. 20, 961 - 974

Biorthogonalization of the Principal Vectors for the Matrices A and A^* with Application to the Computation of the Explicit Representation of the Solution $x(t)$ of $\dot{x} = A\,x,\ x(t_0) = x_0$

L. Kohaupt

TFH Berlin, FB II, Luxemburger Str. 10
D-13353 Berlin, Germany
kohaupt@tfh-berlin.de

Abstract

In this paper, as opposed to the results of an earlier paper, the biorthogonalization of the principal vectors for the matrices A and A^* is achieved without the condition of distinct eigenvalues λ_i in the Jordan blocks $J_i(\lambda_i)$ of matrix A. This allows one to derive an explicit representation of the solution $x(t)$ of the initial value problem $\dot{x} = A\,x,\ x(t_0) = x_0$, that is computed in a new, elegant and effective way for general matrices A. The results are of major interest in *Computational Mathematics* as well as in *Computational Engineering* such as *Computational Mechanics*, where A is the system matrix of a vibration problem.

Keywords: Biorthogonalization method; Principal vectors; Explicit representation; Initial value problem; Vibration problem; Computational Engineering

1. Introduction

In this paper,

- the biorthogonalization of the principal vectors for the matrices A and A^* is achieved, where the derivation is based on the *algebraic eigenspaces* of A and A^* instead of the Jordan blocks therefore *needing not* the hypothesis of distinct eigenvalues in the Jordan blocks of A; thus earlier results are generalized, cf. [6].

- Further, we apply the *new computation method* to derive an explicit representation of the solution $x(t)$ of $\dot{x} = A\,x,\ x(t_0) = x_0$, as well as an explicit representation of the associated fundamental matrix $\Phi(t, t_0) = \Phi(t - t_0, 0)$ in an elegant, straightforward, and effective way.

The paper is structured as follows.

In Section 2, an important relation between the eigenprojections $P_\lambda(A)$ and $P_{\overline{\lambda}}(A^*)$ is derived. In Section 3, the biorthogonalization by a Gram-Schmidt process of principal vectors of A and A^* corresponding to a fixed eigenvalue $\lambda(A)$ and $\lambda(A^*) = \overline{\lambda(A)}$ is applied. In Section 4, the results of Sections 2 and 3 are employed to derive an explicit representation of the solution $x(t)$ of $\dot{x} = A\,x,\ x(t_0) = x_0$, and an explicit representation of the associated fundamental matrix $\Phi(t, t_0) = \Phi(t - t_0, 0)$ based on the algebraic eigenspaces of A and A^*. Since, in this way, the canonical Jordan form is avoided, one *needs not* the hypothesis of distinct eigenvalues in the Jordan blocks of matrix A, thus generalizing earlier results. New is the method of computation, not the respective representation. In Section 5, conclusions are drawn. The references that are not cited are given because they may be useful to the reader.

2. Connection between the eigenprojections $P_\lambda(A)$ and $P_{\overline{\lambda}}(A^*)$

In this section, we prove a lemma and a corollary that will be of fundamental importance in Section 3.

Lemma 1:

Let $X = \mathbb{C}^n$, let $A \in \mathbb{C}^{n\times n}$, and let $\lambda_l,\ l = 1, \cdots, s$ be the distinct eigenvalues of A. Further, let $P_\lambda(A) : X \mapsto X_{\lambda_l}$ be the eigenprojections of A corresponding to the eigenvalue $\lambda_l = \lambda_l(A)$ of A mapping X onto the algebraic eigenspace $X_{\lambda_l} = X_{\lambda_l}(A)$; likewise, let $P_{\overline{\lambda}}(A^*) : X^* = X \mapsto X_{\overline{\lambda_l}}(A^*)$ be the eigenprojection of A^* corresponding to the eigenvalue $\overline{\lambda}$ of A^*. Then,

$$[P_{\lambda_l}(A)]^* = P_{\overline{\lambda}_l}(A^*)$$

for $l = 1, \cdots, s$.

