Optimal Pricing with Loyalty

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Abstract

Market prices vary along the time dimension. In some cases prices change monotonously and, in others they fluctuate in a zigzag pattern. The latter is due to marketers having to face the extremely complex task of setting profit or sales maximizing product prices on the one hand, while maintaining consumer loyalty on the other. This paper suggests an analytical approach for optimally setting product prices with respect to the interdependency between different prices and products, as well as habit formation in terms of both location and brand loyalties. We demonstrate that cyclic pricing policies of harmonic form become optimal when the management of a store is prepared to compromise its net profit goal in order to maintain an image of an affordable store, for the sake of keeping both location and brand loyalties steady. Furthermore, we show that the lower the weight assigned to these loyalties, the greater the optimal frequency of the harmonic zigzags.

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1. Introduction

A review of the behavior of large marketing networks in recent years reveals extremely sophisticated marketing methods that are designed to increase either firm profits or market shares. These marketing techniques include advertising, bundling old products with new products, sample-size packaging (where an initial introductory price can be followed with a
sustained percentage price change over time), and, above all, discounts and seasonal sales. The economic and marketing literature has dealt extensively with these aspects of marketing.

When we observe chain store behavior, particularly, supermarkets or department stores in various countries, we note that there are almost always several products on sale on a cyclical basis for varying periods of time (usually from a week up to even a full month). These products carry a low (and even negative) profit margin, while the prices of the non-sale items remain at normal levels. Sometimes we observe prices steadily increasing over time, while on other occasions we observe prices decreasing over time. However, we often observe a third possibility, that of prices “zigzagging” over time. Since modern market conditions are rapidly changing, marketers are faced with an extremely complex task of setting such product prices that will maximize profits or sales while keeping consumer loyalty steady. Clearly, without a rigorous mathematical model that accounts for dynamic changes in prices over time, one can only reach a limited intuitive solution. Such a solution is likely to be far from optimal.

In the present paper we delineate the specific conditions that would justify zigzag pricing policies that are so frequently driven merely by manager and salesperson intuition. Furthermore, in contrast to intuitive price fluctuations, we identify exact zigzag types, shapes, and frequencies. These results are based both qualitatively and quantitatively on three factors:

(i) Habit formation in terms of the store reputation with respect to the price level.

(ii) Habit formation in terms of location loyalty, where people get used to shopping in a specific location.

(iii) Habit formation in terms of brand loyalty, where people adapt or are addicted to certain brand names regardless of the actual purchasing location.

Clearly adopting an objective function of maximizing the net profits of the firm, would often indicate higher prices (or price index), even at the cost of jeopardizing customer loyalty. Therefore management may feel compelled to compromise its profits by keeping the store price index under control, thereby creating “store loyalty”. Consequently, our goal is to provide an optimal trade-off between the net profit of the store and its price index by factoring in location and brand loyalties of the store customers.

Determining pricing policy need not be left to guesswork or intuition. Instead, we present a model which can guide marketing people and allow them to determine rigorously
which marketing policy (whether a consistent policy of raising or lowering prices overtime, or a zigzag policy whereby prices are intermittently lowered and raised) is optimal.

This kind of analysis is different from other theoretical models in economics. Previous research has studied either the pricing interactions between two products in a two-period model as we discuss below, or pricing policies for a single product over time. A seminal work in the latter direction is that of S. Kalish (1983). His work investigates different forms of the interaction between price and cumulative sales for a single product over a continuous time horizon with the goal of maximizing monopoly profits. Kalish shows that a single price fluctuation ("zigzag") can be optimal when a positive effect of sales on demand alternates with a negative one. In such a case, price is initially low to stimulate demand by high market penetration ("diffusion effect"). The price then monotonically increases to the point where this diffusion effect diminishes. On the other hand, in the case of a negative saturation effect on sales, price is decreasing monotonically over time. According to Kalish, such a switching of positive and negative effects can occur at most once when saturation dominates over diffusion. In the present paper we show that location and brand loyalties can both stimulate and saturate the demand, resulting in a similar effect on sales and thus on optimal pricing. Moreover, if in contrast to standard models (including Kalish (1983)), the monopolist is concerned not only with his profit level, but also with store loyalty (reputation), then positive effects on sales will repeatedly alternate with negative ones resulting in multiple price zigzags. The zigzag effect is due to the fact that the longer the focus is on profit generation (with prices accordingly set high) the lower the loyalties, while a persistent focus on loyalties jeopardizes profits.

