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Bi-Parametric Optimal Partition Sensitivity Analysis for Perturbed Linear Optimization

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Abstract

In bi-parametric linear optimization (LO), perturbation occurs in both the right hand side (RHS) and objective function coefficient (OFC) data that are different nonzero parameters. In this paper, the bi-parametric LO problem is considered, we want to find the region of the parameters variation where the perturbed problem has still an optimal solution with the same optimal partition for parameters values. We are interested in identifying the regions where optimal partition invariant for LO problem. These regions are referred to as invariancy regions. It is proved that invariancy regions are mesh-like area and are separated by vertical and horizontal lines. We present computable LO problems to identify the associated optimal partition regions for LO problem. The behavior of the optimal value function on these regions is investigated too.

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1 Introduction

Let us consider the bi-parametric perturbed primal LO problem:

$$LP(\triangle b, \triangle c, \epsilon, \lambda) \qquad \min\{(c+\lambda\Delta c)^T x | Ax = b+\epsilon\Delta b, x \geq 0\},$$

where, $A \in R^{m\times n}$, and vectors $c, \triangle c \in R^n$ and $b, \triangle b \in R^m$ are fixed data and $x \in R^n$ is unknown vector and ϵ, λ are real parameters.

We refer to $\triangle b$ and $\triangle c$ as perturbation vectors. In special cases, one of the vectors $\triangle b$ and $\triangle c$ might be zero, or all but one of the components are zero. For parameter value $\epsilon = \lambda = 0$, problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$ is an unperturbed primal LO problem and is denoted shortly by $LP = LP(\triangle b, \triangle c, 0, 0)$. *Standard* LO *problem* refers to the fact that the primal and dual LO problems are in standard form.

The dual of $LP(\triangle b, \triangle c, \epsilon, \lambda)$ is defined as:

$$LD(\triangle b, \triangle c, \epsilon, \lambda) \qquad \max\{(b+\epsilon\Delta b)^T y | A^T y + s = c+\lambda\Delta c, s \geq 0\},$$

where $y \in R^m$, and $s \in R^n$, are unknown vectors. For the parameter values $\epsilon = 0$ and $\lambda = 0$ we denote it shortly by $LD = LD(\triangle b, \triangle c, 0, 0)$. Answering to the question" What happens to optimal solutions when such perturbation occurs in input data?" was one of the first preoccupations of optimizers soon after the simplex method was introduced. The related studies area is known as *parametric programming* and *sensitivity analysis*. A classification of sensitivity analysis for LP was introduced by Koltai and Terlaky [7]. We discuss the optimal partition sensitivity analysis for standard form of LP problem containing two parameters, one in objective function and the other in the right hand side of the constraints. Any vector $x \geq 0$ satisfying the constraints of LP is called a *primal feasible solution* and any vector (y, s) with $s \geq 0$ satisfying the constraints of LD is called a *dual feasible solution*. We refer to the index set $\{1, 2, \ldots, n\}$ as *variables index* set.

In this way, primal and dual feasible solutions can be denoted by x and (y, s), respectively. For any primal-dual feasible solution (x, y, s), the *weak duality* property $b^T y \leq c^T x$ holds. If $b^T y = c^T x$ (*strong duality*), then the feasible solutions x and (y, s) are primal and dual optimal solutions of problems LP and LD, respectively. Consequently, for a primal-dual optimal solution (x^*, y^*, s^*), we have $s^{*T} x^* = 0$. Considering the nonnegativity of variables x^*, and s^* the optimality property can be rewritten as ${s_j}^* {x_j}^* = 0$ for $j \in \{1, 2, \ldots, n\}$. Clearly speaking, for a primal-dual optimal solution (x^*, y^*, s^*), the vectors x^*, y^* and s^*, are complementary.