Proof: According to [5, p.39, (5.22)], one has

$$P_{\lambda_l}(A) = -\frac{1}{2\pi i}\int_{\Gamma_l} R_A(\zeta)d\zeta$$

with

$$R_A(\zeta) = (A - \zeta)^{-1}$$

and

$$\Gamma_l = \{\zeta \,|\, |\zeta - \lambda_l| = \delta_l\} \qquad \text{(oriented positively)},$$

$l = 1, \cdots, s$, where δ_l is so small that $\Gamma_l \cap \Gamma_k = \emptyset,\ l \neq k,\ l, k = 1, \cdots, s$.
This entails

$$[P_{\lambda_l}(A)]^* = \overline{-\frac{1}{2\pi i}} \int_{-\Gamma_l^*} [R_A(\zeta)]^* d\overline{\zeta},$$

where $-\Gamma_l^* = \{\overline{\zeta} \,|\, |\overline{\zeta} - \overline{\lambda_l}| = \delta_l\} \qquad \text{(oriented negatively, see Fig.1)}.$

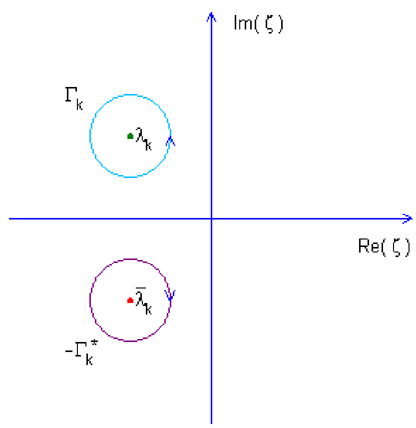

Fig.1: Reflection of Γ_k at real axis

Now,

$$\overline{-\frac{1}{2\pi i}} = \frac{1}{2\pi i},$$

and

$$[R_A(\zeta)]^* = (A^* - \overline{\zeta})^{-1} = [R_{A^*}(\overline{\zeta})].$$

Thus,

$$[P_{\lambda_l}(A)]^* = \frac{1}{2\pi i} \int_{-\Gamma_l^*} [R_{A^*}(\overline{\zeta})] d\overline{\zeta} = -\frac{1}{2\pi i} \int_{\Gamma_l^*} [R_{A^*}(\overline{\zeta})] d\overline{\zeta} = P_{\overline{\lambda}_l}(A^*).$$

◇

Corollary 2:

With the denotations of Lemma 1, one has the orthogonality relations

$$(p, v^*) = 0,\ p \in X_{\lambda_j},\ v^* \in X_{\overline{\lambda}_k}(A^*), \quad j \neq k, \quad j, k = 1, \cdots, s.$$

Proof: According to [5, p.21, (3.28)],

$$P_{\overline{\lambda}_j}(A^*)\,P_{\overline{\lambda}_k}(A^*) = \delta_{jk}\,P_{\overline{\lambda}_k}(A^*)$$

and thus

$$\begin{aligned}(p, v^*) &= (P_{\lambda_j}(A)p, v^*) = (p, [P_{\lambda_j}(A)]^* v^*) = (p, P_{\overline{\lambda}_j}(A^*)v^*) \\ &= (p, P_{\overline{\lambda}_j}(A^*)P_{\overline{\lambda}_k}(A^*)v^*) = (p, \delta_{jk}P_{\overline{\lambda}_k}(A^*)v^*) \\ &= (p, \delta_{jk}v^*) = (p, 0\,v^*) = 0,\end{aligned}$$

since $j \neq k$. ◇

3 Biorthogonalization of the principal vectors of A and A^*

Let us start with some numerical examples. For the computations, MATLAB routine *jordan* has been used.

(i) Numerical examples with $J_i(\lambda_i) \neq J_j(\lambda_j)$, but $\lambda_i = \lambda_j$

Example 1: Diagonalizable matrix A with $\lambda_i = \lambda_j,\ i \neq j$

Let

$$A = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = E \in \mathbb{C}^{n\times n}.$$

Then, $A^* = E$, and $\lambda_i = 1,\ i = 1, \cdots, n$; $p_i = u_i^* = e_i$ =ith unit vector for $i = 1, \cdots, n$, where p_i is the ith eigenvector of A and u_i^* is the ith eigenvector of A^*. We have

$$(p_j, u_k^*) = \delta_{jk},\ j, k = 1, \cdots, n$$

even though $\lambda_i = \lambda_j,\ i \neq j,\ i, j = 1, \cdots, n.$

Example 2: Nondiagonalizable matrix A with $J_i(\lambda_i) \neq J_j(\lambda_j)$, but $\lambda_i = \lambda_j,\ i \neq j$

