Weak Convergence Theorems
for Asymptotically Nonexpansive Mappings

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Abstract. Let $K$ be a nonempty closed convex subset of a uniformly convex
Banach space $E$ and $T : K \rightarrow K$ be an asymptotically nonexpansive mapping
with nonempty fixed points set $F(T)$. The $\{\alpha_n\}, \{\beta_n\}$ are two real sequences
in $[0, 1]$. The purpose of this article is to study the modified Ishikawa iteration
process $\{x_n\}$ of $T$, for any initial guess $x_1 \in K$, defined by

\[x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n,\]
\[y_n = (1 - \beta_n)x_n + \beta_n T^n x_n.\]

Not only weak and strong convergence are obtained but also the restriction
$0 < a \leq \alpha_n \leq b < 1$ on $\{\alpha_n\}$ are relaxed. The results of this article extend
and improve the results of many authors.

Keywords: Asymptotically nonexpansive; Iterative scheme; Fixed point;
Weak convergence; Strong convergence; Opial’s condition

1. INTRODUCTION AND PRELIMINARIES

Let $K$ be a nonempty subset of a Banach space $E$, a mapping $T : K \rightarrow K$
is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n \to \infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $n \geq 1$
and for all $x, y \in E$. This class of mappings, as a natural extension to that of
nonexpansive mappings, was introduced by Goebel and Kirk[1] in 1972.

Recall that $T$ is said to be uniformly L-Lipschitzian mappings, if $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $n \geq 1$ and for all $x, y \in E$, where $L > 0$ is a constant.
It is obvious that, every asymptotically nonexpansive mapping is also uniformly \(L\)-Lipschitzian mapping.

Recall that a Banach space \(E\) is said to satisfy Opial’s condition if, whenever \(\{x_n\}\) is a sequence in \(E\) which converges weakly to \(x\), then
\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \quad \forall \ y \in E \ y \neq x.
\]

Since Schu’s results [2,3], the modified Mann and Ishikawa iteration schemes have been studied extensively by various authors to approximate fixed points of asymptotically nonexpansive mappings (see [2-8] and references therein).

Tan and Xu [4] extended Schu’s result [2,3] from Hilbert spaces to the case of uniformly convex Banach spaces, and from the modified Mann iteration process to the modified Ishikawa iteration process defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n \\
y_n = (1 - \beta_n)x_n + \beta_n T^n x_n.
\] (1.1)

In the present convergence theorems [1-8], the condition \(0 < a \leq \alpha_n \leq b < 1\) is must needed. The purpose of this article is still to study the modified Ishikawa iteration process (1.1), not only weak and strong convergence theorems are obtained but also the restriction \(0 < a \leq \alpha_n \leq b < 1\) on \(\{\alpha_n\}\) are relaxed. The results of this article extend and improve the results of many authors.

In order to prove our theorems, the following lemmas will be useful.

**Lemma 1.1.** [2] Let \(K\) be a nonempty convex subset of a linear normed space and \(T : K \to K\) be a uniformly \(L\)-Lipschitzian mappings. For any given \(x_0 \in E\) and real sequences \(\{\alpha_n\}, \{\beta_n\}\) in \([0, 1]\), the sequence \(\{x_n\}\) is the modified Ishikawa iteration sequence defined by (1.1). Then
\[
\|Tx_n - x_n\| \leq c_n + c_{n-1}L(1 + 3L + 2L^2) \quad \forall n \geq 2
\]
where \(c_n = \|T^n x_n - x_n\|\).

**Lemma 1.2.** [3] Let \(E\) be a uniformly convex Banach space, \(\{t_n\}\) a real sequence such that \(0 < a \leq t_n \leq b < 1, \forall n \geq 1\) for some \(a, b \in (0, 1)\), and \(\{x_n\}, \{y_n\}\) are sequences in \(E\) such that
\[
\limsup_{n \to \infty} \|x_n\| \leq d, \quad \limsup_{n \to \infty} \|y_n\| \leq d
\]
\[
\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = d,
\]
then \(\lim_{n \to \infty} \|x_n - y_n\| = 0\), where \(d \geq 0\) is a constant.