The *support set* of a nonnegative vector ν is defined as $\sigma(\nu) = \{i | \nu_i > 0\}$. Considering this notation, the strong duality property implies the following equalities:

$$\sigma(x^*) \cap \sigma(s^*) = \emptyset \qquad (1)$$

where (x^*, y^*, s^*) is a primal-dual optimal solution of problems LP and LD. A complementary (optimal) solution (x^*, y^*, s^*) is *primal-dual strictly complementary*, if $s^{*T}x^* = 0$ with $s^* + x^* > 0$. Clearly speaking, for a strictly complementary optimal solution (x^*, y^*, s^*), the following relations hold:

$$\sigma(x^*) \cup \sigma(s^*) = \{1, 2, \ldots, n\}. \qquad (2)$$

By the Goldman-Tucker Theorem [4], the existence of strictly complementary optimal solutions of problems LP and LD is guaranteed if these problems are feasible.

Let $\mathcal{LP}(\triangle b, \triangle c, \epsilon, \lambda)$ and $\mathcal{LD}(\triangle b, \triangle c, \epsilon, \lambda)$ be feasible sets of problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$, respectively. Further, let $\mathcal{LP}^*(\triangle b, \triangle c, \epsilon, \lambda)$ and $\mathcal{LD}^*(\triangle b, \triangle c, \epsilon, \lambda)$ denote their optimal solution sets, correspondingly. When $\epsilon = 0$ and $\lambda = 0$ we denote them shortly by $\mathcal{LP} = \mathcal{LP}(\triangle b, \triangle c, 0, 0)$ and $\mathcal{LD} = \mathcal{LD}(\triangle b, \triangle c, 0, 0)$. Analogously, we let $\mathcal{LP}^* = \mathcal{LP}^*(\triangle b, \triangle c, 0, 0)$ and $\mathcal{LD}^* = \mathcal{LD}^*(\triangle b, \triangle c, 0, 0)$, i.e.,

$$\mathcal{LP}^* = \{x^* | x^* \text{ is an optimal solution in } \mathcal{LP}\}$$
$$\mathcal{LD}^* = \{(y^*, s^*) | (y^*, s^*) \text{ is an optimal solution in } \mathcal{LD}\}.$$

Considering (1) and (2), one can define the following partition:

$$\mathcal{B}_V^x = \{j : x_j^* > 0, \forall j \in \{1, 2, \ldots, n\} \text{ for some } x^* \in \mathcal{LP}^*\}$$
$$\mathcal{N}_V^s = \{j : s_j^* > 0, \forall j \in \{1, 2, \ldots, n\} \text{ for some } (y^*, s^*) \in \mathcal{LD}^*\}.$$

Roughly speaking, in any primal optimal solution x^*, whose index is in $\mathcal{N}_V^s$, is always zero. We denote this partition by $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ refer to it as *variables optimal partition*.

The uniqueness of these partitions is a direct consequence of the convexity of optimal solution sets $\mathcal{LP}^*$ and $\mathcal{LD}^*$. (see e.g.,[3],[8]). In this paper, we survey the outlined results in optimal partition invariancy sensitivity analysis for LP problem with two parameters.

Interior Point Methods solve LO problem in polynomial time [8]. They start from a feasible (or an infeasible) interior point of the positive orthant and generate an interior solution nearby the optimal solution. By using a simple rounding procedure [5], a strictly complementary solution of the LO problem can be obtained in strongly polynomial time and strictly complementary optimal solution of the LO problem provides the optimal partitions too.

Associated with the perturbed problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$, let ϕ denote the *optimal value function* that is defined as:

$$\phi(\triangle b, \triangle c, \epsilon, \lambda) = (c + \lambda \triangle c)^T x^*(\epsilon, \lambda) = (b + \epsilon \triangle b)^T y^*(\epsilon, \lambda)$$

where $(x^*(\epsilon, \lambda), y^*(\epsilon, \lambda), s^*(\epsilon, \lambda)$ is a primal-dual optimal solution of problems LP and LD. Further, we define:

$$\phi(\triangle b, \triangle c, \epsilon, \lambda) = +\infty \text{ if } \mathcal{LP}^*(\triangle b, \triangle c, \epsilon, \lambda) = \emptyset$$
$$\phi(\triangle b, \triangle c, \epsilon, \lambda) = -\infty \text{ if } \mathcal{LP}^*(\triangle b, \triangle c, \epsilon, \lambda) = \emptyset \text{ and it is unbounded.}$$