This work combines both price interactions and multi-period aspects into one theoretical framework, by studying multiple products within a continuous-time horizon as the consequence of both profit-maximizing behavior and maintaining consumer loyalties.

Over the last two decades the subject of inter-temporal purchases has been discussed extensively in the economic literature. The literature has focused on the durable products market (e.g. computers, calculators) and includes such works as Stokey (1979 and 1981), Conlick, Gerstner and Sobel (1984), Lansberger and Meilijson (1985), Van Cayseele (1991), in Van Praag and Bode (1992), Calem (1997), and Epstein (1998).

In Epstein (1998), a zigzag policy is maintained under the following conditions: the consumer purchases a product which is either durable or fashionable in only one of many given
periods. The purchase of a product in a particular store affects the consumer’s tendency to purchase other products in the same store only during the current period. In other words, there is no inter-temporal effect on consumption. Yet, location interdependency does exist, this being due to transaction costs of moving from one location to another. In general, the Epstein paper does not address the issue of combined “habit formation” and “brand loyalty” (i.e., maintaining current locked-in customers).

One of the reasons for inter-temporal effects is “habit formation” which is central to the studies undertaken by Pollak and Wales (1992) and Heien and Durham (1991). It is shown in these studies, both theoretically and empirically, that the habit-formation component is a highly significant component of demand. As a result, individuals’ consumption in previous periods affects the current demand. We combine "habit formation" for a specific product with both "brand loyalty" and with "addiction" as previously done by Becker and Murphy (1988), Becker, Grossman and Murphy (1994), McKenzie (1991) and Lee and Kreutzer (1982), and in some sense also by Bergemann and Valimaki (1996).

The inter-relationships between products within a certain time period can be positive or negative. In Paroush and Spiegel (1995), the positive dependence was related only to the relationships between the demand for a specific product in one period and the demand for the same product in the second period. In this sense, Paroush and Spiegel follow the idea of inter-temporal pricing and goodwill as presented by Tirole (1993 p. 71). In his model, Tirole considers just one good that is sold in two consecutive periods, where goodwill is generated by lowering the first-period price, thereby raising the quantity demanded both in the first period and in the second period. A further step in developing the idea of “habit formation” and inter-temporal effects has been developed by Klemperer (1995), who used very similar arguments for suggesting a rationale for multi-product firms.

A negative inter-relationship between products within a time period can be illustrated by the following example of market rivalry. Suppliers may not wish to sell cosmetics and other exclusive products in a supermarket that stocks necessities such as meat and milk. The supplier will prefer to sell the expensive products at a different location and thus prevent consumers from perceiving his products as inexpensive, inferior, or basic.

This paper extends the scope of the above research and examines optimal pricing theory for the case where the sales person has greater flexibility over a continuous time
planning scale. This flexibility arises from the fact that there are products being marketed in multi-time periods, thus allowing a wider range of pricing policies depending upon the nature of the relationship between the products and the time periods.

In our paper we integrate the above models into a single model where the phenomenon of locked-in customers is generated simultaneously by brand and location loyalty. The pricing policy of a firm should take into consideration these two aspects of customer loyalty, to arrive at the optimal mix of prices that fully exploit customer behavior. For example, reducing the price of a product increases the quantity demanded. In the presence of brand loyalty, the increase in quantity demanded acts as a catalyst for future demand for this product. After sowing the seeds of brand and location loyalty, the optimal pricing policy may allow for a future calculated change in the price of some products, thus producing either a consistent or a zigzag pricing policy. The firm can exploit this phenomenon fully and will even go as far as to generate the phenomenon itself, i.e., creating customer loyalties by setting up and maintaining desired store loyalty or reputation levels.

