Approximation Methods for Equilibrium Problems and Common Solution for a Finite Family of Inverse Strongly-Monotone Problems in Hilbert Spaces

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Abstract

The purpose of the paper is to investigate approximation methods for finding an element that is not only a solution of an equilibrium problem but also a common solution for a finite family of inverse strongly-monotone problems in Hilbert spaces.

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1. Introduction

Let $H$ be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle ., . \rangle$ and $\| . \|$, respectively, let $C$ be a nonempty closed (in the norm) and convex subset of $H$, and let $F_0$ be a bifunction from $C \times C$ to $\mathbb{R}$. The equilibrium problem for $F_0$ is to find $u^* \in C$ such that

$$F_0(u^*, v) \geq 0 \quad \forall v \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F_0)$. Assume that the bifunction $F_0$ satisfies the following set of standard properties.

Condition 1.1 The bifunction $F$ is such that:
(A1) $F(u, u) = 0$  \(\forall u \in C\).

(A2) $F(u, v) + F(v, u) \leq 0$  \(\forall (u, v) \in C \times C\).

(A3) For every $u \in C$, $F(u, .) : C \to \mathbb{R}$ is lower semicontinuous and convex.

(A4) $\lim_{t \to +0} F((1 - t)u + tz, v) \leq F(u, v)$  \(\forall (u, z, v) \in C \times C \times C\).

Let $T_i, i = 1, \ldots, N$ be a finite family of $k_i$-strictly pseudo-contractions from $C$ into $C$ with the nonempty set of fixed points $F(T_i)$ (i.e., $F(T_i) = \{x \in C : x = T_i x\}$). Assume that

$$\tilde{S} := \bigcap_{i=1}^{N} F(T_i) \cap EP(F_0) \neq \emptyset.$$ 

The problem of finding an element

$$u^* \in \tilde{S} \quad (1.2)$$

is studied intensively in [1]-[6], [9]-[11], and [13]-[25].

Recall that a mapping $T$ in $H$ is said to be a $k$-strictly pseudo-contraction in the terminology of Browder and Petryshyn [7] if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in D(T)$, the domain of definition of $T$, where $I$ is the identity operator in $H$. Clearly, when $k = 0$, $T$ is nonexpansive, i.e.,

$$\|T(x) - T(y)\| \leq \|x - y\|.$$ 

It means that the class of $k$-strict pseudo-contractions strictly includes the class of nonexpansive mappings.

In the case $T_i \equiv I$, (1.2) is the equilibrium problem (1.1) and shown in [5], [21] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, as well as certain fixed point problems (see also [12]). For finding approximative solutions of (1.1) there exist several approaches: the regularization approach in [9], [11], [13], [22], the gap-function approach in [13], [14], [16], and iterative procedure approach in [1]-[4], [6], [10], [17]-[20].

In the case $F_0 \equiv 0$ and $N = 1$, (1.2) is a problem of finding a fixed point for a $k$-strictly pseudo-contraction in $C$ and studied in [15] where it is proved

Theorem 1.1. Let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a $k$-strict pseudo-contraction for some $0 \leq k < 1$ and assume that the fixed point set $F(T)$ of $T$ is nonempty. Let $\{x_n\}$ be the sequence generated by the
Let $x_0 \in C$ chosen arbitrarily, 
\[
y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
C_n = \{ z \in C : \| y_n - z \|^2 \leq \| x_n - z \|^2 + (1 - \alpha_n)(k - \alpha_n)\| x_n - Tx_n \|^2 \}, \\
Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n}x_0.
\]

Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\alpha_n < 1$ for all $n$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, the projection of $x_0$ onto $F(T)$.

In the case $F_0 \equiv 0$ and $N > 1$, (1.2) is a problem of finding a common fixed point for a finite family of $k_i$-strictly pseudo-contraction $T_i$ in $C$ and studied in [25] where the following algorithm is constructed:

Let $x_0 \in C$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, and $\{u_n\}$ be a sequence in $C$. Then the sequence $\{x_n\}$ generated by

\[
x_1 = \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\
x_2 = \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\
\vdots \\
x_N = \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\
x_{N+1} = \alpha_{N+1} x_N + \beta_{N+1} T_1 x_{N+1} + \gamma_{N+1} u_{N+1}, \\
\vdots
\]

is called the implicit iteration process with mean errors for a family of strictly pseudo-contractions $\{T_i\}_{i=1}^N$.