By fixing $\triangle b$ and $\triangle c$ that are nonzero vectors, ϕ is bi-variate function of ϵ, λ. Remember that, perturbation occurs in the RHS and/or the OFC data. If perturbation in the RHS and the OFC data happens with identical parameter, the problem is referred to as *uni-parametric programming* problem and if these data vary independently, the problem is referred to as *bi- parametric programming* problem.

There are different approaches in parametric programming. One of them is so-called *optimal partition invariancy* sensitivity analysis. In this approach, one wants to identify the range of parameters variation where the optimal partition remains invariant. The first study with this point of view for optimal partition was started by Adler and Monteiro [1]. The cases when $\triangle b$ or $\triangle c$ is zero, have been studied in [8]. Further, In these cases, the range of parameter variation is an interval of the real line and was referred to as *invariancy interval* and the points that distinguish these intervals as *transition points*. All of these studies are considered in uni-parameters [2]. There is only a simple illustrative example in [6] that the authors have considered independently two as parameters and calculated the invariancy region.

In this paper, we consider the problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$, when $\triangle b$ and $\triangle c$ are nonzero vectors and ϵ and λ are not necessarily equal.

We refer to this region as *invariancy region*. It will be proved that the region is a rectangle (if it is not a singleton or a line segment) and the neighboring regions are rectangles as well. It means that all invariancy regions altogether generate a mesh-like area in R^2 constructed by vertical and horizontal (half-) line segments.

Let us refer to the lines outlined here as transition lines and the region between obtained transition lines as *Optimal(Variables)Partitions Invariancy* (OPI) region. Thus, any transition line is a proper OPI region (a singleton or a line segment). The *actual* OPI region is the one which contains the actual parameter values $\epsilon = \lambda = 0$. It should be mentioned that it might be the singleton $\{(0,0)\}$ when $\epsilon_l = \epsilon_u = 0$ and $\lambda_l = \lambda_u = 0$.

The paper is organized as follows: Section 2 contains some necessary concepts and the convexity of invariancy regions is proved. The simultaneous perturbation case, when variation occurs in both the Right Hand Side (RHS) and the Objective Function Coefficient (OFC) data of LP, is considered and the behavior of the optimal value function on this region is studied. Auxiliary LO problems are presented in this section that allows us to identify the associated regions. The interrelation of these regions are studied as well. A simple example is presented in Section 3 to illustrate the results.

2 Invariancy Regions

Let us introduce some simplifying concepts and notations. Let $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the optimal partitions of the index set $\{1, 2, \ldots, n\}$ for problems LP and LD. Thus, the invariancy region, where the optimal partition of this set for problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b,\ \triangle c, \epsilon, \lambda)$ at any point (ϵ, λ) in this region is identical with $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$, is not empty and the origin $(0, 0)$ belongs to this region. We refer to the invariancy region which contains the origin as *actual invariancy region* and denote it by $\mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda)$.

Bear in mind that for $\triangle c = 0$, problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$ reduce to the following problems, respectively:

$LP(\triangle b, \epsilon)$ $\quad \min\{c^T x \mid Ax = b + \epsilon\triangle b, x \geq 0\}$

$LD(\triangle b, \epsilon)$ $\quad \max\{(b + \epsilon\triangle b)^T y \mid A^T y + s = c, s \geq 0\}$.

Let $\mathcal{IR}(\triangle b, \epsilon)$ denote the invariancy region for problem $LP(\triangle b, \epsilon)$. It was proved that the dual optimal solution set $\mathcal{LD}^*(\triangle b, \epsilon)$ is invariant on the invariancy region $\mathcal{IR}(\triangle b, \epsilon)$ (see Theorem IV.56 in [8]).The following lemma presents auxiliary LO problems for identifying the end points of this interval.