Let

$$A = \left[\begin{array}{cc|ccc} 2 & & & & \\ 1 & 2 & & & \\ \hline & & 2 & & \\ & & 1 & 2 & \\ & & & 1 & 2 \end{array}\right].$$

Here, the eigenvalues are $\lambda_1 = \lambda_2 = 2$, and the Jordan blocks $J_i(\lambda_i)$ have dimensions $m_i \times m_i$ with $m_1 = 2$, $m_2 = 3$. One has

$$T = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] = [p_1^{(1)}, p_2^{(1)}; p_1^{(2)}, p_2^{(2)}, p_3^{(2)}],$$

where $p_k^{(i)}$ is the principal vector of stage k corresponding to the eigenvalue $\lambda_i(A)$, and

$$T^{-1}AT = J = \left[\begin{array}{cc|ccc} 2 & 1 & & & \\ & 2 & & & \\ \hline & & 2 & 1 & \\ & & & 2 & 1 \\ & & & & 2 \end{array}\right] = \left[\begin{array}{c|c} J_1(2) & \\ \hline & J_2(2) \end{array}\right].$$

Further,

$$A^* = \left[\begin{array}{cc|ccc} 2 & 1 & & & \\ & 2 & & & \\ \hline & & 2 & 1 & \\ & & & 2 & 1 \\ & & & & 2 \end{array}\right].$$

From this,

$$S = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = [u_1^{(1)^*}, u_2^{(1)^*}; u_1^{(2)^*}, u_2^{(2)^*}, u_3^{(2)^*}],$$

where $u_k^{(i)^*}$ is the principal vector of stage k corresponding to the eigenvalue $\lambda_i(A^*)$, and

$$S^{-1}A^*S = \left[\begin{array}{cc|ccc} 2 & 1 & & & \\ & 2 & & & \\ \hline & & 2 & 1 & \\ & & & 2 & 1 \\ & & & & 2 \end{array}\right].$$

Here, setting $v_l^{(j)^*} := u_{m_j-l+1}^{(j)^*}$, $l = 1, \cdots, m_j$, $j = 1, 2$, we obtain

$$(p_k^{(i)}, v_l^{(j)^*}) = \delta_{kl}\, \delta_{ij}$$

even though $J_i(\lambda_i) \neq J_j(\lambda_j),\ \lambda_i = \lambda_j,\ i \neq j$. In the sequel, it will be shown that this result can be generalized to arbitrary square matrices A.

Remark: Above, $\lambda_1 = \lambda_2 (= 2)$, $m_1 = 2$, $m_2 = 3$, and the systems $\{p_1^{(1)}, p_1^{(2)}; p_2^{(1)}, p_2^{(2)}; p_3^{(2)}\}$, $\{u_2^{(1)^*}, u_3^{(2)^*}; u_1^{(1)^*}, u_2^{(2)^*}; u_1^{(2)^*}\}$ are biorthogonal. Here, we have arranged the first set such that first the principal vectors of stage 1 come, then the principal vectors of stage 2 and finally the principal vector of stage 3. $\diamond$

Now, we turn to general matrices $A \in \mathbb{C}^{n\times n}$. We choose $X = \mathbb{C}^n$ with the usual scalar product $(\cdot,\cdot)$ on $\mathbb{C}^n$.

(i) Diagonalizable matrix A

Here, one has

Theorem 3: (Biorthogonalization without $\lambda_i \neq \lambda_j$, $i \neq j$, $i, j = 1, \cdots, n$)

Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable, let λ_i, $i = 1, \cdots, n$ be the eigenvalues of A and p_i, $i = 1, \cdots, n$ be the associated eigenvectors; further, let u_i^*, $i = 1, \cdots, n$ be the eigenvectors of the adjoint matrix $A^* = \overline{A}^T \in \mathbb{C}^{n\times n}$ corresponding to the eigenvalues $\overline{\lambda}_i$, $i = 1, \cdots, n$, that is, let

$$Ap_i = \lambda_i p_i,\ i = 1, \cdots, n$$

and

$$A^* u_i^* = \overline{\lambda}_i u_i^*,\ i = 1, \cdots, n.$$

Then, the sets $\{p_1, \cdots, p_n\}$ and $\{u_1^*, \cdots, u_n^*\}$ can be determined such that the biorthogonality relations

$$(p_j, u_k^*) = \delta_{jk},\ j, k = 1, \cdots, n$$

hold.