In Section 2, we formulate a general continuous-time dynamic model of pricing with store brand and location loyalty. Optimality conditions for the problem are presented in Section 3. Two general optimal pricing regimes and respective forms of pricing policies are identified and formally proven in Section 4. In particular, in this section we show that optimal prices fluctuate in either an exponential or harmonic way, with the frequency proportional to a ratio between store, brand and location loyalties. Section 5, suggests major insights in setting optimal prices over an entire time range. A numerical example illustrates the approach in Section 6. Main results are summarized in Section 7.

2. The Model
We begin with the basic assumptions of our model. We assume a store manager who desires to maximize profits (or revenue), $J$, from sales while keeping the prices reasonable with respect to the store reputation. We assume that our salesperson continuously sells $N$ product types over planning time horizon $T$.

Notations
We will use the following notations in our model:

- $t$ - current time (independent variable), time units, $0 \leq t \leq T$;
$n$ – product index, dimensionless, $n=1,2,\ldots,N$;

$p_n(t)$ - price of product type $n$ at time $t$ (a decision or control variable), monetary units;

$P_n$ – basic price of product type $n$ expected by customers, monetary units;

$q_n(t)$ - quantity of products of type $n$ demanded at time $t$ (a state variable), product units (packs, grams and so on) per time unit;

$q_n^0$ - initial demand of products of type $n$, product units per time unit;

$\sum_{n=1}^{N} a_n p_n(t)$ - price index of the store, where each product $n$ is characterized by its average weight, $a_n$ (dimensionless), in the buyer’s basket, $\sum_{n=1}^{N} a_n = 1$;

$b_n$ – the maximum effect of unit change in price of product type $n$ on the quantity demanded;

$l_n$ – “brand loyalty” coefficient (dimensionless), $0 \leq l_n \leq 1$, which implies that $(1-l_n)b_n$ is the true effect of the unit change in price of product type $n$ on the quantity demanded;

$c_n$ – the maximum effect of the products of type $n$ sold per time unit on the change of the quantity demanded of this product type;

$d$ – “location loyalty” coefficient (dimensionless), reflects the portion of customers who return to the store and increase quantity demanded at time $t$ irrespective of the current price change, $0 \leq d \leq 1$; thus $dc_n$ is the true effect of the products of type $n$ sold at time $t$ on the change of the quantity demanded of this product type;

$r$- psychological factor (dimensionless), reflects psychological effect of price change on the customers and, thus, on the store reputation

$g$ – “store loyalty” coefficient (measured per time unit with respect to factor $r$, for example, if $r=2$, then $g$ is measured per monetary and time units), $g>0$, reflects willingness of the store management to compromise its profit for the sake of the customers in order to keep its reputation at a certain level with respect to the price index, $\sum_{n=1}^{N} a_n p_n(t)$; clearly, the stronger the compromise the greater the location loyalty that develops, i.e., $d$ is a function of $g$; $d(g)=e+f(g)$, $d(0)=e$ and $\frac{\partial f(g)}{\partial g} \geq 0$;
The Demand for the Products

Let us now specify the demand functions for the different products over time. We assume that prices are net of production or purchasing costs and therefore represent the net profit from selling one unit of these products. The dynamic selling and thus the change in demand process we model is due to two major factors. One factor reflects the difference between the basic price for a product that the customers anticipate, $P_n$, and the true price for this product that they encounter at time $t$, $p_n(t)$. Namely, when the true price at $t$, $p_n(t)$, exceeds the anticipated price, $P_n$, the demand, $q_n(t)$, decreases. In the reverse case it increases. The maximum effect of this fluctuation on demand, $q_n(t)$, for product $n$ is $b_n(P_n - p_n(t))$. This effect is weakened by the brand loyalty, $l_n$, for the product as $(1-l_n)b_n(P_n - p_n(t))$. Specifically, the greater the brand loyalty, $l_n$, for the product, the smaller the effect of the price gap, $P_n - p_n(t)$, on the demand.