Lemma 2.1 *Consider the primal and dual problems LP and LD, respectively. Further, let $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the variables optimal partition of problems LP and LD, respectively. Then, the optimal partition Π_V is invariant for any $\epsilon \in (\epsilon_l^{LP}, \epsilon_u^{LP})$, where ϵ_l^{LP} and ϵ_u^{LP} are obtained by minimizing and maximizing ϵ on the following set:*

$$\{\epsilon \mid Ax - \epsilon\triangle b = b, x_{\mathcal{B}} \geq 0, x_{\mathcal{N}} = 0\} \tag{3}$$

proof:The proof is similar to Theorem IV.73 in [8]

Now for $\triangle b = 0$, we have the following reduced primal and dual LO problems:

$LP(\triangle c, \lambda)$ $\quad \min\{(c + \lambda\triangle c)^T x \mid Ax = b, x \geq 0\}$

$LD(\triangle c, \lambda)$ $\quad \max\{b^T y \mid A^T y + s = c + \lambda\triangle c, s \geq 0\}$

Let $\mathcal{IR}(\triangle c, \lambda)$ denote the invariancy region for general problem $LP(\triangle c, \lambda)$. It was proved that the primal optimal solution set $\mathcal{LP}^*(\triangle c, \lambda)$ is invariant on the invariancy interval $\mathcal{IR}(\triangle c, \lambda)$ (see Theorem IV.60 in [8]).The following lemma presents auxiliary LO problems for identifying the end points of this region.

Lemma 2.2 *Consider the primal and dual problems LP and LD, respectively. Further, let $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the variable optimal partition of problems LP and LD, respectively. Then, the optimal partitions Π_V is invariant for any $\lambda \in (\lambda_l^{LD}, \lambda_u^{LD})$, where λ_l^{LD} and λ_u^{LD} are obtained by minimizing and maximizing $\lambda^{(1)}$ on the following set:*

$$\{\lambda \mid A^T y + s - \lambda \triangle c = c, s_{\mathcal{N}} \geq 0, s_{\mathcal{B}} = 0\} \qquad (4)$$

proof: The proof is similar to Theorem IV.75 in [8]

remark: *Observe that the actual (region which contains the origin) invariancy interval* $\mathcal{IR}(\triangle b, \epsilon)$ *might be the singleton* $\{0\}$. *This situation occurs when solving auxiliary LO problem* (3) *leads to* $\epsilon_l^{LP} = \epsilon_u^{LP} = 0$. *Moreover, if one of these problems is unbounded, the actual invariancy interval* $\mathcal{IR}(\triangle b, \epsilon)$ *is unbounded too. Analogous argument is valid for the actual invariancy interval* $\mathcal{IR}(\triangle c, \lambda)$. *Furthermore, all auxiliary LO problems* $(3)-(4)$ *can be solved in polynomial time.*

2.1 Fundamental properties

The following lemma proves the convexity of the actual invariancy region. Analogous reasoning is valid for other invariancy regions.

Lemma 2.3 *The actual invariancy region* $\mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda)$ *is a convex set.*

proof: Without the loss of generality, one may assume that the actual invariancy region is not the singleton $\{(0,0)\}$. Let (ϵ_1, λ_1) and (ϵ_2, λ_2) be two arbitrary points in $\mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda)$. Further, let $(x_1(\epsilon_1, \lambda_1), y_1(\epsilon_1, \lambda_1), s_1(\epsilon_1, \lambda_1))$ and $(x_2(\epsilon_2, \lambda_2), y_2(\epsilon_2, \lambda_2), s_2(\epsilon_2, \lambda_2))$ be strictly complementary optimal solutions of problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and LD $(\triangle b, \triangle c, \epsilon, \lambda)$ at (ϵ_1, λ_1) and (ϵ_2, λ_2), respectively. For an arbitrary point (ϵ, λ) on the line segment between two points (ϵ_1, λ_1) and (ϵ_2, λ_2), there is a $\theta \in (0, 1)$, such that:

$$\epsilon = \theta\epsilon_1 + (1-\theta)\epsilon_2 \text{ and } \lambda = \theta\lambda_1 + (1-\theta)\lambda_2$$

