An Explicit Iteration Method for Convex Feasibility Problems in Hilbert Spaces

Nguyen Buong and Pham Van Son

Vietnamese Academy of Science and Technology
Institute of Information Technology
18, Hoang Quoc Viet, q. Cau Giay, Ha Noi, Vietnam
nbuong@ioit.ac.vn

Abstract

The purpose of this note is to present an explicit iteration method that converges strongly for solving convex feasibility problems in Hilbert spaces.

Mathematics Subject Classification: 90C25, 90C30, 49J40, 46M37

Keywords: Monotone operators, Lipschitz continuous, Tikhonov regularization, convex feasibility problem, projection operator

1. Introduction

The convex feasibility problem is the problem of computing points laying in the intersection of a finite family of closed convex subsets $C_j, j = 1, ..., N$, of a Hilbert space $H$. This problem appears in various fields of applied mathematics. The theory of Optimization [2], Image Reconstruction from projections [13] and Game Theory [10] are some examples.

Often these constraint sets are closed subspaces in $H$, an immensely successful algorithm for solving this problem is the method of cyclic projections, where a sequence is generated by projecting cyclically onto the constraint subspaces. The fundamental result is due to von Neumann and Halperin: suppose $C_1, C_2, ..., C_N$ are intersecting closed affine subspaces with corresponding projections $P_1, P_2, ..., P_N$. If $x_0 \in H$, then the sequence

$$x_0, x_1 = P_1x_0, x_2 = P_2x_1, ..., x_N = P_Nx_N, x_{N+1} = P_1x_N,...$$

converges in norm to the projection of $x_0$ onto $C := \cap_{j \in J}C_j$, where $J = \{1, 2, ..., N\}$.

In 1933, von Neumann [26] proved this result for two closed subspaces. It was later extended to finitely many closed subspaces by Halperin [20]. An
impressive survey on applications of the method of cyclic projections for intersecting closed affine subspaces can be found in Deutsch [15]. In particular, if each \( C_j \) are hyperplane, then one obtains the well-known method of Kaczmarz [24] for solving systems of linear equations.

Dykstra [16] suggested an algorithm which solves the problem for closed convex cones in Euclidian space. Boyle and Dykstra [5] showed that Dykstra’s algorithm, which coincides with Neumann’s algorithm for closed subspaces, solves the problem for general closed convex sets in a Hilbert space: Let \( q_{-(N-1)} := \ldots = q_0 := 0 \). Denote the mod \( N \) function with values in \( \{1, \ldots, N\} \) by \([.]\). Set \( C_n := C_{[n]} \). Generate the sequence \( \{x_n\}, \{q_n\} \) in \( H \) by

\[
x_n := P_n(x_{n-1} + q_{n-N}) \quad \text{and} \quad q_n := x_{n-1} + q_{n-N} - x_n,
\]

for every \( n \geq 1 \). They proved that the sequence \( \{x_n\} \) converges to \( P_C(x_0) \) in the case that \( H \) is finite-dimensional. A clever product approach, due to Pierra [28] and developed by Flam and Zow [17] and Iusem and De Pierro [22], is as follows: Given strictly positive weights \( \lambda_1, \ldots, \lambda_N \), i.e., \( \sum_{j \in J} \lambda_j = 1 \), define

\[
H := \Pi_{j \in J}(H, \lambda_j, \langle \cdot, \cdot \rangle), \quad A := \{(x_1, \ldots, x_N) \in H : x_1 = \ldots = x_N \in H\}, \quad B := \{(x_1, \ldots, x_N) \in H : x_j \in C_j, j \in J\}.
\]

For the distance of two points \( y = (y_1, \ldots, y_N), z = (z_1, \ldots, z_N) \in H \) we have

\[
\|y - z\|^2 = \sum_{j \in J} \lambda_j\|y_j - z_j\|^2,
\]

wich implies

\[
I := \inf_{y \in H} \sum_{j \in J} d^2(y, C_j) = d^2(A, B).
\]

