The Nearest Points in Normed Linear Spaces

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Abstract

The purpose of this paper is to introduce and discuss the concept of best approximation and best coapproximation in normed linear spaces. We will generalize some previous results about closed convex subsets to closed subsets.

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1. Introduction

We know that a point \( g_0 \in M \) is said to be a best approximation (resp. best coapproximation) for \( x \in X \) if and only if \( \| x - g_0 \| = \| x + M \| = \text{dist}(x, M) \) (resp. \( \| g_0 - g \| \leq \| x - g \| \ \forall \ g \in M \)). It can be easily proved that \( g_0 \) is a best approximation (resp. best coapproximation) for \( x \in X \) if and only if \( x - g_0 \in M \) (resp. \( x - g_0 \in \tilde{M} \)). The set of all best approximations (resp. best coapproximations) of \( x \in X \) in \( M \) is shown by \( P_M(x) \) (resp. \( R_M(x) \)). In other words,

\[
P_M(x) = \{ g_0 \in M : x - g_0 \in \hat{M} \}
\]

and

\[
R_M(x) = \{ g_0 \in M : x - g_0 \in \tilde{M} \}.
\]

If \( P_M(x) \) (resp. \( R_M(x) \)) is non-empty for every \( x \in X \), then \( M \) is called an Proximinal (resp. coproximinal) set. The set \( M \) is Chebyshev (resp. cochebyshev) if \( P_M(x) \) (resp. \( R_M(x) \)) is a singleton set for every \( x \in X \). (see [2-6])
Lemma 1.1 ([7]). Let $X$ be a normed linear space and $M$ be a subspace of $X$. Then

1) $P_M(x)$ is a bounded set.
2) If $M$ is a closed subset of $X$, then $P_M(x)$ is also a closed subset of $X$ for every $x \in X$.
3) If $M$ is a convex subset of $X$, then $P_M(x)$ is also a convex subset of $X$ for all $x \in X$.
4) Every proximinal subset of $X$ is closed.

As usual we denote by $\partial(M)$ and $\text{Int}(M)$ the boundary and interior of $M$, respectively. So $\text{Int}(M) \cup \partial(M)$ will be the closure of $M$. Also the smallest closed convex set containing $M$ is denoted by $\overline{\text{co}}(M)$. Moreover $B(x_0, r)$ denotes the open disk centered at $x_0$ with radius $r > 0$.

The space $X$ is said to a strictly convex, if the relations

$$\|x + y\| = \|x\| + \|y\| \quad 0 \neq x, 0 \neq y \in X$$

imply the existence of a $c \neq 0$ such that $y = cx$.

Lemma 1.2 ([7]). Let $X$ be a normed linear space. Then

1) A non-empty closed convex subset of $X$ is a Chebyshev set if and only if the space $X$ is strictly convex and reflexive.
2) If $X$ is a strictly convex and finite dimensional space, then each Chebyshev subset of $X$ is a convex set (Also see [1-6]).

2. Main Results

Now we can state and prove some theorems about the best approximations and proximinal sets.

Theorem 2.1. Let $X$ be a normed linear space. If $X$ is a finite dimensional space, then each non-empty closed convex subset of $X$ is a proximinal set.

Proof. Suppose that $M$ is a non-empty closed subset of $X$. For any $x_0 \in X \setminus M$, put $r_0 = d(x_0, M) \neq 0$. If $r > r_0$, then there exists a $y \in M$ such that $\|x_0 - y\| < r$. Therefore $y \in B(x_0, r) \cap M$. It follows that $B(x_0, r) \cap M \neq \emptyset$. If $B(x_0, r) = \{y \in X : \|y - x_0\| \leq r\}$ and $B_n = B(x_0, r_0 + 1/n) \cap M$, then it is clear that $B_n$ is a non-empty compact subset of $X$ and $B_{n+1} \subset B_n$ for all $n \geq 1$. Therefore there exists $y_0 \in X$ such that $y_0 \in \bigcap_{n=1}^{\infty} B_n$. Now we have $\|y_0 - x_0\| \leq r_0 + 1/n$ for all $n \geq 1$. Since $r_0 = d(x_0, M)$ we have $\|y_0 - x_0\| = r_0 = d(x_0, M)$. Thus $y_0$ is a best approximation for $x_0$ and therefore $M$ is a proximinal set. ■
Theorem 2.2. Let $X$ be a normed linear space and $M$ be a subspace of $X$. If $M$ is closed and $y_0 \in M$ is a best approximation for $x_0 \in X \setminus M$, then $y_0 \in \partial(M)$.