Let us define

$$x(\epsilon, \lambda) = \theta x_1(\epsilon_1, \lambda_1) + (1-\theta)x_2(\epsilon_2, \lambda_2), \qquad (5)$$
$$y(\epsilon, \lambda) = \theta y_1(\epsilon_1, \lambda_1) + (1-\theta)y_2(\epsilon_2, \lambda_2), \qquad (6)$$
$$s(\epsilon, \lambda) = \theta s_1(\epsilon_1, \lambda_1) + (1-\theta)s_2(\epsilon_2, \lambda_2). \qquad (7)$$

It is easy to verify the $x(\epsilon, \lambda) \in \mathcal{LP}(\triangle b, \triangle c, \epsilon, \lambda)$ and $(y(\epsilon, \lambda), s(\epsilon, \lambda)) \in \mathcal{LD}(\triangle b, \triangle c, \epsilon, \lambda)$. On the other hand, $\sigma(x(\epsilon, \lambda)) = \sigma(x_1(\epsilon_1, \lambda_1)) \cup \sigma(x_2(\epsilon_2, \lambda_2)) = \mathcal{B}$ and $\sigma(s(\epsilon, \lambda) = \sigma(s_1(\epsilon_1, \lambda_1)) \cup \sigma(s_2(\epsilon_2, \lambda_2)) = \mathcal{N}$ that prove the optimality of these solutions for problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$, as well as the invariancy of the optimal partition $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ at (ϵ, λ). The proof is complete.

According to Lemma 2.4 to identify an invariancy region, it is enough to identify its border. Observe that the invariancy region might be unbounded.

2.2 Identifying the invariancy regions

Now, we present a fundamental theorem that talks about a relationship between the actual invariancy region $\mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda)$ and two actual invariancy regions $\mathcal{IR}(\triangle b, \epsilon)$ and $\mathcal{IR}(\triangle c, \lambda)$. This relationship plays a significant role in identifying the actual invariancy region $\mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda)$ and speaks of the fact that this identification can be done in polynomial time. Analogous statement can be used to identify all possible invariancy regions.

Theorem 2.4 *Consider the bi-parametric LO problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$. Let $\mathcal{IR}(\triangle b, \epsilon)$ be the actual invariancy interval of problems $LP(\triangle b, \epsilon)$ and $LD(\triangle b, \epsilon)$. Moreover, let $\mathcal{IR}(\triangle c, \lambda)$ be the actual invariancy interval of problems $LP(\triangle c, \lambda)$ and $LD(\triangle c, \lambda)$. Then*

$$\mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda) = \mathcal{IR}(\triangle b, \epsilon) \times \mathcal{IR}(\triangle c, \lambda).$$

proof: let $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the optimal partition of index $\{1, 2, ..., n\}$ for problems LP and LD. Moreover, let (x^*, y^*, s^*) be a strictly complementary optimal solution of these problems. Consequently, $\sigma(x^*) = \mathcal{B}_V^x$ and $\sigma(s^*) = \mathcal{N}_V^s$.

First we prove the inclusion

$$\mathcal{IR}(\triangle b, \epsilon) \times \mathcal{IR}(\triangle c, \lambda) \subseteq \mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda).$$

