On the Existence of Solutions of
Boundary Value Problems for a Class of
High Order Operator-Differential Equations

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Abstract

In the paper we give the sufficient conditions on the existence and
uniqueness of generalized solutions a boundary value problems of one
class of high order operator operator-differential equations at which the
equation describes the process of corrosive fracture of metals in ag-
gressive media and the principal part of the equation has a multiple
characteristics.

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differential equation

1. Introduction

Many problems of mechanics and mathematical physics are connected
with the investigation of solvability of operator-differential equations. As an
example, we can show the following papers.

As is known, stress-strain state of a plate may be separated into internal
and external layers [1-4]. Construction of a boundary layer is related with
sequential solution of plane problems of elasticity theory in a semi-strip. In
Papkovich’s paper [5] and in others a boundary value problem of elasticity
theory in a semi-strip $x > 0, |y| \leq 1$ is reduced to the definition of Airy
biharmonic functions that is found in the form

$$u = \sum_{\text{Im } \sigma_k > 0} C_k \varphi_k(y) e^{i\sigma_k x},$$
where \( \varphi_k \) are Papkovich functions \([5-6]\), \( \sigma_k \) are corresponding values of a self-adjoint boundary value problem, \( C_k \) are unknown coefficients. In this connection in \([6]\) there is a problem on representation of a pair of functions \( f_1 \) and \( f_2 \) in the form

\[
\sum_{k=1}^{\infty} C_k P_k \varphi_k = f_1, \quad \sum_{k=1}^{\infty} C_k Q_k \varphi_k = f_2, \quad (1)
\]

where \( P_k, Q_k \) are differential operators defined by boundary conditions for \( x = 0 \). In the paper \([7,8]\) some sufficient conditions of convergence of expansion \((1)\) is given for the cases when the coefficients \( C_k \) are obviously defined with the help of generalized orthogonality.

In \([9]\) the coefficients \( C_k \) are uniquely defined by the boundary values of a biharmonic function and its derivatives. The trace problem for a two-dimensional domain with piecewise smooth boundary was studied in the paper \([10]\). The paper \([11]\) deals with differential properties of solutions of general elliptic equations in the domains with canonical and corner points. Some new results for a biharmonic equation are in \([12]\). Investigation of behaviour of solution of problems of elasticity theory in the vicinity of singular points of the boundary is in the papers \([13-14]\). M.B. Orazov \([15]\) and S.S. Mirzoyev \([16]\) studied the problem when a principal part of the equation is of the form:

\[
(-1)^m \frac{d^m}{dx^m} + A^{2m}
\]

where \( A \) is a self-adjoint operator pencil and it has a multiple characteristics, that differs it from above-mentioned papers.

\section{Problem statement}

Let \( H \) be a separable Hilbert space, \( A \) be a positive definite self-adjoint operator in \( H \) with domain of definition \( D(A) \). By \( H_\gamma \) we denote a scale of Hilbert spaces generated by the operator \( A \), i.e. \( H_\gamma = D(A^{\gamma}) \), \( (\gamma \geq 0) \), \( (x, y)_\gamma = (A^\gamma x, A^\gamma y), x, y \in D(A^\gamma) \). By \( L_2 ((a, b); H) \) \((\infty \leq a < b \leq \infty)\) we denote a Hilbert space of vector-functions \( f(t) \), determined in \((a, b)\) almost everywhere with values in \( H \), measurable, square integrable in Bochner sense

\[
\|f\|_{L_2((a, b); H)} = \left( \int_a^b \|f\|_2^2 \, dt \right)^{1/2}.
\]

Then we determine a Hilbert space for natural \( m \geq 1 \) \([17]\)

\[
W_2^m ((a, b); H) = \{ u/u^{(m)} \in L_2 ((a, b); H), A^m u \in L_2 ((a, b); H_m) \}
\]
with norm
\[ \|u\|_{W^m_2((a,b);H)} = \left( \|u^{(m)}\|_{L^2((a,b);H)}^2 + \|A^m u\|_{L^2((a,b);H)}^2 \right)^{1/2}. \]