The other factor which affects the dynamic change in demand, $q_n(t)$, is the well-known “birth and death” process, where any change in a population is proportional to the population itself. This implies that the greater the demand, the greater the expected increase, $dc_nq_n(t)$, if the store has location loyalty, $d$. This is because the loyal customers return to the store and, moreover, become catalysts, sometimes unintentionally, by advertising it to neighbors, relatives, and friends.

Summarizing these two effects, the dynamic change in the state of the demand for a product of type $n$ is:

$$
\dot{q}_n(t) = (1-l_n)b_n(P_n - p_n(t)) + dc_nq_n(t), \quad q_n(0) = q^0_n, \quad n=1,2,...,N.
$$

The Explicit Level of Prices

The typical way of keeping customers and of maintaining a particular reputation level is by limiting prices with respect to the interdependent products, which of course has the effect of comprising profits. In our model this is accomplished in two ways: explicit and implicit. The implicit way is formalized by introducing the price index, $\sum_{n=1}^{N} a_n p_n(t)$, into the objective function in addition to the standard profit-based term, as discussed below. This can have a positive psychological effect on customers but it cannot guarantee that the price index will
never reach very high levels. Therefore, to prevent such high levels from being reached, with their associated shocking effect on customers, we impose an explicit upper bound $M$ on the price index:

$$
\sum_{n=1}^{N} a_n p_n(t) \leq M.
$$

**The Objective Function**

From equation (1), one can readily observe that the dynamics adopted in our model are similar to those suggested by Kalish (1983) for the case of a single product. The difference lies in the objective function and the multiple products accounted for. In contrast to pure profit maximization in Kalish, our objective is more complex. We use a utility function that takes two conflicting goals into account: profit and reputation.

The idea to depart from the standard microeconomic model of profit maximization and to also consider other factors such as fairness, reputation, loyalty, location, and brand loyalty was introduced in several papers during the 80’s. In one of these papers, Nobel Prize Laureate Daniel Kahneman and his colleagues (Kahneman et al, 1986) claim that

"the absence of considerations of fairness and loyalty from standard economic theory is one of the most striking contrasts between this body of theory and other social sciences – and also between economic theory and lay intuitions about human behavior"

This claim states, in effect, that we should be maximizing a utility function rather than net profit. It turns out that this extension is crucial. Once the monopolist is willing to limit his potential profit for the sake of his reputation, the optimal pricing policy takes on a zigzag form so frequently observed in real life.

Cobb-Douglas (CD), $A^B$, Constant Elasticity of Substitution (CES), $\left( A^{\frac{1}{\gamma}} + B^{\frac{1}{\gamma}} \right)^\frac{1}{\gamma}$, and Quadratic utility (QU), $A - C^2B^2$, are among the most popular utility functions which have been employed to allow a compromise between price and profit on the one hand, and reputation or quality on the other. These functions involve at least two terms combined in a multiplicative (CD function), additive (QU functions), or more complex (CES function) manner. From the seller's viewpoint, one term normally presents profit or revenue, while the other a non-cash cost, interpreted as the seller's reputation (for similar utility functions see, for
example, Shubik and Levitan (1980), Singh and Vives (1984)). In this paper, we illustrate our approach by use of the two-term utility function, where traditionally the first term is
determined by the company profit, \( p_n(t)q_n(t) \), for each product \( n \), and the second term by the
company reputation in terms of the psychological effect of the selected or weighted product
prices, \( a_n[p_n(t)]' \) on the customers. Since, our approach can be used to solve for any
psychological effect \( r \) (which in CD functions is frequently referred to as the “importance” of
the problem), we first present our objective function in its most general form and then select a
specific value for \( r \). Our goal is to find such a set of pricing policies
\( \{p_n(t), n = 1,\ldots,N; 0 \leq t \leq T\} \) that maximizes utility over the entire planning horizon, \( T \),
subject to constraints (1)-(2):

\[
J = \int_0^T \sum_{n=1}^N \left( p_n(t)q_n(t) - ga_n[p_n(t)]' \right) dt .
\]

