

Eigenvalue Estimates for Hill's Equation with Periodic Coefficients¹

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Abstract

In this paper, by using the new form of discriminant, a set of new eigenvalue estimates of $x'' + \lambda p(t)x = 0$ is obtained.

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1 Introduction

We consider the real differential equation

$$x'' + \lambda p(t)x = 0 \quad (1.1)$$

where $p \in C^1$, $p(t) > 0$, $p(t+T) = p(t)$, λ is a real parameter.

According to Liapunoff's oscillation theorems ([1]-[3]), there is a sequence of eigenvalues

$$0 = \Lambda_0 < \lambda_1 \leq \Lambda_1 < \lambda_2 \leq \Lambda_2 < \dots \rightarrow 0$$

such that (1.1) is stable for $\lambda \in (\Lambda_i, \lambda_{i+1})$ ($i = 0, 1, 2, \dots$) and (1.1) is unstable for $\lambda \in (\lambda_i, \Lambda_i)$ ($i = 1, 2, 3, \dots$).

In 1975, under the assumption $p(t) \geq 0$, Hell^[1] gave the lower estimates for λ_n and Λ_n as follows

$$\lambda_n \geq \frac{4n^2}{T \int_0^T p(t)dt}, \quad \Lambda_n \geq \frac{4(n+1)^2 \cos^2 \frac{\pi}{2} \cdot \frac{1}{n+1}}{T \int_0^T p(t)dt}. \quad (1.2)$$

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For more works on this direction, one could refer to [4-17] and the references cited therein.

In [9], by developing some new analysis technique, Shi obtained a new form of discriminant for (1.1). By applying this new form of discriminant, Shi *et al* [10] further obtained a calculation method for characteristic multiplier of Hill's equation with high convergence rate and small error. The aim of this paper is, by using the new form of discriminant of [9], to obtain a set of more accurate eigenvalue estimates of equation (1.1).

2 The discriminant of $x'' + \lambda p(t)x = 0$ and estimate of remained term

We first consider the system (1.1) with periodic $T = \pi$, at the end of next section, we will deal with the case $T \neq \pi$.

Let's consider equation

$$x'' + \lambda p(t)x = 0 \quad (2.1)$$

where $p \in C^1, p(t) > 0, p(t + \pi) = p(t)$, λ is a real parameter.

Let $x_1(t), x_2(t)$ denote the solutions of (2.1) with initial conditions $x_1(0) = 1, x_1'(0) = 0, x_2(0) = 0, x_2'(0) = 1$, then the eigenvalue of matrix $\begin{pmatrix} x_1(\pi) & x_2(\pi) \\ x_1'(\pi) & x_2'(\pi) \end{pmatrix}$ is the characteristic multiplier of (2.1). Denote $\Delta = x_1(\pi) + x_2'(\pi)$, since

$$\det \begin{pmatrix} x_1(\pi) & x_2(\pi) \\ x_1'(\pi) & x_2'(\pi) \end{pmatrix} = \det \begin{pmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{pmatrix} = 1,$$

its characteristic multipliers ρ_1, ρ_2 satisfy the following equation

$$\rho^2 - \Delta\rho + 1 = 0. \quad (2.2)$$

Obviously

$$\rho_{1,2} = \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 - 1}, \quad (2.3)$$

therefore, if $|\Delta| < 2$, then $|\rho_1| = |\rho_2| = 1$, and Eq.(2.1) is stable; if $|\Delta| > 2$, then $|\rho_1| > 1, |\rho_2| < 1$, and Eq.(2.1) is unstable. Δ is called the discriminant of equation (2.1).

In [9], by means of successive approximation method, Shi showed that the discriminant Δ (see also (1.5) of [10,P.58]) takes the form

$$\Delta = 2 \cos \int_0^\pi \sqrt{\lambda p(t)} dt + \sum_{n=1}^{\infty} \frac{1}{2^{4n-1}} \int_0^\pi \int_0^{t_1} \cdots \int_0^{t_{2n-1}} \cos \Phi(t_1 \cdots t_{2n}) \cdot \prod_{i=1}^{2n} \frac{p'(t_i)}{p(t_i)} dt_{2n} \cdots dt_1 \stackrel{\text{def}}{=} 2 \cos \int_0^\pi \sqrt{\lambda p(t)} dt + \sum_{n=1}^{\infty} \Delta_n, \quad (2.4)$$

where

$$\Phi(t_1 \cdots t_{2n}) = \int_0^\pi \sqrt{\lambda p(t)} dt - 2 \int_{t_2}^{t_1} \sqrt{\lambda p(s)} ds - \cdots - 2 \int_{t_{2n}}^{t_{2n-1}} \sqrt{\lambda p(s)} ds. \tag{2.5}$$