Here and in sequel, derivatives are understood in the distributions theory sense [17]. Assume
\[ L^2((0, \infty); H) \equiv L^2(\mathbb{R}^+; H), \quad L^2((-\infty, \infty); H) \equiv L^2(\mathbb{R}; H), \]
\[ W^m_2((0, \infty); H) \equiv W^m_2(\mathbb{R}^+; H), \quad W^m_2((-\infty, +\infty); H) \equiv W^m_2(\mathbb{R}; H). \]

Then we determine the spaces
\[ W^m_2(\mathbb{R}^+; H; \{\nu\}_{\nu=0}^{m-1}) = \{ u | u \in W^m_2(\mathbb{R}^+; H), u^{(\nu)}(0) = 0, \nu = 0, m-1 \}. \]

Obviously, by the trace theorem [17] the space \( W^m_2(\mathbb{R}^+; H; \{\nu\}_{\nu=0}^{m-1}) \) is a closed subspace of the Hilbert space \( W^m_2(\mathbb{R}^+; H) \).

Let’s define a space of \( D([a, b]; H_\gamma) \)-times infinitely differentiable functions for \( a \leq t \leq b \) with values in \( H_\gamma \) having a compact support in \([a, b]\). As is known a linear set \( D([a, b]; H_\gamma) \) is everywhere dense in the space \( W^m_2((a, b); H), [17] \).

It follows from the trace theorem that the space
\[ D(\mathbb{R}^+; H_m; \{\nu\}_{\nu=0}^{m-1}) = \{ u | u \in D(\mathbb{R}^+; H_m), u^{(\nu)}(0) = 0, \nu = 0, m-1 \} \]
and also everywhere is dense in the space.

In the Hilbert space \( H \) we consider the boundary value problem

\[ \left( -\frac{d^2}{dt^2} + A^2 \right)^m u(t) + \sum_{j=1}^m A_j u^{(m-j)}(t) = 0, \quad t \in R_+ = (0, +\infty), \]

\[ u^{(\nu)}(0) = \varphi_\nu, \quad \nu = 0, m-1, \varphi_\nu \in H_{m-\nu-1/2}. \]

Here we assume that the following conditions are fulfilled:
1) \( A \) is a positive-definite self-adjoint operator with completely continuous inverse \( C = A^{-1} \in \sigma_\infty; \)
2) The operators
\[ B_j = A^{-j/2} A_j A^{-j/2} \quad (j = 2k, k = 1, m) \]
and
\[ B_j = A^{-(j-1)/2} A_j A^{-(j-1)/2} \quad (j = 2k - 1, k = 1, m - 1); \]
3) The operators \((B + E_m)\) are bounded in \( H \).

Equation (2) describes a process of corrosion fracture in aggressive media that were studied in the paper [18].
3. Some definition and auxiliary facts

Denote

\[ P_0 \left( \frac{d}{dt} \right) u(t) \equiv \left( -\frac{d^2}{dt^2} + A^2 \right)^m u(t), \quad u(t) \in D(R_+; H_m), \quad (4) \]

\[ P_1 \left( \frac{d}{dt} \right) u(t) \equiv \sum_{j=1}^{m-1} A_j u^{(m-j)}(t), \quad u(t) \in D(R_+; H_m), \quad (5) \]

**Lemma 1.** Let \( A \) be a positive-definite self-adjoint operator, the operators \( B_j = A^{-j/2} A_j A^{-j/2} \) \((j = 2k, k = \frac{1}{2}, \ldots, m)\) and \( B_j = A^{-(j+1)/2} A_j A^{-(j-1)/2} \) \((j = 2k-1, k = 1, \ldots, m-1)\) be bounded in \( H \). Then a bilinear functional

\[ P_1(u, \psi) \equiv (P_1(d/dt)u, \psi)_{L_2(R_+; H)} \]

determined for all vector-functions \( u \in D(R_+; H_m) \) and \( \psi \in D(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}) \) continues by continuity on the space \( W_2^m (R_+; H) \oplus W_2^m (R_+; H_m; \{\nu\}_{\nu=0}^{m-1}) \) up to bilinear functional \( P_1(u, \psi) \) acting in the following way

\[ P_1(u, \psi) = \sum_{j=1}^{m-1} (-1)^{m-j/2} (A_j u^{(m-j)/2}, \psi^{(m-j)/2})_{L_2} + \]

\[ + \sum_{j=2k-1}^{m-1} (-1)^{m-(j+1)/2} (A_j u^{(m-(j-1)/2)}, \psi^{(m-(j-1)/2)})_{L_2}. \]

Here in the first term, the summation is taken over even \( j \), in the second term over odd \( j \).