**Lemma 1.3.** [5] Let \(\{a_n\}, \{b_n\}\) be sequences of nonnegative real numbers satisfying the inequality
\[
a_{n+1} \leq (1 + b_n)a_n, n \geq 1,
\]
Asymptotically nonexpansive mapping

2. Main results

Theorem 2.1 Let $E$ be a uniformly convex Banach space, $K$ a nonempty closed convex subset of $E$ and $T : K \to K$ an asymptotically nonexpansive mapping with nonempty fixed points set $F(T)$ and with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < +\infty$. \{\alpha_n\}, \{\beta_n\} are real sequences in $[0, 1]$ and there exists a subsequence $\{n_k\}_{k=0}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that, $0 < a \leq \alpha_{n_k} \leq b < 1, 0 \leq \beta_{n_k} \leq b < 1$, for any given constants $a, b \in (0, 1)$. Then for any given $x_1 \in K$, we have

$$\lim_{n \to \infty} \|x_{n_k} - Tx_{n_k}\| = 0,$$

where $\{x_n\}$ is modified Ishikawa iteration sequence defined by (1.1).

Proof For any given $p \in F(T)$, we have

$$\|x_{n+1} - p\| = \|(1 - \alpha_n)(x_n - p) + \alpha_n(T^n y_n - p)\|$$

$$\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T^n y_n - p\|$$

$$\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n k_n\|y_n - p\|. \quad (2.1)$$

$$\|y_n - p\| = \|(1 - \beta_n)(x_n - p) + \beta_n(T^n x_n - p)\|$$

$$\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|T^n x_n - p\|$$

$$\leq (1 - \beta_n)\|x_n - p\| + \beta_n k_n\|x_n - p\|. \quad (2.2)$$

Substituting (2.2) into (2.1), we get

$$\|x_{n+1} - p\| \leq [1 + \alpha_n(1 + k_n \beta_n)(k_n - 1)]\|x_n - p\|$$

by the convergence of $\{k_n\}$ and $\alpha_n, \beta_n \in [0, 1]$, then there exists some $M > 0$ such that

$$\|x_{n+1} - p\| \leq [1 + M(k_n - 1)]\|x_n - p\|.$$

Therefore, by condition $\sum_{n=1}^{\infty} (k_n - 1) < +\infty$ and Lemma 1.3, we know that, the limit $\lim_{n \to \infty} \|x_n - p\|$ exists. Setting $\lim_{n \to \infty} \|x_n - p\| = d$, then we have

$$\lim_{n \to \infty} \|x_{n_k} - p\| = d. \quad (2.3)$$

Next, it follows from (2.2) that

$$\|T^{n_k} y_{n_k} - p\| \leq k_{n_k}\|y_{n_k} - p\|$$

$$\leq k_{n_k}(1 - \beta_{n_k})\|x_{n_k} - p\| + k_{n_k}^2 \beta_{n_k}\|x_{n_k} - p\|$$

$$\leq k_{n_k}^2\|x_{n_k} - p\|.$$

Thus

$$\limsup_{k \to \infty} \|T^{n_k} y_{n_k} - p\| \leq \limsup_{k \to \infty} k_{n_k}^2 \|x_{n_k} - p\| = d. \quad (2.4)$$

Since

$$\lim_{k \to \infty} \|\alpha_{n_k}(T^{n_k} y_{n_k} - p) + (1 - \alpha_{n_k})(x_{n_k} - p)\| = \lim_{k \to \infty} \|x_{n_k+1} - p\| = d. \quad (2.5)$$
By condition of theorem 2.1, lemma 1.2 and (2.3)(2.4)(2.5) we obtain that
\[
\lim_{k \to \infty} \|T^{n_k}y_{n_k} - x_{n_k}\| = 0. \tag{2.6}
\]
It follows from the condition \(0 \leq \beta_{n_k} \leq b < 1\) that
\[
\|T^{n_k}x_{n_k} - x_{n_k}\| \leq \|T^{n_k}x_{n_k} - T^{n_k}y_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\|
\leq k_{n_k}\|x_{n_k} - y_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\|
\leq k_{n_k}\beta_{n_k}\|T^{n_k}x_{n_k} - x_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\|
\leq k_{n_k}b\|T^{n_k}x_{n_k} - x_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\|
\]
which implies that
\[
(1 - k_{n_k}b)\|T^{n_k}x_{n_k} - x_{n_k}\| \leq \|T^{n_k}y_{n_k} - x_{n_k}\|.
\]
Therefore, it is easy to see that
\[
\lim_{n \to \infty} \|T^{n_k}x_{n_k} - x_{n_k}\| = 0,
\]
using lemma 1.1, we obtain that
\[
\lim_{n \to \infty} \|x_{n_k} - Tx_{n_k}\| = 0.
\]
This completes the proof.