Thus, objective (3) implies a trade-off or compromise between the two conflicting
targets represented by the utility components. The trade-off is controlled by two system
parameters, the store loyalty to the customers, \( g \), and the psychological factor, \( r \). Specifically,
the greater the store loyalty coefficient the more the store management or the sales person is
willing to compromise in terms of profits for the sake of customer satisfaction (and, thus, its
own reputation). The second parameter, \( r>1 \), takes into account the price level first of all for the
more expensive products which customers are more quick to take note of. The larger this factor
is, the greater the psychological effect on the trade-off between profits and store loyalties.
When \( r=2 \), the utility function (QU) reflects risk aversion that underlies the loyalty. This
observation is due to the well-known property of quadratic utility frequently referred to as
IARA (increasing absolute risk aversion) and defined by the Arrow-Pratt measure of absolute
risk-aversion. Consequently, we illustrate the approach by choosing \( r=2 \). To simplify the
upcoming presentation, we introduce the following two notations:

\[
L_n = (1-ln)b_n
\]
\[
D_n = dc_n.
\]

Then the state equation (1) takes the following form

\[
\dot{q}_n(t) = L_n(P_n - p_n(t)) + D_n q_n(t) .
\]
3. Optimality Conditions

To study optimal behavior of the dynamic system (1′)-(3), we use the maximum principle (Maimon et al, 1998). This is accomplished by constructing the Hamiltonian:

$$H(t)= \sum_{n=1}^{N} [p_n(t)q_n(t) - ga_n p_n^3(t)] + \sum_{n=1}^{N} \psi_n(t) [L_n(P_n - p_n(t)) + D_n q_n(t)],$$  \hspace{1cm} (6)

where the co-state variable $\psi_n(t)$ measures the marginal cost of product $n$, i.e., the change in the objective function value due to a unit change of the demand for $n$ and satisfies the following co-state equation and boundary condition

$$\dot{\psi}_n(t) = -\frac{\partial H(t)}{\partial q_n(t)} = -p_n(t) - D_n \psi_n(t), \quad \psi_n(T) = 0.$$  \hspace{1cm} (7)

According to the maximum principle, the optimal value of the decision variables, $p_n(t)$, $n=1,2,...,N$ maximizes the Hamiltonian, $H$, at each point of time, $t$, as a function of only these variables. Thus, by combining only $p_n(t)$ dependent terms of $H(t)$, we obtain the following dual optimization problem,

$$H(p_n(t)) = \sum_{n=1}^{N} [p_n(t)q_n(t) - ga_n p_n^3(t)] - \sum_{n=1}^{N} \psi_n(t) L_n p_n(t) \rightarrow \max$$  \hspace{1cm} (8)

subject to (2).

Based on this dual formulation we now determine the following optimal prices as a function of the state $q_n(t)$ and co-state $\psi_n(t)$ variables.

**LEMMA 1.** Given primal (1)-(3) and dual (7)-(8) problems, optimal price of product $n$ at time $t$ satisfies the following condition:

- if $\sum_{n=1}^{N} a_n p_n(t) < M$, then $p_n(t) = \frac{q_n(t) - L_n \psi_n(t)}{2ga_n}$ otherwise,

- if $\sum_{n=1}^{N} a_n p_n(t) = M$, then $p_n(t) = \frac{q_n(t) - L_n \psi_n(t) + \lambda(t)a_n}{2ga_n}$,

where $\lambda(t) = \frac{2gM - \sum_{n=1}^{N} [q_n(t) - L_n \psi_n(t)]}{\sum_{n=1}^{N} a_n} \geq 0$.