**Proof.** Let \( u \in D(R_+; H_m), \psi \in D(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}) \). After integration by parts we have

\[ P_1(u, \psi)_{L_2} \equiv (P_1(d/dt)u, \psi)_{L_2} = \sum_{j=0}^{m} (A_j u^{(m-j)}, \psi)_{L_2} = \]

\[ = \sum_{j=2k}^{m-j/2} (-1)^{m-j/2} (A_j u^{(m-j/2)}, \psi^{(m-j/2)})_{L_2} + \]

\[ + \sum_{j=(2k-1)}^{m-(j+1)/2} (-1)^{m-(j+1)/2} (A_j u^{(m-(j-1)/2)}, \psi^{(m-(j-1)/2)})_{L_2}. \]

Since

\[ P_1(u, \varphi) = \sum_{j=2k} (-1)^{m-j/2} (B_j A^{j/2} u^{(m-j/2)}, A^{j/2} \psi^{(m-j/2)})_{L_2} + \]
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\[ \sum_{j=(2k-1)} (-1)^{m-(j+1)/2} \left( B_j A_j u^{(m-(j-1)/2)}, A^{(j-1)/2} \psi^{(m-(j-1)/2)} \right)_{L^2}, \]

from belongless of \( u \in D \left( R_+; H_m \right) \) and \( \psi \in D \left( R_+; H_m; \{ \nu \}_{\nu=0}^{m-1} \right) \) by intermediate derivatives theorem [17] it follows that

\[ |P_1(u, \varphi)| \leq \sum_{(j=2k)} \| B_j \| \left\| A^{j/2} u^{(m-j/2)} \right\|_{L^2} \left\| A^{j/2} \psi^{(m-j/2)} \right\|_{L^2} + \]

\[ + \sum_{j=(2k-1)} \left( B_j \| A^{(j+1)/2} u^{(m-(j-1)/2)} \|_{L^2} \left\| A^{(j-1)/2} \psi^{(m-(j-1)/2)} \right\|_{L^2} \leq \right. \]

\[ \leq \text{const} \left\| u \right\|_{W^m_2(\mathbb{R}_+; H)} \left\| \psi \right\|_{W^m_2(\mathbb{R}_+; H)}, \]

i.e. \( P_1(u, \varphi) \) is continuous in the space \( D(R_+; H_m) \oplus D \left( R_+; H_m; \{ \nu \}_{\nu=0}^{m-1} \right) \) therefore it continues by continuity on the space \( W^m_2(R_+; H) \oplus W^m_2 \left( R_+; \{ \nu \}_{\nu=0}^{m-1} \right) \).

The lemma is proved.

**Definition.** The vector-function \( u(t) \in W^m_2(R_+; H) \) is said to be a generalized solution of (2), (3), if

\[ \lim_{t \to 0} \left\| u^{(\nu)}(t) - \varphi_\nu \right\|_{H_{m-\nu-1/2}} = 0, \quad \nu = 0, m-1 \]

and for any \( \psi(t) \in W^m_2 \left( R_+; H; \{ \nu \}_{\nu=0}^{m-1} \right) \) it is fulfilled the identity

\[ \langle u, \psi \rangle = \langle u, \psi \rangle_{W^m_2(R_+; H)} + \sum_{p=1}^{m-1} C^p_m \left( A^p u^{(m-p)}, A^p \psi^{(m-p)} \right)_{L^2(R_+; H)} + P_1(u, \psi) = 0, \]

where

\[ C^p_m = \frac{m(m - 1) \ldots (m - p + 1)}{p!} = \binom{m}{p}. \]