Proof: Let

$$T^{-1}AT = J = diag(\lambda_i)_{i=1,\cdots,n}$$

and

$$T = [a_1, \cdots, a_n].$$

Then, the eigenvector of A corresponding to λ_i is given by a_i for $i = 1, \cdots, n$, and the vectors $a_1, \cdots, a_n$ are linearly independent.

Further, let

$$S^{-1}A^*S = diag(\overline{\lambda}_i)_{i=1,\cdots,n}$$

and

$$S = [e_1, \cdots, e_n].$$

Then, the eigenvector of A^* corresponding to $\overline{\lambda}_i$ is given by e_i for $i = 1, \cdots, n$ and the vectors $e_1, \cdots, e_n$ are linearly independent.

It will suffice to prove Theorem 1 for the case of two distinct eigenvalues. For example, let

$$\lambda_1 = \cdots = \lambda_{n_1} \neq \lambda_{n_1+1} = \cdots = \lambda_n.$$

Then, according to [9, Section 2], the systems $\{a_1, \cdots, a_{n_1}\}$, $\{e_1, \cdots, e_{n_1}\}$ can be biorthogonalized by a Gram-Schmidt process giving the systems $\{p_1, \cdots, p_{n_1}\}$, $\{u_1^*, \cdots, u_{n_1}^*\}$ with the properties

$$(P1) \qquad (p_j, u_k^*) = \delta_{jk},\ j, k = 1, \cdots, n_1$$

$$(P2) \quad [p_1, \cdots, p_{n_1}] = [a_1, \cdots, a_{n_1}], \quad [u_1^*, \cdots, u_{n_1}^*] = [e_1, \cdots, e_{n_1}].$$

Likewise, the systems $\{a_{n_1+1}, \cdots, a_n\}$, $\{e_{n_1+1}, \cdots, e_n\}$ can be biorthogonalized giving the systems $\{p_{n_1+1}, \cdots, p_n\}$, $\{u_{n_1+1}^*, \cdots, u_n^*\}$ with the properties

$$(P1') \qquad (p_j, u_k^*) = \delta_{jk},\ j, k = n_1 + 1, \cdots, n$$

$$(P2') \quad [p_{n_1+1}, \cdots, p_n] = [a_{n_1+1}, \cdots, a_n], \quad [u_{n_1+1}^*, \cdots, u_n^*] = [e_{n_1+1}, \cdots, e_n].$$

The property

$$(P1'') \qquad (p_j, u_k^*) = 0,\ j \in J_1,\ k \in J_2 \quad \text{or} \quad j \in J_2,\ k \in J_1$$

with $J_1 = \{1, \cdots, n_1\}$, $J_2 = \{n_1 + 1, \cdots, n\}$ follows from Corollary 2 in Section 2. $\diamond$

(ii) General square matrix A

In this case, we have

Theorem 4: (Biorthogonalization without $\lambda_i \neq \lambda_j,\ i \neq j$ for the Jordan blocks)

Let $A \in \mathbb{C}^{n\times n}$ and $\lambda_l,\ l = 1, \cdots, s$ be the distinct eigenvalues of A, let $X_{\lambda_l},\ l = 1, \cdots, s$ be the associated algebraic eigenspaces. Moreover, let $X_{\lambda_l} = [a_1^{(l)}, \cdots, a_{n_l}^{(l)}],\ l = 1, \cdots, s$ where $a_1^{(l)}, \cdots, a_{n_l}^{(l)}$ is a basis of X_{λ_l}. Similarly, let $X_{\overline{\lambda}_l}(A^*) = [e_1^{(l)}, \cdots, e_{n_l}^{(l)}],\ l = 1, \cdots, s$ be the algebraic eigenspaces corresponding to the eigenvalues $\overline{\lambda}_l$ of A^* for $l = 1, \cdots, s$. We assume that $a_1^{(l)}, \cdots, a_{n_l}^{(l)}$ are arranged such that first the principal vectors of stage 1 appear, then those of stage 2, and finally those of largest stage k. Then, the biorthogonalization *method delivers biorthogonal systems with the following properties* $(P1)-(P4)$*:*