Theorem 2.2 Let \(E\) be a uniformly convex Banach space, \(K\) a nonempty closed convex subset of \(E\) and \(T : K \to K\) a completely continuously asymptotically nonexpansive mapping with nonempty fixed points set and with sequence \(\{k_n\}\) such that \(\sum_{n=1}^{\infty}(k_n - 1) < +\infty\). Let \(\alpha_n\), \(\beta_n\) are real sequences in \([0, 1]\) and there exists a subsequence \(\{n_k\}_{k=0}^{\infty}\) of \(\{n\}_{n=1}^{\infty}\) such that,
\[
0 < a \leq \alpha_{n_k} \leq b < 1, 0 \leq \beta_{n_k} \leq b < 1, \text{ for some } a, b \in (0, 1).
\]
Then for any given \(x_1 \in K\), the modified Ishikawa iteration sequence (1.1) converges strongly to a fixed point of \(T\).

Proof From the proof of theorem 2.1 we know the \(\{x_n\}\) is bounded, since \(T\) is completely continuous, then there exists subsequence \(\{Tx_{n_{k_i}}\}\) of \(\{Tx_{n_k}\}\) such that \(\lim_{i \to \infty} Tx_{n_{k_i}} = p_0\). Thus it follows from Theorem 2.1 that \(\lim_{i \to \infty} x_{n_{k_i}} = p_0\), then it is easy to see that \(Tp_0 = p_0\), that is \(p_0 \in F(T)\), where \(F(T)\) denote the fixed points set of \(T\). Because \(\lim_{n \to \infty} \|x_n - p_0\|\) exists, consequently, we have \(\lim_{n \to \infty} x_n = p_0\). This completes the proof.

Theorem 2.3 Let \(E\) be a uniformly convex Banach space which satisfies the Opial’s condition, \(K\) a nonempty closed convex subset of \(E\) and \(T : K \to K\) be an asymptotically nonexpansive mapping with nonempty fixed points set \(F(T)\), if \(\sum_{n=1}^{\infty}(k_n - 1) < \infty\) and there exists a subsequence \(\{n_k\}_{k=1}^{\infty}\) of \(\{n\}_{n=1}^{\infty}\) such that \(0 < a \leq \alpha_{n_k} \leq b < 1, \beta_{n_k} \leq b < 1\) for any given constants \(a, b \in (0, 1)\), then subsequence \(\{x_{n_k}\}\) defined by (1.1) converges weakly to a fixed point of \(T\).

Proof It follows from the proof of Theorem 2.1 that \(\{x_n\}\) is bounded, since \(E\) is uniformly convex, every bounded subset of \(E\) is weakly compact, so that
there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \{x_{n_k}\} \) converges weakly to a point \( q \in K \). Therefore, it follows from theorem 2.1 that
\[
\lim_{n \to \infty} \|x_{n_k} - Tx_{n_k}\| = 0.
\]

By lemma 1.4, we know \( I - T \) is demi-closed, so that \( q \in F(T) \).

Finally, we prove that the sequence \( \{x_{n_k}\} \) converges weakly to \( q \). In fact, suppose this is not true, then there must exists a subsequence \( \{x_{n_kj}\} \subset \{x_{n_k}\} \) such that \( \{x_{n_kj}\} \) converges weakly to another \( q_1 \in K, q_1 \neq q \). Then, by the same method given above, we can also prove that \( q_1 \in F(T) \).