First of all we consider the problem

\[ P_0 \left( \frac{d}{dt} \right) u(t) = \left( -\frac{d^2}{dt^2} + A^2 \right)^m u(t) = 0, \quad t \in R_+ = (0, +\infty), \quad (7) \]

\[ u^{(\nu)}(0) = \varphi_\nu, \quad \nu = 0, m-1. \quad (8) \]

It holds
Theorem 1. For any collection $\varphi_\nu \in H_{m-\nu-1/2}$ ($\nu = 0, m - 1$) problem (7), (8) has a unique generalized solution.

Proof. Let $c_0, c_1, \ldots, c_{m-1} \in H_{m-\nu-1/2}$ ($\nu = 0, m - 1$), $e^{-At}$ be a holomorphic semi-group of bounded operators generated by the operator $(-A)$. Then the vector-function

$$u_0(t) = e^{-tA} \left( c_0 + \frac{t}{1!} A c_1 + \ldots + \frac{t^{m-1}}{(m-1)!} A^{m-1} c_{m-1} \right)$$

belongs to the space $W_2^m(R_+; H)$. Really, using spectral expansion of the operator $A$ we see that each term

$$\frac{t^{m-\nu}}{(m-\nu)!} A^{m-\nu} e^{-tA} \in W_2^m(R_+; H) \quad \text{for} \quad c_\nu \in H_{m-1/2} \quad (\nu = 0, m - 1).$$

Then it is easily verified that $u_0(t)$ is a generalized solution of equation (7), i.e. it satisfies the relation

$$(u_0, \varphi)_{W_2^m} + \sum_{p=1}^{m-1} C_p^m \left( A^{m-p} u_0^{(p)}, A^{m-p} \varphi^{(p)} \right) = 0$$

for any $\varphi \in W_2^m \left( R_+; H; \{ \nu \}_{\nu=0}^{m-1} \right)$.

Show that $u^{(0)}(0) = \varphi_\nu$, $\nu = 0, m - 1$. For this purpose we must determine the vectors $c_\nu$ ($\nu = 0, m - 1$) from condition (8). Obviously, in order to determine the vectors $c_\nu$ ($\nu = 0, m - 1$) from condition (8) we get a system of equations with respect to the vectors

$$\begin{pmatrix}
E & 0 & \cdots & 0 \\
-E & E & \cdots & 0 \\
E & -E & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{m-1} \begin{pmatrix} 1 \\ m-1 \end{pmatrix} E & (-1)^{m-2} \begin{pmatrix} 2 \\ m-2 \end{pmatrix} E & \cdots & E
\end{pmatrix} \times \begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_{m-1}
\end{pmatrix} = \begin{pmatrix}
\varphi_0 \\
A^{-1} \varphi_1 \\
A^{-2} \varphi_2 \\
\vdots \\
A^{-(m-1)} \varphi_{m-1}
\end{pmatrix},$$

where $E$ is a unique operator in $H$ and $\begin{pmatrix} p \\ m-s \end{pmatrix}$ = $C_p^{m-s}$. Since the principal operator determinant is invertible, we can uniquely determine $c_\nu$ ($\nu = 0, m - 1$).
Obviously, for any $\nu$ the vector $A^{-(m-\nu)}\varphi_\nu \in H_{m-1/2}$, since $\varphi_\nu \in H_{m-\nu-1/2}$. As the vector at the right hand side of the equation (9) belongs to the space
\[
\bigoplus_{m \text{ times}} H_{m-1/2} = \left(H_{m-1/2}\right)^m,
\]
then taking into account the fact that the principal operator matrix $\tilde{E}$ as a product of the invertible scalar matrix by matrix where $\tilde{E}$ is a unique matrix in $(H_{m-1/2})^m$, then it is unique. Therefore, each vector $c_\nu$ ($\nu = 0, m-1$) is determined uniquely and belongs to the space $H_{m-1/2}$. The theorem is proved.