Define \( 2N + 1 \) sequences \( \{a_n\}, \{b_n^j\}, \{q_n^j\} \) by

\[
b_0^j := x, \quad q_0^j := 0,
\]

\[
a_{n+1} := \sum_{j \in J} \lambda_j b_n^j, \quad b_{n+1}^j := P_i(a_{n+1} + q_n^j), \quad q_{n+1}^j := a_{n+1} + q_n^j - b_{n+1}^j,
\]

for \( n \geq 0 \) and \( j \in J \). Then, \( \sum_{j \in J} \lambda_j\|b_n^j - a_n\|^2, \sum_{j \in J} \|b_n^j - a_{n+1}\|^2 \to I \) and \( a_n/n, b_n^1/n, b_n^2/n, \ldots, b_n^N/n \to 0 \). Moreover,

(i) If \( I \) is not attained, then \( \|a_n\|, \max\{\|b_n^1\|, \ldots, \|b_n^N\|\} \to +\infty \).

(ii) If \( I \) is attained, then

\[
a_n \to P_C(x), b_n^j \to P_j P_C(x),
\]

for \( j \in J \). Bauschke and Lewis [3] discovered the close relationship between the Dykstra’s algorithm with Bregman projections and the very general and powerful algorithm of Tseng [30], namely the Dual block coordinate ascent method. Bregman, Censor and Reich showed [6] that Dykstra’s algorithm
with Bregman projections, is actually the nonlinear extension of Bregman’s primal-dual, dual coordinate ascent, row-action minimization algorithm.

Another iterative method named block-iterative projection algorithm, which is a parallel algorithm and proposed in [1] in the Euclidian space $\mathbb{R}^n$ setting and further studied in [11], [14], iteratively generates a sequence as follows. Choose an initial point $x_0 \in H$ and for each $k$, do

$$x^{k+1} = x^k - \beta_k \sum_{j \in J} \alpha_k(j) A_j(x^k),$$

where $A_j(x_k) = (I - P_j)(x_k)$, $I$ is the identity operator in $H$ and $P_j(x_k)$ is the orthogonal projection of $x^k$ onto the set $C_j$, $\alpha_k(j) : J \to \mathbb{R}_+$ is a weight function (i.e., $\sum_{j \in J} \alpha_k(j) = 1$) and $\beta_k$ are relaxation parameters. When for all $k$, $\alpha_k(j) = \alpha(j), j \in J$ and $\alpha(.)$ is some fixed weight function, the block iterative projection method (1.1) is called simultaneous [21]. The block-iterative projection method is called sequential if all weight functions $\alpha_k$ are Kronecker vectors in $\mathbb{R}^N$. The first strong convergence result for the nonsequential block iterative projection method, due to Pierra [28], shows that block-iterative projection methods with uniformly distributed weight functions (i.e., with $\alpha_k(j) = 1/N, j \in J$) and appropriately chosen relaxor parameters $\beta_k$ strongly converge in Hilbert spaces provided that convex feasibility problem satisfies conditions similar to those in [19]. De Pierro and Isem [14] extended Pierra result by showing that simultaneous block-iterative projection method with constant sequences of relaxation parameters strongly converges to a point in $C$ when one of the sets $C_j$ is compact.

A block iterative projection method is called almost simultaneous [11] if the sequence $\{\alpha_k\}$ has a convergent subsequence whose limit in $\mathbb{R}^J$ is a positive weight function $\alpha_*$. In [12] it proved the following result.

**Theorem 1.1** If a block-iterative projection method is almost simultaneous with relaxation parameters

$$0 < \beta_1 \leq \beta_k \leq \beta_2 < 2,$$

and if any of the following conditions is satisfied:

(A) there exists $j_0 \in J$ such that $C_{j_0} \cap \text{Int}[\cap_{j \neq j_0} C_j] \neq \emptyset$;

(B) all, except for possibly one, of the sets $C_j$ are uniformly convex;

(C) each $C_j$ is halfspace (i.e., $C_j = \{x \in H : \langle x, c_j \rangle \leq b_j\}$ for some $c_j \in H$ and for some $b_j \in \mathbb{R}$);

(D) at least one set $C_j$ is boundedly compact;

(E) $H$ is finite-dimensional;

then any sequence $\{x^k\}$ generated by this method is strongly convergent in $H$ to a point in $C$.