Let $(\overline{\epsilon}, \overline{\lambda}) \in \mathcal{IR}(\triangle b, \epsilon) \times \mathcal{IR}(\triangle c, \lambda)$ be fixed parameter. Since the dual optimal solution set $\mathcal{LD}^*(\triangle b, \epsilon)$ is invariant on $\mathcal{IR}(\triangle b, \epsilon)$, one might consider $(x^*(\overline{\epsilon}), y^*, s^*)$ as a strictly complementary optimal solution of problems $LP(\triangle b, \overline{\epsilon})$ and $LD(\triangle b, \overline{\epsilon})$. Consequently, equality $\sigma(x^*(\overline{\epsilon})) = \mathcal{B}_V^x$ holds. Since the primal optimal solution set $\mathcal{LP}^*(\triangle c, \lambda)$ is invariant on $\mathcal{IR}(\triangle c, \lambda)$, one might consider $(x^*, y^*(\overline{\lambda}), s^*(\overline{\lambda}))$ as a strictly complementary optimal solution of problems $LP(\triangle c, \overline{\lambda})$ and $LD(\triangle c, \overline{\lambda})$. Consequently, equality $\sigma(s^*(\overline{\lambda})) = \mathcal{N}_V^s$ holds, too. It is obvious that $(x^*(\overline{\epsilon}), y^*(\overline{\lambda}), s^*(\overline{\lambda}))$ is feasible solution of problems $\mathcal{LP}^*(\triangle b\ , \triangle c, \overline{\epsilon}, \overline{\lambda}) \times \mathcal{LD}^*(\triangle b, \triangle c, \overline{\epsilon}, \overline{\lambda})$. Moreover, equality $x^*(\overline{\epsilon})^T s^*(\overline{\lambda}) = 0$ holds, that proves the optimality of this solution. Then $(\overline{\epsilon}, \overline{\lambda}) \in \mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda)$.

Now, let $(\overline{\epsilon}, \overline{\lambda}) \in \mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda)$ be given and $(x^*(\overline{\epsilon}), y^*(\overline{\lambda}), s^*(\overline{\lambda}))$ be a strictly complementary optimal solution of problems $LP(\triangle b, \triangle c, \overline{\epsilon}, \overline{\lambda})$ and $LD(\triangle b, \triangle c, \overline{\epsilon}, \overline{\lambda})$. Thus, $\sigma(x^*(\overline{\epsilon})) = \mathcal{B}_V^x$ and $\sigma(s^*(\overline{\lambda})) = \mathcal{N}_V^s$. Therefore, $(x^*(\overline{\epsilon}), y^*, s^*)$ and $(x^*, y^*(\overline{\lambda}), s^*(\overline{\lambda}))$ are strictly complementary optimal solution of problems $LP(\triangle b, \overline{\epsilon})$, $LD(\triangle b, \overline{\epsilon})$, $LP(\triangle c, \overline{\lambda})$ and $LD(\triangle c, \overline{\lambda})$. The inclusion $\mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda) \subseteq \mathcal{IR}(\triangle b, \epsilon) \times \mathcal{IR}(\triangle c, \lambda)$ follows of $\sigma(x^*(\overline{\epsilon})) = \mathcal{B}_V^x$ and $\sigma(s^*(\overline{\lambda})) = \mathcal{N}_V^s$. The proof is complete.

Corollary 2.5 *Consider the bi-parametric LO problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$. Let $\mathcal{IR}(\triangle b, 0)$ be the actual invariancy interval of problems $LP(\triangle b, 0)$ and $LD(\triangle b, 0)$. Moreover, let $\mathcal{IR}(\triangle c, 0)$ be the actual invariancy interval of problems $LP(\triangle c, 0)$ and $LD(\triangle c, 0)$. Then*

$$\mathcal{IR}(\triangle b, \triangle c, 0, 0) = \mathcal{IR}(\triangle b, 0) \times \mathcal{IR}(\triangle c, 0).$$

2.3 Optimal value function on an invariancy region

In this subsection, we investigate the behavior of the optimal value function on invariancy regions.