The scheme (1.3) can be expressed in the compact form as

\[
x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n,
\]

where $T_n = T_{n \mod N}$. It is proved the following

**Theorem 1.2.** Let $C$ be a nonempty closed convex subset of $H$. Let $\{T_i\}_{i=1}^N$ be $N$ strictly pseudo-contractive selfmaps of $C$ such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$ and $\{u_n\}$ be a bounded sequence in $C$, let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three sequences in $[0,1]$ satisfying the following conditions:

(i) $\{\alpha_n\} + \{\beta_n\} + \{\gamma_n\} = 1, \forall n \geq 1,$

(ii) there exist constants $\sigma_1, \sigma_2$ such that $0 < \sigma_1 \leq \beta_n \leq \sigma_2 < 1, \forall n \geq 1,$

(iii) $\sum_{n=1}^\infty \gamma_n < \infty.$

Then the implicit iterative sequence $\{x_n\}$ defined by (1.3) converges weakly to a common fixed point of the maps $\{T_i\}_{i=1}^N$. Moreover, in addition if there exists $i_0 \in \{1,2,\ldots,N\}$ such that $T_{i_0}$ is demicompact then $\{x_n\}$ converges strongly.
In the case $F_0 \neq 0$ and $N = 1$, (1.2) is a problem of finding a fixed point for a $k$-strictly pseudo-contraction in $C$ which is an equilibrium point for $F$, and studied in [24] where it is proved the following theorem.

**Theorem 1.3.** Let $C$ be a nonempty closed convex subset of $H$. Let $F$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $S$ be a nonexpansive mapping of $C$ into $H$ such that $F(S) \cap \text{EP}(F) \neq \emptyset$. Let $f$ be a contraction of $H$ into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$
\begin{align*}
F(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) &\geq 0, \quad \forall y \in C, \\
x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) Su_n,
\end{align*}
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,
$$

$$
\lim_{n \to \infty} \inf r_n > 0, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.
$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap \text{EP}(F)$, where $z = P_{F(S) \cap \text{EP}(F)} f(z)$.

Set $A_i = I - T_i$. Obviously, $A_i$ are $\lambda_i$ inverse strongly-monotone, i.e.,

$$
\langle A_i(x) - A_i(y), x - y \rangle \geq \lambda_i \|A_i(x) - A_i(y)\|^2 \quad \forall x, y \in D(A_i), \lambda_i = \frac{1 - k_i}{2}.
$$

From now on, let $\{A_i\}_{i=1}^{N}$ be a finite family of $\lambda_i$ inverse strongly-monotone operators in $H$ with $C \subset \cap_{i=1}^{N} D(A_i)$ and $\lambda_i > 0, i = 1, ..., N$.

Set $S = \cap_{i=1}^{N} S_i$, where $S_i = \{x \in C : A_i(x) = 0\}$ is called the solution set of $A_i$ in $C$.

Assume that $\text{EP}(F_0) \cap S \neq \emptyset$.

Our problem of investigation is to find an element

$$
u^* \in \text{EP}(F_0) \cap S.
$$

(1.4)

Because every nonexpansive mapping is $1/2$ inverse strongly-monotone, the problem of finding an element $u^* \in C$ that is not only a solution of an inverse strongly-monotone problem but also a fixed point of a nonexpansive mapping is a particular case of (1.4) when $F_0 \equiv 0, N = 2$ and studied in [23] where it is proved the following theorem.

**Theorem 1.4.** Let $C$ be a nonempty closed convex subset of $H$. Let $\lambda > 0$. Let $A$ be $\lambda$ inverse strongly-monotone mapping of $C$ into $H$, and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap \text{VI}(C, A) \neq \emptyset$ where
VI$(C, A)$ denotes the solution set of the following variational inequality: find $x_* \in C$ such that
\[ \langle A(x_*), x - x_* \rangle \geq 0, \quad \forall x \in C. \]

Let $\{x_n\}$ be a sequence generated by
\[
x_0 \in C,
\]
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n A(x_n)),
\]
for every $n = 0, 1, ..., $ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\lambda)$ and $\{\alpha_n\} \subset (c, d)$ for some $c, d \in (0, 1)$. Then, $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(C, A)$, where
\[ z = \lim_{n \to \infty} P_{F(S) \cap VI(C, A)} x_n. \]

In this paper, on the base of idea in [8] we present two methods of regularization which are the Tikhonov regularization and the regularization inertial proximal point algorithm for solving (1.4) where $F_0 \neq 0$ and $\{A_i\}_{i=1}^N$ are $\lambda_i(\lambda_i > 0)$ inverse strongly-monotone with that condition (A3) is replaced by

(A3') For every $u \in C$, $F_0(u, .) : C \to \mathbb{R}$ is weakly lower semicontinuous and convex.