$(P1)$ $(p_j^{(l)}, v_k^{(l)^*}) = \delta_{jk},\ j,k = 1,\cdots,n_l,\ l = 1,\cdots,s$

$(P2)$ $(p_j^{(l)}, v_k^{(i)^*}) = 0,\ l \neq i,\ j = 1,\cdots,n_l,\ k = 1,\cdots,n_i,\ l,i = 1,\cdots,s$

$(P3)$ $[p_1^{(l)},\cdots,p_{n_l}^{(l)}] = [a_1^{(l)},\cdots,a_{n_l}^{(l)}] = X_{\lambda_l},\ l = 1,\cdots,s$

$[v_1^{(l)^*},\cdots,v_{n_l}^{(l)^*}] = [e_1^{(l)},\cdots,e_{n_l}^{(l)}] = X_{\overline{\lambda}_l}(A^*)$

$(P4)$ $\{p_1^{(l)},\cdots,p_{n_l}^{(l)}\}$ is arranged in a corresponding way to that of $\{\mathrm{a}_1^{(\mathrm{l})},\cdots,\mathrm{a}_{\mathrm{n_l}}^{(\mathrm{l})}\}$, that is, first the principal vectors of stage 1 appear, then those of stage 2, and finally those of stage $\mathrm{k_l}$; as opposed to this, every vector $\{\mathrm{v}_1^{(\mathrm{l})^*},\cdots,\mathrm{v}_{\mathrm{n_l}}^{(\mathrm{l})^*}\}$ may be a principal vector of stage $\mathrm{n_l}$ for $\mathrm{l} = 1,\cdots,\mathrm{s}$.

Proof: The proof of $(P1)$, $(P3)$, $(P4)$ follows directly from the biorthogonalization method, and $(P2)$ is a consequence of Corollary 2. ◇

Connection between the spaces $M_{\lambda_i},\ i = 1,\cdots,r$ and $X_{\lambda_k},\ k = 1,\cdots,s$

Let

$$T^{-1}A\,T = diag(J_i(\lambda_i))_{i=1,\cdots,r}$$

be the canonical Jordan form with

$$J_i(\lambda_i) = tridiag[0,\lambda_i,1] \in \mathbb{C}^{m_i\times m_i},\ i = 1,\cdots,r.$$

Let

$$A\,b_k^{(l)} = \lambda_k\,b_k^{(l)} + b_{k-1}^{(l)},\ k = 1,\cdots,m_l,\ l = 1,\cdots,r$$

with

$$b_0^{(l)} = 0,\ l = 1,\cdots,r.$$

Then,

$$T = [b_1^{(1)},\cdots,b_{m_1}^{(1)};\cdots;b_1^{(r)},\cdots,b_{m_r}^{(r)}].$$

Here, $b_k^{(l)}$ is a principal vector of stage k corresponding to the eigenvalue λ_l.

Likewise, let

$$S^{-1}A^*\,S = diag(J_i(\overline{\lambda}_i))_{i=1,\cdots,r},$$

and let

$$A\,f_k^{(l)} = \overline{\lambda}_k\,f_k^{(l)} + f_{k-1}^{(l)},\ k = 1,\cdots,m_l,\ l = 1,\cdots,r$$

with

$$f_0^{(l)} = 0,\ l = 1,\cdots,r.$$

Then,

$$S = [f_1^{(1)}, \cdots, f_{m_1}^{(1)}; \cdots ; f_1^{(r)}, \cdots, f_{m_r}^{(r)}].$$

Further, let

$$M_{\lambda_i} = [b_1^{(i)}, \cdots, b_{m_i}^{(i)}], \qquad M_{\overline{\lambda}_i}(A^*) = [f_1^{(i)}, \cdots, f_{m_i}^{(i)}],$$

$i = 1, \cdots, r$.

We discuss the connection between the M_{λ_i} and X_{λ_i} by an *example*. For this, let

$$J = \left[\begin{array}{c|c} J_1(\lambda_1) & \\ \hline & J_2(\lambda_2) \end{array}\right].$$

Case 1: $\lambda_1 \neq \lambda_2$.