Theorem 2.6 *The optimal value function $\phi(\triangle b, \triangle c, \epsilon, \lambda)$ is a bivariate quadratic function on actual invariancy region $\mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda)$.*

proof: When the actual invariancy region is the $\{(0,0)\}$, there is nothing to prove. Let the actual invariancy region be a nontrivial one containing the origin. Further, let (ϵ_1, λ_1), (ϵ_2, λ_2) and (ϵ_3, λ_3) be three arbitrary points in the actual invariancy region . Let (x_1, y_1, s_1), (x_2, y_2, s_2) and (x_3, y_3, s_3) be primal-dual optimal solutions for these three points, respectively. Let (ϵ, λ) be a point in the interior of the triangle made of these three points as vertices. Therefore, there are $\theta_1, \theta_2 \in (0,1)$ with $0 < \theta_1 + \theta_2 < 1$ such that

$$\epsilon = \epsilon_3 - \theta_1(\Delta\epsilon_1 + \Delta\epsilon_2) - \theta_2\Delta\epsilon_2 \quad (8)$$
$$\lambda = \lambda_3 - \theta_1(\Delta\lambda_1 + \Delta\lambda_2) - \theta_2\Delta\lambda_2 \quad (9)$$

where $\Delta\epsilon_1 = \epsilon_2 - \epsilon_1$, $\Delta\epsilon_2 = \epsilon_3 - \epsilon_2$, $\Delta\lambda_1 = \lambda_2 - \lambda_1$ and $\Delta\lambda_2 = \lambda_3 - \lambda_2$. Let us define:

$$x^*(\epsilon, \lambda) = x_3 - \theta_1(\Delta x_1 + \Delta x_2) - \theta_2\Delta x_2 \quad (10)$$
$$y^*(\epsilon, \lambda) = y_3 - \theta_1(\Delta y_1 + \Delta y_2) - \theta_2\Delta y_2 \quad (11)$$
$$s^*(\epsilon, \lambda) = s_3 - \theta_1(\Delta s_1 + \Delta s_2) - \theta_2\Delta s_2 \quad (12)$$

where $\Delta x_j = x_{j+1} - x_j$, $\Delta y_j = y_{j+1} - y_j$, and $\Delta s_j = s_{j+1} - s_j$, with $j = 1, 2$. It is easy to verify that $(x^*(\epsilon, \lambda), y^*(\epsilon, \lambda), s^*(\epsilon, \lambda))$ is a primal-dual optimal solution of problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$. With replacing (8) and (9) and (11) in

$$\phi(\triangle b, \triangle c, \epsilon, \lambda) = (b + \epsilon\triangle b)^T y^*(\epsilon, \lambda)$$

implies:

$$\phi(\triangle b, \triangle c, \epsilon, \lambda) = a_0 - a_1\theta_1 - a_2\theta_2 - a_3\theta_1\theta_2 - a_4\theta_1^2 - a_5\theta_2^2 \quad (13)$$

where

$$\begin{aligned}
a_0 &= (b + \epsilon_3\Delta b)^T y_3 \\
a_1 &= (b + \epsilon_3\Delta b)^T(\Delta y_1 + \Delta y_2) + (\Delta\epsilon_1 + \Delta\epsilon_2)\Delta b^T y_3 \\
a_2 &= (b + \epsilon_3\Delta b)^T\Delta y_2 + (\Delta\epsilon_2\Delta b)^T y_3 \\
a_3 &= (\Delta\epsilon_1 + \Delta\epsilon_2)\Delta b^T\Delta y_2 + \Delta\epsilon_2\Delta b^T(\Delta y_1 + \Delta y_2) \\
a_4 &= (\Delta\epsilon_1 + \Delta\epsilon_2)\Delta b^T(\Delta y_1 + \Delta y_2) \\
a_5 &= \Delta\epsilon_2\Delta b^T\Delta y_2 + \Delta\epsilon_2\Delta b^T\Delta y_2
\end{aligned} \quad (14)$$

On the other hand, solving equations (8) and (9) for θ_1 and θ_2 gives:

$$\theta_1 = \alpha_1 + \beta_1\epsilon + \gamma_1\lambda \quad (15)$$
$$\theta_2 = \alpha_2 + \beta_2\epsilon + \gamma_2\lambda \quad (16)$$

where

$\alpha_1 = \frac{\epsilon_3\Delta\lambda_2 - \lambda_3\Delta\epsilon_2}{G}$, $\beta_1 = -\frac{\Delta\lambda_2}{G}$, $\gamma_1 = \frac{\Delta\epsilon_2}{G}$

$\alpha_2 = \frac{\lambda_3(\Delta\epsilon_1 + \Delta\epsilon_2) - \epsilon_3(\Delta\lambda_1 + \Delta\lambda_2)}{G}$, $\beta_2 = \frac{\Delta\lambda_1 + \Delta\lambda_2}{G}$, $\gamma_2 = -\frac{\Delta\epsilon_1 + \Delta\epsilon_2}{G}$.