Proof. If $y_0 \in \text{Int}(M)$, then there exists $r > 0$ such that $B(y_0, r) \subset M$. If $d(x_0, M) = \|x_0 - y_0\| = s$, we set

$$y_1 = \frac{s}{s + r_1}y_0 + \frac{r_1}{s + r_1}x_0.$$ 

Then we have

$$\|y_1 - y_0\| = \frac{rs}{s + r} < r, \quad \|y_1 - x_0\| = \frac{s^2}{s + r} < s = \|x_0 - y_0\|.$$ 

Since $y_1 \in B(y_0, r) \subset M$, it contradicts with the best approximativity of $y_0$. Thus $y_0 \in \partial(M)$. \qed

Theorem 2.3. Let $X$ be a normed linear space and $M$ be a subset of $X$. If $g_0 \in R_M(x)$ and $(1 - \lambda)x + \lambda g_0 \in M$ for some $\lambda$. Then $(1 - \lambda)x + \lambda g_0 \in R_M(x)$.

Proof. Since $g_0 \in R_M(x)$ we have $\|g - g_0\| \leq \|x - g\|$ for all $g \in M$. Therefore

$$\|g - (1 - \lambda)x - \lambda g_0\| \leq \|(1 - \lambda)g + \lambda g - (1 - \lambda)x - \lambda x\|$$

$$\leq (1 - \lambda)\|g - x\| + \lambda\|g - g_0\|$$

$$\leq \|x - g\|.$$ 

For all $g \in M$. \qed

Example 2.4. Suppose $X = R^2$ with the norm $\|(x, y)\| = |x| + |y|$ and $M = \{(x, y) : x \geq 0, y \geq 0\}$ is a subset of $X$. Then $g_0 = (0, 1) \in R_M(-1, 1)$ and for $\lambda = 2$ we have $(1, 1) \in R_M(-1, 1)$. Therefore $(1, 1) \notin \partial(M)$.

Theorem 2.5. Let $X$ be a normed linear space and $M$ be a subspace of $X$. If $g_0 \in R_M(x)$, then $g_0 \in \partial(M)$.

Proof. If $g_0 \notin \partial(M)$, then for all $\lambda$, $(1 - \lambda)x + \lambda g_0 \in M$. Since $M$ is a subspace, Therefore $x \in M$ and is a contradiction. \qed

Corollary 2.6. Let $X$ be a normed linear space and $M$ be a subspace of $X$. Let $M$ be non-empty and closed. If $x_0 \in X \setminus M$ has a best approximation in $M$, then $d(x_0, M) = d(x_0, \partial(M))$.

Proof. Since $\partial(M) \subset M$, we have

$$d(x_0, \partial(M)) = \inf\{\|x_0 - z\| : z \in \partial(M)\}$$

$$\geq \inf\{\|x_0 - z\| : z \in M\} = d(x_0, M).$$
Also, suppose that $y_0 \in M$ is a best approximation for $x_0$. Then $d(x_0, M) = \|x_0 - y_0\|$ and thus from Theorem 2.2, $y_0 \in \partial(M)$, and

$$d(x_0, \partial(M)) = \inf \{\|x_0 - z\| : z \in \partial(M)\} \leq \|x_0 - y_0\| = d(x_0, M).$$

It follows that $d(x_0, \partial(M)) = d(x_0, M)$. ■

**Theorem 2.7.** Let $X$ be a normed linear space. If for every non-empty closed set $M$ of $X$ such that $\partial(\overline{co}(M)) = \partial(M)$, the set $P_M(x)$ is a singleton for all $x \in X \setminus \partial(\overline{co}(M))$, then the space $X$ is strictly convex and reflexive.

**Proof.** Suppose $M$ is any non-empty closed convex subset of $X$. Since $\overline{co}(M) = M$, we have $\partial(\overline{co}(M)) = \partial(M)$. By hypothesis $P_M(x)$ is a singleton for all $x \in X \setminus \partial(\overline{co}(M)) = X \setminus M$. Thus $M$ is a Chebyshev set. Now from Lemma 1.4 it follows that the space $X$ is strictly convex and reflexive. ■

**References**


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