Clearly,

$$y \in C_j \iff y = P_j(y) \iff A_j(y) = 0.$$
It means that $y$ is a fixed point of the operator projection $P_j$. Note that $P_j$ is nonexpansive, i.e.,
\[\|P_j(x) - P_j(y)\| \leq \|x - y\|\]

Therefore, the convex feasibility problem is the problem of finding a common fixed point for a finite family of nonexpansive mappings that is studied intensively in [23], [25], [27], [29], [31]-[33] for the last years. The approximative methods in those works are not belong to a class of parallel algorithms. So, we do not present them in this paper.

In this paper, on the our idea [8], [9], we propose an explicit method that is a parallel algorithm and converges strongly without any condition from (A) to (E) in theorem 1.1. On the base of constructing an operator method of regularization of the type
\[
\sum_{j \in J} A_j(x) + \alpha_n(x - \overline{x}) = 0,
\]

depending on the positive regularization parameter $\alpha_n$ that tens to zero as $n \to +\infty$, where $\overline{x}$ is an element in $H$ that does not belong to $C$, we investigate the explicit iteration method where $z_{n+1}$ is defined by
\[
z_{n+1} = z_n - \beta_n \left( \sum_{j \in J} A_j(z_n) + \alpha_n(z_n - \overline{x}) \right), z_0 \in H,
\]

and \{\beta_n\} also is a sequence of positive numbers.

In Section 2 we shall prove that \{z_n\} converges strongly to an element $x_* \in C$ as $n \to +\infty$ under the suitable choice of the sequences \{\beta_n\} and \{\alpha_n\}.

Below, the symbols $\rightharpoonup$ and $\to$ denote the weak convergence and convergence in the norm, respectively.

2. Main result

We need the following lemmas.

**Lemma 2.1** Let \{a_n\}, \{b_n\}, \{c_n\} be the sequences of positive numbers satisfying the conditions
\[
(i) \quad a_{n+1} \leq (1 - b_n)a_n + c_n, b_n < 1,
(ii) \quad \sum_{n=0}^{\infty} b_n = +\infty, \quad \lim_{n \to +\infty} \frac{c_n}{b_n} = 0.
\]

Then, $\lim_{n \to +\infty} a_n = 0$.

**Lemma 2.2** (Demiclosedness Principle) [18] If $D$ is a closed convex subset of $H$, $T : D \to H$ is nonexpansive, \{x_n\} is a sequence in $D$ such that $x_n \rightharpoonup x \in D$ and $x_n - Tx_n \to 0$, then $x - Tx = 0$.

We have the following results.

**Theorem 2.1** (i) For each $\alpha_n > 0$, problem (1.2) has a unique solution $x_n$.
(ii) $\lim_{n \to +\infty} x_n = x_*, x_* \in S, \|x_* - \overline{x}\| \leq \|y - \overline{x}\| \quad \forall y \in C$. 
(iii) \[ \|x_n - x_p\| \leq M \frac{\alpha_n - \alpha_p}{\alpha_n}, \]

where \( M = \|x_* - \overline{x}\| \).

**Proof.** (i) It is not difficult to see that \( A_j \) is monotone and Lipschitz continuous with the Lipschitz constant \( L_j = 2 \). Therefore, the operator \( \sum_{j \in J} A_j \) also is monotone and Lipschitz continuous with the Lipschitz constant \( L = 2N \) with the domain \( H \). Consequently [7], \( \sum_{j \in J} A_j \) is maximal monotone, i.e., equation (1.2) possesses a unique solution \( x_n \) for each \( \alpha_n > 0 \).

(ii) From (1.2) it follows

\[
\sum_{j \in J} \langle A_j(x_n), x_n - y \rangle + \alpha_n \langle x_n - \overline{x}, x_n - y \rangle = 0 \quad \forall y \in C.
\]

(2.1)

Since for \( y \in C \) we have \( A_j(y) = 0, j = 1, \ldots, N \). So,

\[
\sum_{j \in J} \langle A_j(y), y - x_n \rangle = 0.
\]

The last equality, (2.1) and the monotone property of \( A_j \) give

\[
\langle x_n - \overline{x}, x_n - y \rangle \leq 0
\]

or

\[
\langle x_n - \overline{x}, x_n - \overline{x} \rangle \leq \langle x_n - \overline{x}, y - \overline{x} \rangle \quad \forall y \in C.
\]