$G = \Delta\epsilon_1\Delta\lambda_2 - \Delta\epsilon_2\Delta\lambda_1$

Replacing (14) – (16) in (13) gives the following representation of the optimal value function:

$$\phi(\triangle b, \triangle c, \epsilon, \lambda) = b_0 + b_1\epsilon + b_2\lambda + b_3\epsilon\lambda + b_4\epsilon^2 + b_5\lambda^2 \quad (17)$$

where

$b_0 = a_0 - a_1\alpha_1 - a_2\alpha_2 + a_3\alpha_1\alpha_2 + a_4\alpha_1^2 + a_5\alpha_2^2,$
$b_1 = -a_1\beta_1 - a_2\beta_2 + a_3\alpha_2\beta_1 + a_3\alpha_1\beta_2 + 2a_4\alpha_1\beta_1 + 2a_5\alpha_2\beta_2,$
$b_2 = -a_1\gamma_1 - a_2\gamma_2 + a_3\alpha_2\gamma_1 + a_3\alpha_1\gamma_2 + 2a_4\alpha_1\gamma_1 + 2a_5\alpha_2\gamma_2,$
$b_3 = a_3\beta_1\gamma_2 + a_3\beta_2\gamma_1 + 2a_4\beta_1\gamma_1 + 2a_5\beta_2\gamma_2,$
$b_4 = a_3\beta_1\beta_2 + a_4{\beta_1}^2 + a_5{\beta_2}^2,$
$b_5 = a_3\gamma_1\gamma_2 + a_4{\gamma_1}^2 + a_5{\gamma_2}^2,$

that is a quadratic function of ϵ and λ. The proof is complete.

3 Illustrative examples

In this section, we apply the results of the previous sections and express an example to illustrate the *LO* problems.

Example : Consider the following LP problem in the form:

$$\begin{array}{lccl} \min & x_1 & +x_2 & \\ s.t & x_1 & +x_2 & \geq 4 \\ & x_1 & & \geq 2 \\ & x_1 & ,x_2 & \geq 0 \end{array}$$

Its dual is

$$\begin{array}{lccl} \max & 4y_1 & +2y_2 & \\ s.t & y_1 & +y_2 & \leq 1 \\ & y_1 & & \leq 1 \\ & y_1 & ,y_2 & \geq 0 \end{array}$$

It is easy to verify that the primal problem has multiple optimal solutions, while its dual problem has a unique solution (y^*, s^*), where $y^* = (1,0)^T$ and $s^* = (0,0)^T$. It is easy to verify that the optimal partition of the index set $\{1,2\}$ is $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s) = \{\{1,2\}, \phi\}$.

A strictly complementary optimal solution of (LP) and (LD) is:

$$x^* = (2,2),\ r^* = (0,0),\ y^* = (1,0) \text{ and } s^* = (0,0).$$

Let $\Delta b = (-1,1)$ and $\Delta c = (-1,1)$ be perturbing directions. We have then $\mathcal{IR}(\triangle b, \epsilon) = [0,1]$ and $\mathcal{IR}(\triangle c, \lambda) = [-\infty, 1]$. Thus

$$\mathcal{IR}(\triangle b, \triangle c, \epsilon, \lambda) = [0,1] \times (-\infty, 1]$$

4 Conclusion

In this paper we introduced the concept of bi-parametric optimal partition invariancy sensitivity analysis for *LO*. We presented auxiliary LO problem that enables us to identify the associated regions. We are interested in developing the results of this study to the bi(tetra)-parametric support set (expansion[9]) sensitivity analysis for (general [9]) *LO* problem and CQO, as well.

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