In this case, $M_{\lambda_1} = X_{\lambda_1}$, $M_{\lambda_2} = X_{\lambda_2}$, that is, here the space M_{λ_i} generated by the principal vectors corresponding to the eigenvalue λ_i is equal to the algebraic eigenspace X_{λ_i}.

Case 2: $\lambda_1 = \lambda_2$.

In this case, $X_{\lambda_1} = X_{\lambda_2} = [M_{\lambda_1} \cup M_{\lambda_2}]$, that is, here $X_{\lambda_1} = X_{\lambda_2}$ is the space generated by the principal vectors corresponding to the Jordan blocks $J_1(\lambda_1)$ and $J_2(\lambda_2)$ ◇

4. Applications

In paper [6], the explicit representations of the solution $x(t)$ of $\dot{x} = A\,x$, $x(t_0) = x_0$, and of the associated fundamental matrix $\Phi(t, t_0)$ are based on the Jordan decomposition $J = diag(J_i(\lambda_i)_{i=1,\cdots,r})$. There, we had to assume that $\lambda_i \neq \lambda_j,\ i \neq j,\ i, j = 1, \cdots, r$. In this paper, the explicit representations of $x(t)$ and $\Phi(t, t_0)$ are derived without this condition. This is achieved by using the algebraic eigenspaces X_{λ_i} resp. $X_{\overline{\lambda}_i}(A^*)$ of A resp. A^* corresponding to the distinct eigenvalues $\lambda_i,\ i = 1, \cdots, s$ resp. $\overline{\lambda}_i,\ i = 1, \cdots, s$ and by the Gram-Schmidt biorthogonalization method.

4.1 Explicit representation of the solution

First, we discuss the case of a diagonalizable matrix A and then the case of a general square matrix.

(i) Diagonalizable matrix A

We have

Theorem 5: (Explicit representation of $x(t)$)

Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable. Let $p_1, \cdots, p_n$ and $v_1^*, \cdots, v_n^*$ be biorthogonal eigenvectors corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A and $\overline{\lambda}_1, \cdots, \overline{\lambda}_n$ of A^*. Then, the solution of the initial value problem

$$\dot{x} = A\,x,\ t \geq t_0,\ x(t_0) = x_0,$$

is given by

$$x(t) = \sum_{k=1}^{n} (x_0, v_k^*)\, p_k\, e^{\lambda_k\,(t-t_0)}\, x_0,\ \ t \geq t_0.$$

Proof: One has

$$x(t) = \sum_{i=1}^{n} c_i x_i(t) = \sum_{i=1}^{n} c_i p_i e^{\lambda_k\,(t-t_0)}.$$

Set $t = t_0$. Then,

$$x_0 = \sum_{i=1}^{n} c_i p_i.$$

Application of $(\cdot, v_k^*)$ on both sides delivers

$$(x_0, v_k^*) = (\sum_{i=1}^{n} c_i p_i, v_k^*) = \sum_{i=1}^{n} c_i (p_i, v_k^*) = c_k$$

since $(p_i, v_k^*) = \delta_{ik}$. From this, the assertion follows. ◇

(ii) General square matrix A

Here, we have

Theorem 6: (Representation of $x(t)$)

Let $A \in \mathbb{C}^{n\times n}$, and let λ_l, $l = 1, \cdots, s$ be the distinct eigenvalues of A. Further, let X_{λ_l} be the algebraic eigenspace of A corresponding to the eigenvalue λ_l as well as $X_{\overline{\lambda}_l}(A^*)$ be the algebraic eigenspace of A^* corresponding to the eigenvalue $\overline{\lambda}_l$ for $l = 1, \cdots, s$. Let

$$X_{\lambda_l} = [p_1^{(l)}, \cdots, p_{n_l}^{(l)}], \qquad X_{\overline{\lambda}_l}(A^*) = [{v_1^{(l)}}^*, \cdots, {v_{n_l}^{(l)}}^*],$$

where $p_1^{(l)}, \cdots, p_{n_l}^{(l)}$ and ${v_1^{(l)}}^*, \cdots, {v_{n_l}^{(l)}}^*$ are biorthogonal vectors computed by the described method with the properties $(P1) - (P4)$.