The strong and weak convergences of any sequence are denoted by $\to$ and $\rightharpoonup$, respectively.

2. Main results.

We formulate the following facts in [5], [21] which are necessary in the proof of our results.

**Proposition 2.1** (i) If $F(., v)$ is hemicontinuous for each $v \in C$ and $F$ is monotone, i.e., satisfies (A2) in condition 1.1, then $U^* = V^*$, where

$U^*$ is the solution set of $F(u^*, v) \geq 0$ $\forall v \in C$,

$V^*$ is the solution set of $F(u, v^*) \leq 0$ $\forall u \in C$,

and it is convex and closed.

(ii) If $F(., v)$ is hemicontinuous for each $v \in C$ and $F$ is strongly monotone, i.e., there exists a positive constant $\tau$ such that
\[ F(u, v) + F(v, u) \leq -\tau \|u - v\|^2, \]
then $U^*$ contains a unique element.

**Lemma 2.1** Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be the sequences of positive numbers satisfying the conditions:

(i) $a_{n+1} \leq (1 - b_n) a_n + c_n, b_n < 1$,

(ii) $\sum_{n=0}^{\infty} b_n = +\infty, \lim_{n \to +\infty} \frac{c_n}{b_n} = 0$.

Then, $\lim_{n \to +\infty} a_n = 0$.

Let $S_A$ be a solution set of an inverse strongly-monotone operator $A$. 

Lemma 2.2 Let $C_1$ be a closed convex subset of $C$ with the property $S_A \cap C_1 \neq \emptyset$. Then, the solution set of the following variational inequality
\[
\langle A(\tilde{y}), x - \tilde{y} \rangle \geq 0 \quad \forall x \in C_1, \tilde{y} \in C_1, \tag{2.1}
\]
is coincided with $S_A \cap C_1$.

Proof. Obviously, every element in $S_A \cap C_1$ is a solution of (2.1). Let $\tilde{y}$ be an arbitrary solution of (2.1). We have to prove that $A(\tilde{y}) = 0$. Let $\tilde{x}$ be an element in $S_A \cap C_1$. Since $\tilde{x}$ is a zero element of the monotone operator $A$ and $\tilde{y}$ is a solution of (2.1), then
\[
0 = \langle A(\tilde{x}), \tilde{x} - \tilde{y} \rangle \geq \langle A(\tilde{y}), \tilde{x} - \tilde{y} \rangle \geq 0.
\]
Hence, $\langle A(\tilde{y}), \tilde{x} - \tilde{y} \rangle = 0 = \langle A(\tilde{y}), \tilde{y} - \tilde{x} \rangle$. Consequently, $\langle A(\tilde{y}) - A(\tilde{x}), \tilde{y} - \tilde{x} \rangle = 0$.

From the inverse strongly-monotone property of $A$ it follows $A(\tilde{y}) = A(\tilde{x}) = 0$. It means that $\tilde{y} \in S_A \cap C_1$. Lemma is proved.

We construct the Tikhonov regularization solution $u_\alpha$ by solving the single equilibrium problem
\[
F_\alpha(u_\alpha, v) \geq 0 \quad \forall v \in C, u_\alpha \in C,
\]
\[
F_\alpha(u, v) := \sum_{i=0}^{N} \alpha^{\mu_i} F_i(u, v) + \alpha \langle u, v - u \rangle, \alpha > 0, \tag{2.2}
\]
\[
F_i(u, v) = \langle A_i(u), v - u \rangle, i = 1, ..., N,
\]
\[
\mu_0 = 0 < \mu_i < \mu_{i+1} < 1, i = 2, ..., N - 1,
\]
and $\alpha$ is the regularization parameter.

We have the following results.

Theorem 2.1. (i) For each $\alpha > 0$, problem (2.2) has a unique solution $u_\alpha$.