In the space $W_2^m \left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$ we define a new norm
\[
\|u\|_{W_2^m (R_+; H)} = \left(\|u\|_{W_2^m (R_+; H)}^2 + \sum_{p=1}^{m-1} C_p^m \|A^{m-p}u(p)\|_{L_2(R_+; H)}^2\right)^{1/2}.
\]

By the intermediate derivatives theorem [17] the norms $\|u\|_{W_2^m (R_+; H)}$ and $\|u\|_{W_2^m (R_+; H)}$ are equivalent in the space $W_2^m \left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$. Therefore, the numbers
\[
N_j \left(R_+; \{\nu\}_{\nu=0}^{m-1}\right) = \sup_{0 \neq u \in W_2^m (R_+; H; \{\nu\}_{\nu=0}^{m-1})} \|A^{m-j}u(j)\|_{L_2(R_+; H)} \|u\|_{W_2^m (R_+; H)}^{-1}, \ j = 0, m.
\]
are finite.

The next lemma enables to find exact values of these numbers.

**Lemma 2.** The numbers $N_j \left(R_+; \{\nu\}_{\nu=0}^{m-1}\right)$ are determined as follows:
\[
N_j \left(R_+; \{\nu\}_{\nu=0}^{m-1}\right) = d_{m,j}^{m/2},
\]
where
\[
d_{m,j} = \begin{cases} 
\left(\frac{j}{m}\right)^m \left(\frac{m-j}{m}\right)^{m-j}, & j = 1, m-1 \\
1, & j = 0, m
\end{cases}
\]

Using the method of the papers [16, 19] the lemma is easily proved.

**The basic results**

Now, let’s prove the principal theorems.
Theorem 2. Let \( A \) be a positive-definite self-adjoint operator, the operators \( B_j = A^{-j/2}A_jA^{-j/2} \) \((j = 2k, k = 0, m)\) and \( B_j = A^{-(j-1)/2}A_jA^{-(j-1)/2}\) \((j = 2k - 1, k = 1, m - 1)\) be bounded in \( H \) and it hold the inequality

\[
\alpha = \sum_{j=1}^{m} C_j \| B_{m-j} \| < 1,
\]

where

\[
C_j = \begin{cases} 
  d_{m,j/2}^{m/2}, & j = 2k, \; k = 0, m \\
  (d_{m,(j-1)/2}d_{m,(j+1)/2})^{m/2}, & j = 2k - 1, \; k = 1, m - 1
\end{cases}
\]

and

\[
d_{m,j} = \begin{cases} 
  \left( \frac{j}{m} \right)^{m/2} \left( \frac{m-j}{m} \right)^{m/2}, & j = 1, m - 1 \\
  1, & j = 0, m
\end{cases}
\]

Then for any \( \varphi_\nu \in D \left( A^{m-n-1/2} \right), \; (\nu = 0, m - 1) \) problem (2), (3) has a unique generalized solution and it holds the inequality

\[
\| u \|_{W^m_2(R_+;H)} \leq \text{const} \sum_{\nu=0}^{m-1} \| \varphi \|_{m-\nu-1/2}.
\]

Proof. Let \( \psi \in D \left( R_+; H; \{ \varphi \}_{\nu=0}^{m-1} \right) \). Then for any \( \psi \)

\[
\text{Re } P(\psi, \psi) = \text{Re } P_0(\psi, \psi) + \text{Re } P_1(\psi, \psi) = \\
= \left( \left( -\frac{d}{dt} + A \right)^m \psi, \left( -\frac{d}{dt} + A \right)^m \psi \right) + \\
+ \text{Re } P_1(\psi, \psi) \geq \left\| \left( -\frac{d}{dt} + A \right)^m \psi \right\|^2 - | \text{Re } P_1(\psi, \psi) | \geq \\
\geq \left\| \left( -\frac{d}{dt} + A \right)^m \psi \right\|_{L_2(R_+;H)}^2 - |P_1(\psi, \psi)|_{L_2(R_+;H)},
\]