**Proof:** see appendix.
4. Optimal Pricing Strategies

Lemma 1 delivers optimal pricing as a function of state $q_n(t)$ and co-state $\psi_n(t)$ variables for two different price index based regimes. One pricing regime, which we term *loose pricing*, refers to time intervals wherein the price index does not reach its upper bound, i.e., constraint (2) is inactive, $\sum_{n=1}^{N} a_n p_n(t) < M$ along an interval of time, or reaches its upper bound only at single points, i.e., $\sum_{n=1}^{N} a_n p_n(t) \leq M$. With respect to Lemma 1, an equivalent and more precise definition of this regime is $\lambda(t) = \frac{2gM - \sum_{n=1}^{N} [q_n(t) - L_n \psi_n(t) ]}{\sum_{n=1}^{N} a_n} = 0$ over an interval of time. The other regime, which we term *tense pricing*, refers to pricing at the maximum allowed level, i.e., constraint (2) is continuously active, $\sum_{n=1}^{N} a_n p_n(t) = M$ over an interval of time, which is equivalent to $\lambda(t) \geq 0$. In the following lemmas, we derive general forms of the optimal solution for the two corresponding regimes, which involve integration constants in place of the state and co-state variables. In particular, Lemma 2 presents optimal pricing strategies over the loose pricing regime.

**Lemma 2.** Let $A_n, B_n, \hat{A}_n, \hat{B}_n, \tilde{A}_n$ and $\tilde{B}_n$ be unknown integration constants and there exists a time interval, $\tau$, such that $\sum_{n=1}^{N} a_n p_n(t) \leq M$ for $t \in \tau$. Then the optimal price for product $n$ is determined as follows:

- if $\frac{(1-l_n)b_n}{ga_n} - dc_n > 0$, then $p_n(t) = A_n \sin w_n t + B_n \cos w_n t + C_n$ (harmonic pricing policy),
- if $\frac{(1-l_n)b_n}{ga_n} - dc_n < 0$, then $p_n(t) = \hat{A}_n e^{\xi t} + \hat{B}_n e^{-\xi t} + C_n$ (exponential pricing policy),
- if $\frac{(1-l_n)b_n}{ga_n} - dc_n = 0$, then $p_n(t) = \frac{dc_n (1-l_n)b_n P_n}{4ga_n} + \tilde{A}_n t + \tilde{B}_n$ (parabolic pricing policy),

where $C_n = \frac{(1-l_n)b_n P_n}{2((1-l_n)b_n - gda_n)}$, $w_n = \sqrt{dc_n \left( \frac{(1-l_n)b_n}{ga_n} - dc_n \right)}$ and $z_n = \sqrt{dc_n \left( dc_n - \frac{(1-l_n)b_n}{ga_n} \right)}$. 


Lemma 2 provides an important insight into optimal pricing strategies when the dynamic system (1)-(3) operates according to the loose regime, that is when the upper bound, $M$, of the price index $\sum_{n=1}^{N} a_n p_n(t)$ is relatively loose, so that $\sum_{n=1}^{N} a_n p_n(t) \leq M$ holds without being imposed. In such a case, optimal pricing depends on the system parameter,