Owing to the formula

$$\int_0^a \int_0^{t_1} \int_0^{t_{k-1}} f(t_1) \cdots f(t_k) dt_k \cdots dt_1 = \frac{1}{k!} \left(\int_0^a f(t) dt \right)^k,$$

we obtain that

$$\begin{aligned} \left| \sum_{n=1}^\infty \Delta_n \right| &\leq \sum_{n=1}^\infty \frac{1}{2^{4n-1}} \int_0^\pi \int_0^{t_1} \cdots \int_0^{t_{2n-1}} \prod_{i=1}^{2n} \frac{|p'(t_i)|}{p(t_i)} dt_{2n} \cdots dt_1 \\ &= \sum_{n=1}^\infty \frac{1}{2^{4n-1}} \cdot \frac{1}{(2n)!} \left(\int_0^\pi \frac{|p'(t)|}{p(t)} dt \right)^{2n}. \end{aligned}$$

Denote

$$\int_0^\pi \frac{|p'(t)|}{p(t)} dt = v. \tag{2.6}$$

It follows from $2 \sum_{n=1}^\infty \frac{x^{2n}}{(2n)!} = e^x + e^{-x} - 2$ that

$$\left| \sum_{n=1}^\infty \Delta_n \right| \leq e^{v/4} + e^{-v/4} - 2. \tag{2.7}$$

Remark 1. For any $p(t)$ which satisfies $p(t) \in C^1$, $p(t) > 0$, $p(t + \pi) = p(t)$, v in (2.6) could be compute directly. For example, if $p(t)$ has two extreme points τ_1 and τ_2 on the $(0, \pi)$, without loss of generality, suppose τ_1 is the maximum point and τ_2 is the minimum point, then

$$\begin{aligned} v &= \int_0^\pi \frac{|p'(t)|}{p(t)} dt = \int_0^{\tau_1} \frac{p'(t)}{p(t)} dt - \int_{\tau_1}^{\tau_2} \frac{p'(t)}{p(t)} dt + \int_{\tau_2}^\pi \frac{p'(t)}{p(t)} dt \\ &= \ln \frac{p(\tau_1)}{p(0)} - \ln \frac{p(\tau_2)}{p(\tau_1)} + \ln \frac{p(\pi)}{p(\tau_2)} = 2 \ln \frac{p(\tau_1)}{p(\tau_2)}. \end{aligned}$$

3 Eigenvalue Estimates for (2.1)

Let v be defined by (2.6) and let

$$\varphi_0 = \frac{1}{2}(4 - e^{v/4} - e^{-v/4}). \tag{3.1}$$

If $v < 5.2$, one could easily verify that $\varphi_0 > 0$. On the other hand, from $e^x + e^{-x} > 2$ one also could deduce $\varphi_0 < 1$.

Theorem 3.1 Assume that $v < 5.2$ and

$$\lambda \in \left(\left(\frac{n\pi + \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2, \left(\frac{(n+1)\pi - \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2 \right), \quad (n = 0, 1, 2, \dots),$$

then Hill's equation (2.1) is stable.

Proof Suppose

$$\left(\frac{n\pi + \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2 < \lambda < \left(\frac{(n+1)\pi - \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2,$$

we obtain that

$$n\pi + \cos^{-1} \varphi_0 < \int_0^\pi \sqrt{\lambda p(t)} dt < (n+1)\pi - \cos^{-1} \varphi_0.$$

It follows from (2.4) and (2.7) that

$$\begin{aligned} |\Delta| &\leq 2 \left| \cos \int_0^\pi \sqrt{\lambda p(t)} dt \right| + \left| \sum_{n=1}^{\infty} \Delta_n \right| \\ &< 2 \cos(\cos^{-1} \varphi_0) + e^{v/4} + e^{-v/4} - 2 = 2. \end{aligned}$$

Therefore, Hill's equation (2.1) is stable. The proof of Theorem 3.1 is completed.

Theorem 3.2 If $v < 5.2$, then the eigenvalues of (2.1) satisfies following estimates

$$\left(\frac{n\pi - \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2 \leq \lambda_n \leq \Lambda_n \leq \left(\frac{n\pi + \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2 \quad (n = 1, 2, \dots)$$

Proof By Theorem 3.1,

$$\begin{aligned} &\left(\left(\frac{n\pi + \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2, \left(\frac{(n+1)\pi - \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2 \right) \subset (\Lambda_n, \lambda_{n+1}), \quad (n = 0, 1, 2, \dots), \\ &\left(\left(\frac{(n-1)\pi + \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2, \left(\frac{n\pi - \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2 \right) \subset (\Lambda_{n-1}, \lambda_n), \quad (n = 1, 2, \dots). \end{aligned}$$

Hence,

$$\left(\frac{n\pi - \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2 \leq \lambda_n \leq \Lambda_n \leq \left(\frac{n\pi + \cos^{-1} \varphi_0}{\int_0^\pi \sqrt{p(t)} dt} \right)^2, \quad (n = 1, 2, \dots).$$

This completes the proof of Theorem 3.2.