(ii) $\lim_{\alpha \to 0} u_\alpha = u^*, u^* \in EP(F_0) \cap S, \|u^*\| \leq \|y\| \quad \forall y \in EP(F_0) \cap S$.

(iii) $\|u_\alpha - u_\beta\| \leq (\|u^*\| + dN) |\alpha - \beta| / \alpha$, $\alpha, \beta > 0$,

where $d$ is a positive constant.

Proof. It is not difficult to verify that $F_i, i = 1, ..., N$, all are the bifunctions. Therefore, $F_\alpha(u, v)$ also is a bifunction, i.e. $F_\alpha(u, v)$ satisfies condition 1.1, and strongly monotone with constant $\alpha > 0$. Hence, (2.2) has a unique solution $u_\alpha$ for each $\alpha > 0$.

Now we shall prove that
\[
\|u_\alpha\| \leq \|y\| \quad \forall y \in EP(F_0) \cap S. \tag{2.3}
\]
Since $y \in EP(F_0) \cap S$, then $F_0(y, u_\alpha) \geq 0$ and $A_i(y) = 0, i = 1, ..., N$. Consequently, $F_i(y, u_\alpha) = 0, i = 1, ..., N$, and
\[
\sum_{i=0}^{N} \alpha^{\mu_i} F_i(y, u_\alpha) \geq 0 \quad \forall y \in EP(F_0) \cap S.
\]
This fact, $u_{\alpha}$ is the solution of (2.2) and the properties of $F_i$ give

$$\langle u_{\alpha}, y - u_{\alpha} \rangle \geq 0 \quad \forall y \in EP(F_0) \cap S,$$

that implies (2.3). It means that $\{u_{\alpha}\}$ is bounded. Let $u_{\alpha_k} \to u^* \in H$, as $k \to +\infty$. Since $C$ is closed in the norm and convex, then $C$ is weak closed. Hence, $u^* \in C$. We prove that $u^* \in EP(F_0)$. From (A2) and (2.2) it follows

$$F_0(v, u_{\alpha_k}) + \sum_{i=1}^{N} \alpha_k^\mu_i F_i(v, u_{\alpha_k}) \leq \alpha_k \langle v, v - u_{\alpha_k} \rangle \quad \forall v \in C.$$

Using the property (A3') we obtain $F_0(v, u^*) \leq 0$ for any $v \in C$. By virtue of the proposition 2.1, we have $u^* \in EP(F_0)$. Now we show that $u^* \in S_i$, $i = 1, \ldots, N$. From (2.2), $F_0(y, u_{\alpha_k}) \geq 0$ for any $y \in EP(F_0)$, and the monotone property of $F_0$, i.e. $F_0(u_{\alpha_k}, y) + F_0(y, u_{\alpha_k}) \leq 0$, it implies that

$$\sum_{i=1}^{N} \alpha_k^\mu_i F_i(u_{\alpha_k}, y) + \alpha_k \langle u_{\alpha_k}, y - u_{\alpha_k} \rangle \geq 0 \quad \forall y \in EP(F_0).$$

By tending $k \to \infty$, we have got

$$F_1(y, u_{\alpha_k}) + \sum_{i=2}^{N} \alpha_k^\mu_i F_i(y, u_{\alpha_k}) \leq \alpha_k \langle y, y - u_{\alpha_k} \rangle \quad \forall y \in EP(F_0).$$

That has the form

$$\langle A_1(y), y - u^* \rangle \geq 0 \quad \forall y \in EP(F_0).$$

The last inequality is equivalent to

$$\langle A_1(u^*), y - u^* \rangle \geq 0 \quad \forall y \in EP(F_0).$$

Since $EP(F_0) \cap F(T_i) \neq \emptyset$ and $A_1$ is an inverse strongly-monotone, from lemma 2.2 it follows $u^* \in S_1$.

Let $\tilde{S}_i = EP(F_0) \cap (\cap_{l=1}^{i} S_l)$. Then, $\tilde{S}_i$ is also closed convex, and $\tilde{S}_i \neq \emptyset$.