Therefore,

\[
\|x_n - \overline{x}\| \leq \|y - \overline{x}\| \quad \forall y \in C.
\]

(2.2)

Hence, \( \{x_n\} \) is bounded. Let \( x_{n_k} \rightharpoonup \overline{x} \in H \), as \( k \to +\infty \). First, we prove that \( \overline{x} \in S_i, i = 1, \ldots, N \). Remember [29] that \( A_j \) is 1/2-inverse strongly monotone, i.e.,

\[
\langle A_j(x) - A_j(y), x - y \rangle \geq \frac{1}{2} \|A_j(x) - A_j(y)\|^2.
\]

Thus, for any \( y \in C \) from (1.2) it implies that

\[
\|A_i(x_{n_k})\|^2 \leq 2 \langle A_i(x_{n_k}), x_{n_k} - y \rangle \\
\leq 2 \sum_{j \in J, j \neq i} \langle A_j(y) - A_j(x_{n_k}), x_{n_k} - y \rangle \\
+ 2\alpha_{n_k} \langle -y, x_{n_k} - y \rangle.
\]

Consequently,

\[
\|A_i(x_{n_k})\| \leq \sqrt{2\alpha_{n_k}} \|y\|.
\]
Hence,
\[ \lim_{k \to \infty} \| A_i(x_{n_k}) \| = 0. \]

By virtue of the demiclosed property of \( A_i \), we have \( A_i(\tilde{x}) = 0 \), i.e., \( \tilde{x} \in S_i \).

From (2.2) with \( y = \tilde{x} \) it follows
\[ \lim_{k \to +\infty} x_{n_k} = \tilde{x} \in C. \]

Since \( C \) is a closed convex subset in Hilbert space \( H \), then it possesses a unique element \( x_* \) with minimal norm. Therefore, all sequence \( \{ x_n \} \) converges to \( x_* \) as \( n \to +\infty \).

(iii) Let \( x_p \) be the solution of (1.2) when \( \alpha_n \) is replaced by \( \alpha_p \). Then, from (1.2) and the monotone property of \( A_j \) it follows
\[ \sum_{j \in J} \langle A_j(x_n) - A_j(x_p), x_n - x_p \rangle + \alpha_n \langle x_n - \bar{x}, x_n - x_p \rangle + \alpha_p \langle x_p - \bar{x}, x_p - x_n \rangle = 0. \]

Hence,
\[ \| x_n - x_p \| \leq \frac{\| x_n - \bar{x} \|}{\alpha_n} \| x_n - \bar{x} \| \leq \frac{\| x_n - \bar{x} \|}{\alpha_n} \| x_* - \bar{x} \|. \]

Theorem is proved now.

**Theorem 2.2** Assume that the following conditions hold:

(i) \( 1 \geq \alpha_n \searrow 0, \beta_n \to 0 \), as \( n \to +\infty \).

(ii) \( \lim_{n \to +\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_n \beta_n} = \lim_{n \to +\infty} \frac{\beta_n}{\alpha_n} = 0. \)

(iii) \( \sum_{n=0}^{\infty} \beta_n \alpha_n = +\infty. \)

Then, \( z_n \to x_* \), as \( n \to +\infty. \)

**Proof.** Let \( \Delta_n = \| z_n - x_n \| \). Obviously,
\[ \Delta_{n+1} = \| z_{n+1} - x_{n+1} \| \\
= \| z_n - x_n - \beta_n \left( \sum_{j \in J} A_j(z_n) + \alpha_n (z_n - \bar{x}) \right) - (x_{n+1} - x_n) \|, \]

where
\[ \| z_n - x_n - \beta_n \left( \sum_{j \in J} A_j(z_n) + \alpha_n (z_n - \bar{x}) \right) \|^2 = \| z_n - x_n \|^2 + \beta_n^2 \left( \sum_{j \in J} A_j(z_n) + \alpha_n (z_n - \bar{x}) \right)^2 - 2\beta_n \left( z_n - x_n \right) \sum_{j \in J} A_j(z_n) \]
Explicit iteration method