Then, the solution of the initial value problem

$$\dot{x} = A\,x,\ t \geq t_0,\ x(t_0) = x_0,$$

is given by

$$x(t) = \sum_{l=1}^{s} \sum_{k=1}^{n_l} (x_0, {v_k^{(l)}}^*)\, x_k^{(l)}(t),$$

where

$$x_k^{(l)}(t) = [p_1^{(l)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p_{k-1}^{(l)}\,(t-t_0) + p_k^{(l)}\,]\, e^{\lambda_l\,(t-t_0)},$$

$k = 1, \cdots, n_l,\ l = 1, \cdots, s.$

Proof: The general solution of $\dot{x} = A\,x$ has the form

$$x(t) = \sum_{l=1}^{s} \sum_{k=1}^{n_l} c_k^{(l)}\, x_k^{(l)}(t).$$

Set $t = t_0$. Then,

$$x_0 = \sum_{l=1}^{s} \sum_{k=1}^{n_l} c_k^{(l)}\, p_k^{(l)}.$$

Now, apply $(\cdot, v_i^{(l)^*})$ on both sides. Then,

$$(x_0, v_i^{(l)^*}) = c_i^{(l)},$$

from which the assertion follows. ◇

4.2 Explicit representation of the fundamental matrix

Again, first we discuss the case of a diagonalizable matrix A and then the case of a general square matrix. The results of this subsection are obtained from those of the preceding one by rewriting them in an appropriate way.

(i) Diagonalizable matrix A

We have

Theorem 7: (Representation of $\Phi(t, t_0)$)

Let the conditions of Theorem 5 be fulfilled. Then, the fundamental matrix $\Phi(t, t_0)$ *of* $\dot{x} = A\,x$ *has the form*

$$\Phi(t, t_0) = \Phi(t - t_0, 0) = \sum_{k=1}^{n} e^{\lambda_k\,(t-t_0)}\, p_k v_k,\ t \geq t_0,$$

where $v_k = (v_k^*)^*,\ k = 1, \cdots, n$. *Here,* p_k *is a column vector,* v_k *a row vector, and thus* $p_k v_k \in \mathbb{C}^{n \times n}$ *a matrix.*

Proof: According to Theorem 5, one has

$$x(t) = \sum_{k=1}^{n} (x_0, v_k^*)\, p_k\, e^{\lambda_k\,(t-t_0)};$$

now,

$$(x_0, v_k^*) = v_k \cdot x_0.$$

Combining both relations, gives

$$x(t) = \sum_{k=1}^{n} [p_k v_k \, e^{\lambda_k\,(t-t_0)}]\, x_0,$$

from which the assertion follows. ◇

(ii) General square matrix A

Here, we obtain

Theorem 8: (Representation of $\Phi(t, t_0)$)

Let the conditions of Theorem 6 be fulfilled. Then, the fundamental matrix $\Phi(t, t_0)$ *of* $\dot{x} = A\,x$ *has the form*

$$\Phi(t, t_0) = \Phi(t - t_0, 0)$$

$$= \sum_{l=1}^{s} e^{\lambda_l\,(t-t_0)} \sum_{k=1}^{n_l} [\frac{(t-t_0)^{k-1}}{(k-1)!}\, p_1^{(l)}\, v_1^{(l)} + \cdots + (t-t_0)\, p_{k-1}^{(l)}\, v_{k-1}^{(l)} + p_k^{(l)}\, v_k^{(l)}],$$

where $v_k^{(l)} = (v_k^{(l)*})^*,\ k = 1, \cdots, n_l,\ l = 1, \cdots, s.$

Proof: Theorem 8 is derived from Theorem 6 in a similar manner as Theorem 7 from Theorem 5. ◇

Remark: We want to establish the relation between the new and the known results. Let $P = [p_1^{(1)}, \cdots, p_{n_1}^{(1)}; \cdots, p_1^{(s)}, \cdots; p_{n_s}^{(s)}]$ and $V^* = [v_1^{(1)^*}, \cdots, v_{n_1}^{(1)^*}; \cdots; v_1^{(s)^*}, \cdots, v_{n_s}^{(s)^*}]$. Then $VP = E$ so that $V = P^{-1}$, and $P^{-1}AP = J$ is the Jordan canonical form of matrix A. It is known that $\Phi(t, t_0) = e^{A\,(t-t_0)} = Pe^{J(t-t_0)}P^{-1}$. Writing this in full, leads also to the above representations. But, the preceding form says nothing about the method of computation, and to the best of our knowledge, the way of computation in Theorems 5 - 8 is new. ◇