Since by lemma 2

\[
\| A^k \psi^{(m-k)} \|_{L_2(R_+;H)} \leq d_{m,m-k}^{m/2} \| u \|_{W^m_2(R_+;H)},
\]

then

\[
|P_1(\psi, \psi)| \leq \]
\[
\leq \left( \sum_{(j=2k)} \|B_{m-j}\| \frac{m/2}{m-k} + \sum_{(j=2k-1)} \|B_{m-j}\| \frac{m/2}{m-k+1} \right) \|\psi\|_{W_m^2}^2.
\]

Here \(d_{0,0} = d_{m,m} = 1\) and
\[
d_{m,k} = \left( \frac{k}{m} \right)^\frac{k}{m} \left( \frac{m-k}{m} \right)^\frac{m-k}{m}, \quad (k = 1, m-1),
\]
thus
\[
|P_1(\psi, \psi)| \leq \sum_{j=1}^m C_j \|B_{m-j}\|,
\]
where
\[
C_j = \left\{ \begin{array}{ll}
\frac{m/2}{m-j/2}: & j = 2k, k = \overline{0,m} \\
\left( \frac{m/2}{d_{m,j+1}/2} \frac{m/2}{d_{m,j-1}/2} \right)^{m/2}: & j = 2k-1, k = \overline{1,m-1}.
\end{array} \right.
\]

Consequently
\[
|P_1(\psi, \psi)| \leq \alpha \|\psi\|_{W_m^2(R_+:H)}^2.
\]

Then
\[
\text{Re } P(\psi, \psi)_{L_2(R_+:H)} \geq (1 - \alpha)P_0(\psi, \psi)_{L_2(R_+:H)}.
\]

(10)

Now we look for a generalized solution of problem (2), (3) in the form
\[
u(t) = \nu_0(t) + \theta(t),
\]
where \(\nu_0(t)\) is a generalized solution of problem (7), (8) and \(\theta(t) \in W_m^m (R_+:H; \{\nu\}_{\nu=0}^{m-1})\). To define \(\theta(t)\) we get relation
\[
\langle \theta; \psi \rangle = \langle \theta, \psi \rangle_{W_m^m(R_+:H)} + \sum_{p=1}^{m-1} C_p^p (A^{m-p} \theta, A^{m-p} \psi) + P_1 (\theta, \psi) = P_1(u_0, \psi).
\]

(11)

Since the right hand side of the equality is a continuous functional in \(W_m^m (R_+:H; \{\nu\}_{\nu=0}^{m-1})\), and the left hand side \(\langle \theta; \psi \rangle\) is a bilinear functional in the space \(W_m^m (R_+:H; \{\nu\}_{\nu=0}^{m-1}) \oplus W_m^m (R_+:H; \{\nu\}_{\nu=0}^{m-1})\), then by inequality (10) it satisfies conditions of Lax-Milgram theorem [20]. Consequently, there exists a unique vector-function \(\theta(t) \in W_m^m (R_+:H; \{\nu\}_{\nu=0}^{m-1})\) that satisfies
equality (11) and \( u(t) = u_0(t) + \theta(t) \) is a generalized solution of problem (2), (3).

Further, by \( J(R_+; H) \) we denote a set of generalized solutions of problem (2), (3) and define the operator \( \Gamma : J(R_+; H) \to \tilde{H} = \bigoplus_{k=0}^{m-1} H_{m-k-1/2} \) acting in the following way \( \Gamma u = (u^{(k)}(0))_{k=0}^{m-1} \). Obviously \( J(R_+; H) \) is a closed set and by the trace theorem \( \| \Gamma u \|_{\tilde{H}} \leq C \| u \|_{W_2^m(R_+; H)} \). Then by the Banach theorem on the inverse operator there exists the inverse operator \( \Gamma^{-1} : \tilde{H} \to J(R_+; H) \). Consequently

\[
\| u \|_{W_2^m(R_+; H)} \leq \text{const} \sum_{k=0}^{m-1} \| \varphi \|_{m-k-1/2}.
\]

The theorem is proved.

**Remark.** From the proof we can show that for \( m = 2 \), \( c_1 = c_3 = 1/2 \), \( c_2 = 1/4 \), \( c_4 = 1 \).

**References**


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