Remark 2. Since $0 < \cos^{-1} \varphi_0 < \frac{\pi}{2}$, it follows that

$$(n\pi - \cos^{-1} \varphi_0)^2 > 4n^2 \text{ for } n \geq 2.$$

Also, from $\cos^2 x = 1 - \sin^2 x < 1 - \left(\frac{2}{\pi}x\right)^2$ it follows that

$$\begin{aligned} 4(n+1)^2 \cos^2 \frac{\pi}{2} \cdot \frac{1}{n+1} &< 4(n+1)^2 \left[1 - \left(\frac{2}{\pi} \cdot \frac{\pi}{2} \cdot \frac{1}{n+1}\right)^2 \right] \\ &< 4(n+1)^2 - 4 < (n\pi - \cos^{-1} \varphi_0)^2 \text{ for } n \geq 3. \end{aligned}$$

On the other hand, by Buniakowski inequality,

$$\left(\int_0^\pi \sqrt{p(t)} dt\right)^2 \leq \int_0^\pi 1^2 dt \cdot \int_0^\pi \left(\sqrt{p(t)}\right)^2 dt = \pi \int_0^\pi p(t) dt.$$

Thus, if $v < 5.2$, for $n \geq 3$, Theorem 3.2 improves the estimates of Hell ([1]).

Now let's consider the general case of system (1.1), i.e., consider system

$$\frac{d^2x(t)}{dt^2} + \lambda p(t)x(t) = 0, \tag{3.2}$$

where $p(t) \in C^1$, $p(t) > 0$, $p(t + T) = p(t)$. Let $t = \frac{T}{\pi}\tau$, then

$$\frac{d^2x\left(\frac{T}{\pi}\tau\right)}{\left(\frac{T}{\pi}\right)^2 d\tau^2} + \lambda p\left(\frac{T}{\pi}\tau\right) x\left(\frac{T}{\pi}\tau\right) = 0. \tag{3.3}$$

Denote $x\left(\frac{T}{\pi}\tau\right) = y(\tau)$, $p\left(\frac{T}{\pi}\tau\right) = q(\tau)$, (3.3) can be rewrite as

$$\frac{d^2y(\tau)}{d\tau^2} + \left(\frac{T}{\pi}\right)^2 \lambda q(\tau)y(\tau) = 0 \tag{3.4}$$

and $q(\tau + \pi) = p\left(\frac{T}{\pi}(\tau + \pi)\right) = p\left(\frac{T}{\pi}\tau + T\right) = p\left(\frac{T}{\pi}\tau\right) = q(\tau)$. Denote

$$v = \int_0^T \frac{|p'(t)|}{p(t)} dt, \tag{3.5}$$

let $t = \frac{T}{\pi}\tau$, then

$$v = \int_0^\pi \frac{|p'\left(\frac{T}{\pi}\tau\right)|}{p\left(\frac{T}{\pi}\tau\right)} \cdot \frac{T}{\pi} d\tau.$$

Since $q(\tau) = p\left(\frac{T}{\pi}\tau\right)$, it follows that

$$q'(\tau) = p'\left(\frac{T}{\pi}\tau\right) \cdot \frac{T}{\pi},$$

therefore,

$$v = \int_0^\pi \frac{|q'(\tau)|}{q(\tau)} d\tau.$$

Let φ_0 be defined by (3.1), by Theorem 3.1, if λ satisfies

$$\left(\frac{n\pi + \cos^{-1}\varphi_0}{\int_0^\pi \sqrt{q(\tau)} d\tau}\right)^2 < \lambda \left(\frac{T}{\pi}\right)^2 < \left(\frac{(n+1)\pi - \cos^{-1}\varphi_0}{\int_0^\pi \sqrt{q(\tau)} d\tau}\right)^2, \quad (3.6)$$

then (3.4) is stable. On the other hand, let $\tau = \frac{\pi}{T}t$,

$$\int_0^\pi \sqrt{q(\tau)} d\tau = \frac{\pi}{T} \int_0^T \sqrt{q\left(\frac{\pi}{T}t\right)} dt = \frac{\pi}{T} \int_0^T \sqrt{p(t)} dt$$

Thus, we had proved

Theorem 3.3. *Suppose $v < 5.2$ and*

$$\lambda \in \left(\left(\frac{n\pi + \cos^{-1}\varphi_0}{\int_0^T \sqrt{p(t)} dt}\right)^2, \left(\frac{(n+1)\pi - \cos^{-1}\varphi_0}{\int_0^T \sqrt{p(t)} dt}\right)^2 \right), \quad (n = 0, 1, 2, \dots),$$

then Hill's equation (3.2) is stable, and the eigenvalues of (3.2) satisfies the estimates

$$\left(\frac{n\pi - \cos^{-1}\varphi_0}{\int_0^T \sqrt{p(t)} dt}\right)^2 \leq \lambda_n \leq \Lambda_n \leq \left(\frac{n\pi + \cos^{-1}\varphi_0}{\int_0^T \sqrt{p(t)} dt}\right)^2.$$

Remark 3 *Obviously, our estimate is more accurate than the results in [1-3].*

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