\[ + \alpha_n(z_n - \bar{x}) - \left( \sum_{j \in J} A_j(x_n) + \alpha_n(x_n - \bar{x}) \right) \]
\[ \leq (1 - 2\alpha_n \beta_n) \| z_n - x_n \|^2 + \beta_n^2 \left| \sum_{j \in J} A_j(z_n) + \alpha_n(z_n - \bar{x}) \right|^2, \]

and

\[ \left\| \sum_{j \in J} A_j(z_n) + \alpha_n(z_n - \bar{x}) \right\|^2 \leq \left( \sum_{j \in J} \| A_j(z_n) - A_j(x_n) \| + \alpha_n \| z_n - \bar{x} \| \right)^2 \]
\[ \leq \left( \sum_{j \in J} 2\| z_n - x_n \| + \alpha_n \| z_n - \bar{x} \| \right)^2 \leq c_1 \| z_n - x_n \|^2 + c_2, \]

where \( c_i, i = 1, 2, \) are positive constants. Therefore,

\[ \Delta_{n+1} \leq \left\{ \Delta_n^2 (1 - 2\beta_n \alpha_n + c_1 \beta_n^2) + c_2 \beta_n^2 \right\}^{1/2} + M \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}. \]

By taking the squares of the both sides for the last inequality, and then applying the elementary estimate

\[ (a + b)^2 \leq (1 + \varepsilon)(a^2 + \frac{1}{\varepsilon} b^2), \varepsilon > 0, \varepsilon = \alpha_n \beta_n, \]

we obtain the inequality

\[ \Delta_{n+1}^2 \leq \left[ \Delta_n^2 (1 - 2\alpha_n \beta_n + c_1 \beta_n^2) + c_2 \beta_n^2 \right] (1 + \alpha_n \beta_n) \]
\[ + M^2 \left( \frac{\alpha_n - \alpha_{n+1}}{\alpha_n \beta_n} \right)^2 \alpha_n \beta_n (1 + \alpha_n \beta_n) \]
\[ \leq \Delta_n^2 (1 - \alpha_n \beta_n + c_3 \beta_n^2) + c_4 \beta_n^2 \]
\[ + M^2 \left( \frac{\alpha_n - \alpha_{n+1}}{\alpha_n \beta_n} \right)^2 \alpha_n \beta_n (1 + \alpha_n \beta_n). \]

where \( c_3, c_4 \) are constants (which may depend on \( z_0 \)). In finall, the proof is finished by using lemma 1.1 with

\[ a_n = \Delta_n^2, \]
\[ b_n = \alpha_n \beta_n (1 - c_3 \frac{\beta_n}{\alpha_n}), \]
\[ c_n = c_4 \beta_n^2 + M^2 \left( \frac{\alpha_n - \alpha_{n+1}}{\alpha_n \beta_n} \right)^2 \alpha_n \beta_n (1 + \alpha_n \beta_n). \]

Remark 1. The sequences \( \beta_n = (1 + n)^{-1/2} \) and \( \alpha_n = (1 + n)^{-p}, 0 < 2p < 1 \) satisfy all conditions in theorem 2.2.
2. We list some cases where the projection is easy calculated [4]:

(i) If \( C = \{ x \in H : \|x\| \leq 1 \} \), then
\[
P_C(x) = \begin{cases} 
  x, & \text{if } \|x\| \leq 1, \\
  x/\|x\|, & \text{otherwise.}
\end{cases}
\]

(ii) If \( C \) is a hyperplane, say \( C = \{ x \in H : \langle a, x \rangle = b \} \) with \( a \in H \setminus \{0\} \), then
\[
P_C(x) = x - ((\langle a, x \rangle - b)/\|a\|^2)a.
\]

(iii) If \( C \) is a halfspace, say \( C = \{ x \in H : \langle a, x \rangle \leq b \} \) with \( a \in H \setminus \{0\} \), then
\[
P_C(x) = x - ((\langle a, x \rangle - b)^+ / \|a\|^2)a.
\]

This work was supported by the Vietnamese Fundamental Research Program in Natural Sciences N. 100506.

References


Explicit iteration method


[30] P. Tseng, Dual cooradinate ascent methods for non-strictly convex mini-

[31] H.K. Xu, An iterative approach to quadratic optimization, J. of Opti-


Received: September 16, 2007