Now, suppose that we have proved that $u^* \in \tilde{S}_i$, and need to show that $u^*$ belongs to $S_{i+1}$. Again, by virtue of (2.2) for $y \in \tilde{S}_i$ we can write

$$F_{i+1}(y, u_{\alpha_k}) + \sum_{l=i+2}^{N} \alpha_k^\mu_{l-1} F_l(y, u_{\alpha_k}) \leq \alpha_k \langle y, y - u_{\alpha_k} \rangle \quad \forall y \in \tilde{S}_i.$$
After passing $k \to \infty$, we obtain
\[ F_{i+1}(y, u^*) \leq 0 \quad \forall y \in \tilde{S}_i. \]

By virtue of $\tilde{S}_i \cap S_{i+1} \neq \emptyset$, $u^*$ also is an element of $S_{i+1}$, i.e., $F_{i+1}(y, u^*) \leq 0 \quad \forall y \in \tilde{S}_i$. Inequality (2.3) and the weak convergence of $\{u_{\alpha_k}\}$ to $u^* \in EP(F_0) \cap S$, which is a closed convex subset in $H$, give the strong convergence of $\{u_{\alpha_k}\}$ to $u^*$: $\|u^*\| \leq \|y\| \quad \forall y \in EP(F_0) \cap S$.

From (2.2) and the properties of $F_i(u, v)$, for each $\alpha, \beta > 0$ it follows
\[ \sum_{i=0}^{N} (\alpha^\mu_i - \beta^\mu_i) F_i(u_\alpha, u_\beta) + \alpha \langle u_\alpha, u_\beta - u_\alpha \rangle + \beta \langle u_\beta, u_\alpha - u_\beta \rangle \geq 0 \]
or
\[ \|u_\alpha - u_\beta\| \leq \frac{\|u_\alpha - u_\beta\|}{\alpha} + \frac{1}{\alpha} \sum_{i=1}^{N} |\alpha^\mu_i - \beta^\mu_i| |F_i(u_\alpha, u_\beta)|, \]

because $\mu_0 = 0$. All $F_i, i = 1, \ldots, N$, are bounded, because the operators $A_i$ all are Lipschitzian with Lipschitz constants $L_i = 1/\lambda_i$. Using (2.3), the boundedness of $F_i$ and the Lagrange’s mean-value theorem for the function $\alpha(t) = t^\mu, 0 < \mu < 1, t \in [1, +\infty)$, on $[\alpha, \beta]$ if $\alpha < \beta$ or $[\beta, \alpha]$ if $\beta < \alpha$ we have conclusion (iii). Theorem is proved now.

**Remark.** Obviously, if $u_{\alpha_k} \to \bar{u}$, where $u_{\alpha_k}$ is the solution of (2.2) with $\alpha = \alpha_k \to 0$, as $k \to +\infty$, then $EP(F_0) \cap S \neq \emptyset$.

Further, we consider the regularization inertial proximal point algorithm where $z_{n+1}$ is defined by
\[ \tilde{c}_n \left( \sum_{i=0}^{N} \alpha^\mu_n F_i(z_{n+1}, v) + \alpha_n \langle z_{n+1}, v - z_{n+1} \rangle \right) + \langle z_{n+1} - z_n, v - z_{n+1} \rangle - \gamma_n \langle z_n - z_{n-1}, v - z_{n+1} \rangle \geq 0 \quad \forall v \in C, z_0, z_1 \in C, \]  

and $\{\tilde{c}_n\}$ and $\{\gamma_n\}$ are the sequences of positive numbers. Note that in the case $N = 0$ algorithm (2.4) is considered in [18] without the regularized term $\alpha_n \langle z_{n+1}, v - z_{n+1} \rangle$, and the obtained result only is the weak convergence of the sequence $\{z_n\}$ under some condition. By virtue of this term we shall obtain a stronger result.

It is not difficult to verify that the bifunction
\[ \tilde{c}_n \left( \sum_{i=0}^{N} \alpha^\mu_n F_i(u, v) + \alpha_n \langle u, v - u \rangle \right) + \langle u - z_n, v - u \rangle - \gamma_n \langle y_n, v - u \rangle, \]

where $y_n = z_n - z_{n-1}$, is strongly monotone with constant $\tilde{c}_n \alpha_n$. Therefore, (2.4) possesses a unique solution $z_{n+1}$ for each $n$. 
Theorem 2.2 Assume that the parameters $\tilde{c}_n$, $\gamma_n$ and $\alpha_n$ are chosen such that:

(i) $0 < c_0 < \tilde{c}_n < C_0$, $0 \leq \gamma_n < \gamma_0$,

(ii) $\sum_{n=1}^{\infty} b_n = +\infty$, $b_n = \tilde{c}_n \alpha_n / (1 + \tilde{c}_n \alpha_n)$,

(iii) $\sum_{n=1}^{\infty} \gamma_n b_n^{-1} \|z_n - z_{n-1}\| < +\infty$,

(iv) $\lim_{n \to \infty} \alpha_n = 0$, $\lim_{n \to \infty} \frac{\|z_n - z_{n-1}\|}{\|z_n - z_{n-1}\|} = 0$.