5. Conclusion

In this paper, we have computed a biorthogonal system $\{p_1^{(1)}, \cdots, p_{n_1}^{(1)}; \cdots, p_1^{(s)}, \cdots; p_{n_s}^{(s)}\}$ and $\{v_1^{(1)^*}, \cdots, v_{n_1}^{(1)^*}; \cdots; v_1^{(s)^*}, \cdots, v_{n_s}^{(s)^*}\}$ of prinipal vectors, where the Gram-Schmidt biorthogonalization method is applied to the distinct algebraic eigenspaces X_{λ_k} of matrix A and $X_{\overline{\lambda_k}}(A^*)$ of matrix A^* for $k = 1, \cdots, s$. We mention that the application of the Gram-Schmidt biorthogonalization method to the whole system $\{p_1^{(1)}, \cdots, p_{n_1}^{(1)}; \cdots, p_1^{(s)}, \cdots; p_{n_s}^{(s)}\}$ would not produce the same result and would take more time. Further, we have applied the new procedure to effectively compute an explicit representation of the solution $x(t)$ of the initial value problem $\dot{x} = A\,x$, $x(t_0) = x_0$, and an explicit representation of the associated fundamental matrix $\Phi(t, t_0) = \Phi(t - t_0, 0)$. Since the derivation of these results is based on the algebraic eigenspaces of A and A^*, we *need not* the hypothesis of distinct eigenvalues in the Jordan blocks of A; thus, earlier results described in [6] are generalized. Often, as an objection to the use of the Jordan canonical form, it is said that its calculation might not be reliable. This may take place in some cases. But, in dynamical systems from engineering practice, this scarcely will happen. Namely, usually large vibration models are reduced to small-sized ones with system matrix A being even diagonalizable, where no problems at all must be expected. In any case, one could use a numerical integration method to check whether the obtained results are good. For the actual computation of the Jordan canonical form, MATLAB routine *jordan* may be employed. We add that the advantage of the results of Theorems 5 - 8 over those computed by numerical integration is that they enable the user to assess the impact of distinct parts of the solution. The obtained results are very important in *Computational Mathematics* as well as in *Computational Engineering* such as *Computational Mechanics*, where A is the system matrix of a dynamical system. The results of this paper will also serve as the basis of further achievements by the author in the area of dynamical systems. In this context, also numerical examples for vibration models will be presented.

References

[1] F.R. Gantmacher, Matrizentheorie, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1986.

[2] E. Hairer, S.P. Nørset, G. Wanner, Solving Ordinary Differential Equations I, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo Hong Konk Barcelona Budapest,1993.

[3] P.R. Halmos, Finite-Dimensional Vector Spaces, Springer-Verlag, New York Heidelberg Berlin, 1974.

[4] M.R. Hestenes, Inversion of matrices by biorthogonalization and related results, J. Soc. Indust. Appl. Math. 6(1)(1958)51-90.

[5] T. Kato, Perturbation Theory for Linear Operators, Springer, New York, 1966.

[6] L. Kohaupt, Construction of a biorthogonal system of principal vectors for the matrices A and A^* with applications to $\dot{x} = A\,x,\ x(t_0) = x_0$, Journal of Computational Mathematics and Optimization 3(3)(2007)163-192.

[7] P.C. Müller, W.O. Schiehlen, Lineare Schwingungen, Akademische Verlagsgesellschaft, Wiesbaden, 1976.

[8] H. Niemeyer, E. Wermuth, Lineare Algebra, Vieweg, Braunschweig Wiesbaden, 1987.

[9] B.N. Parlett, D.R. Taylor, Z.A. Liu, A look-ahead Lanczos algorithm for unsymmetric matrices, Mathematics of Computation 44(169)(1985)105-124.

[10] A.E. Taylor, Introduction to Functional Analysis, John Wiley & Sons, New York London, 1958.

[11] H. Waller, W. Krings, Matrizenmethoden in der Maschinen- und Bauwerksdynamik, Bibliographisches Institut, Mannheim Wien Zürich, 1975.

[12] R. Zurmühl, S. Falk, Matrizen und ihre Anwendungem, Teil 1, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.

Received: November 14, 2007