Because, we have proved that, for any \( p \in F(T) \), the limit \( \lim_{n \to +\infty} \|x_n - p\| \) exists. Then we can let
\[
\lim_{n \to \infty} \|x_n - q\| = d_1, \quad \lim_{n \to \infty} \|x_n - q_1\| = d_2,
\]
by Opial’s condition of \( E \), we have
\[
d_1 = \limsup_{i \to \infty} \|x_{n_{ki}} - q\| < \limsup_{i \to \infty} \|x_{n_{ki}} - q_1\| = d_2,
\]
\[
= \limsup_{j \to \infty} \|x_{n_{kj}} - q_1\| < \limsup_{j \to \infty} \|x_{n_{kj}} - q\| = d_1.
\]
This is a contradiction, hence \( q = q_1 \). This implies that \( \{x_{n_k}\} \) converges weakly to a fixed point of \( T \), this completes the proof.

**Theorem 2.4** Let \( E \) be a uniformly convex Banach space which satisfies the Opial’s condition, \( K \) a nonempty closed convex subset of \( E \) and \( T : K \to K \) an asymptotically nonexpansive mapping with nonempty fixed points set \( F(T) \) and with sequence \( \{k_n\} \) such that \( \sum_{n=1}^{\infty} (k_n - 1) < +\infty \). \( \{\alpha_n\}, \{\beta_n\} \) are real sequences in \([0, 1]\) which satisfy the following conditions

1. There exists a constant \( d \in (0, 1) \) such that \( 0 \leq \alpha_n \leq d < 1 \);
2. There exists a subsequence \( \{n_k\} \) of \( \{n\} \) such that \( n_{k+1} - n_k \leq N \) for some given constant \( N > 0 \) and \( 0 < a \leq \alpha_{n_k} \leq b < 1 \) for some given constants \( a,b \in (0, 1) \);
3. \( 0 \leq \beta_{n_k} \leq c < 1 \) for some given constant \( c \in (0, 1) \).

Then the Ishikawa iteration sequence \( \{x_n\} \) defined by (1.2) converges weakly to a fixed point of \( T \).

**Proof.** By using the theorem 2.1 we have
\[
\lim_{k \to \infty} \|x_{n_k} - Tx_{n_k}\| = 0. \quad (2.7)
\]
On the other hand, if \( n \neq n_k \), then \( \alpha_n \to 0 \) as \( n \to \infty \), so that
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \alpha_n \|x_n - T^n y_n\| = 0. \quad (2.8)
\]
Therefore, we obtain that
\[
\|x_n - Tx_n\| \leq \|x_n - x_{n_k}\| + \|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tx_n\|
\]
\[
\leq (1 + k_1)\|x_n - x_{n_k}\| + \|x_{n_k} - Tx_{n_k}\|
\]
\[
\leq (1 + k_1) \sum_{i=n}^{n_k-1} \|x_i - x_{i+1}\| + \|x_{n_k} - Tx_{n_k}\| \tag{2.9}
\]
where $0 < n_k - n \leq N$. It follows from (2.7)-(2.9) that
\[
\lim_{n \to \infty, n \neq n_k} \|x_n - Tx_n\| = 0.
\]
Hence this together with (2.7) we have, for all $n \geq 1$, that
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0.
\]
By using the standard method as in the theorem 2.3, we know the Ishikawa iteration sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of $T$. This completes the proof.

Remark. For example: In the theorem 2.4, the parameters $\{\alpha\}, \{\beta_n\}$ can be chosen as follows
\[
\alpha_n = \begin{cases} 
\frac{1}{n+1} & \text{if } n \neq 10m, \\
\frac{1}{2} & \text{if } n = 10m.
\end{cases} \quad m = 1, 2, 3, \ldots
\]
\[
\beta_n = \begin{cases} 
\beta_n \text{ chosen arbitrarily} & \text{if } n \neq 10m, \\
\frac{1}{2} & \text{if } n = 10m.
\end{cases} \quad m = 1, 2, 3, \ldots
\]

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