Then, the sequence $\{z_n\}$ defined by (2.4) converges strongly to the element $u^*$, as $n \to +\infty$.

Proof. Denote by $u_n$ and $u_{n+1}$ the solutions of (2.2) with $\alpha = \alpha_n$ and $\beta = \alpha_{n+1}$, respectively. Then, we have the following inequality

$$\|u_{n+1} - u_n\| \leq (\|u^*\| + dN) \frac{\alpha_n - \alpha_{n+1}}{\alpha_n},$$

$$d = \max_{1 \leq i \leq N} \left\{ \frac{4\|u^*\|^2}{\lambda_i} \right\}.$$ 

On the other hand, (2.4) and (2.2) can be rewritten in the equivalent forms

$$\tau_n \sum_{j=0}^{N} \alpha_n^j F_i(z_{n+1}, v) + \langle z_{n+1}, v - z_{n+1} \rangle \geq \beta_n \langle z_n, v - z_{n+1} \rangle$$

$$+ \beta_n \gamma_n \langle z_n - z_{n-1}, v - z_{n+1} \rangle \quad \forall v \in C,$$

$$\tau_n \sum_{i=0}^{N} \alpha_n^i F_i(u_n, v) + \langle u_n, v - u_n \rangle \geq \beta_n \langle u_n, v - u_n \rangle, \quad \forall v \in C,$$

respectively, where $\tau_n = \tilde{c}_n/\beta_n$, $\beta_n = 1/(1 + \tilde{c}_n \alpha_n)$. Replacing $v = u_n$ and $v = z_{n+1}$ in the last two inequalities, respectively, and then summarizing the results, we obtain the inequality

$$\langle z_{n+1} - u_n, u_n - z_{n+1} \rangle \geq \beta_n \langle z_n - u_n, u_n - z_{n+1} \rangle$$

$$+ \beta_n \gamma_n \langle z_n - z_{n-1}, u_n - z_{n+1} \rangle.$$ 

Consequently,

$$\|z_{n+1} - u_n\| \leq \beta_n \|z_n - u_n\| + \beta_n \gamma_n \|z_n - z_{n-1}\|.$$

Hence,

$$\|z_{n+1} - u_{n+1}\| \leq \|z_{n+1} - u_n\| + \|u_{n+1} - u_n\|$$

$$\leq \beta_n \|z_n - u_n\| + \beta_n \gamma_n \|z_n - z_{n-1}\|$$

$$+ (\|u^*\| + dN) \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}$$

$$\leq (1 - b_n) \|z_n - u_n\| + c_n,$$

$$c_n = \beta_n \gamma_n \|z_n - z_{n-1}\| + (\|u^*\| + dN) \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}.$$
Since the serie in (iii) is convergent, then $\beta_n \gamma_n \|z_n - z_{n-1}\| b_n^{-1} \leq \gamma_n \|z_n - z_{n-1}\| b_n^{-1} \to 0$, as $n \to \infty$. This fact and (iv) follow $\lim_{n \to \infty} \epsilon_n b_n^{-1} = 0$. By using the above lemma with $a_n = \|z_n - u_n\|$ we have

$$\lim_{n \to \infty} \|z_n - u_n\| = 0.$$ 

Since $u_n \to u^*$, then $z_n \to u^*$, as $n \to \infty$. Theorem is proved.

**Remark** The sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ which are defined by

$$\alpha_n = (1 + n)^{-p}, 0 < p < 1/2,$$

$$\gamma_n = (1 + n)^{-\tau} \frac{\|z_n - z_{n-1}\|}{1 + \|z_n - z_{n-1}\|^2},$$

with $\tau > 1 + p$ satisfy all conditions in theorem 2.2.

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**References**


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