$$I_n = dc_n \left( \frac{(1-l_n)b_n}{ga_n} - dc_n \right),$$

which we term the disloyalty index. From (9) we observe that this index compares the ratio between brand “disloyalty”, $1-l_n$, and store loyalty weighted with system parameters against the location loyalty. As location loyalty is frequently a linear function of store loyalty, $f(g) = kg$ and $d = e + kg$, this implies that the lower the brand loyalty (i.e. the greater the brand disloyalty), the location loyalty, and the store loyalty, the greater the disloyalty index. Specifically, if this index is positive, i.e., brand, location and store loyalties are relatively low or moderate, the optimal pricing is cyclic with frequency equal to the square root of the disloyalty index. This sort of behavior accurately reflects real life, since the lower the loyalties, the greater the need for discount pricing, resulting in frequent price fluctuations used by management to attract customers with changes, thereby compensating for the lack of loyalty. Moreover, if store loyalty is negligible, $g=0$, (management is interested only in net profit irrespective as to how high prices are driven), then $d=e$ and the fluctuation frequency tends to infinity, which eventually imitates an average constant price having no fluctuation at all. This means that if the store refuses to compromise its profit level at all (the objective function (3) has only one parameter – net profit), then the optimal solution consists of setting constant prices at a level such that the price index is at its maximum. On the other hand, the stronger the importance of various loyalties in the system the smaller $I_n>0$, that is less frequent zigzags would be required. This is also well justified as the stronger the loyalties, the more indifferent customers are to price fluctuations. In such a case the optimal pricing policy has at most one zigzag which is either of parabolic shape (when the disloyalty index is zero) or of exponential shape (negative disloyalty index). Note, that the disloyalty index $I_n$ depends on product index $n$, i.e., it is possible that the optimal pricing policy within the same store will be harmonic for some.
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products \((l_n > 0)\), while exponential for others \((l_n < 0)\), or parabolic \((l_n = 0)\). It is also worth emphasizing that interval \(\tau\) employed in Lemma 2 does not necessarily cover the entire planning horizon, \([0, T]\). The price index may eventually rise to beyond the upper bound under these policies. Therefore, a change in pricing strategy would be required as, for example, that arising in the case of the tense pricing regime. This also implies that even if loyalties are high thereby causing no price zigzags for the loose pricing regime over one time interval, a change in regime can still necessitate a new pricing strategy, which will have a zigzag with respect to the previous time interval. Thus, with low loyalties price zigzags are cyclic under the loose pricing regime when the planning horizon is long enough to include more than one harmonic period, i.e., \(\sqrt{I_n T} \geq 2\pi\). With high loyalties and a tight upper bound on the price index, price zigzags are possible between consecutive regimes over relatively short planning horizons.

Given \(g, l_n, b_n, d\) and \(c_n\), it is reasonable to assume that \(I_n = d c_n \left(\frac{(1-l_n)b_n}{g a_n} - d c_n\right) \neq 0\) and there is therefore no need to consider parabolic pricing.

We now study the tense pricing regime which is characterized by a tight upper bound, \(M\), imposed on the product index, \(\sum_{n=1}^{N} a_n p_n(t)\). This regime applies in the case where loose pricing results in a violation of the upper bound. To prevent violation of this bound, this regime requires simultaneous price coordination for all products, which significantly complicates the pricing strategy. Therefore, we consider here two special cases which allow for tractable formulations to be derived. The first case is characterized by (i) equal effects of location loyalty on the demand of all products,

\[
(1-l_1)\frac{b_1}{a_1} = (1-l_2)\frac{b_2}{a_2} = \ldots = (1-l_N)\frac{b_N}{a_N} = R,
\]

and (ii) balanced brand loyalties, i.e., equal brand loyalties with respect to the product weights in the price index,

\[d c_1 = d c_2 = \ldots = d c_n = D.\]
LEMMA 3. Let $A_{in}, B_{in}, \hat{A}_{in}$ and $\hat{B}_{in}$ be unknown integration constants and there exists a time interval, $\tau$, such that $\sum_{n=1}^{N} a_n p_n(t) = M$ for $t \in \tau$. Then the optimal price for products is determined as follows:

if $\frac{R}{g} - D > 0$, then $p_i(t) = \frac{\sum_{n=1}^{N} a_i (A_{in} \sin wt + B_{in} \cos wt + C_{in}) - M}{\sum_{i=1}^{N} a_i}$

for $i \neq n$ and $p_n(t) = \frac{\sum_{i=1}^{N} a_i (A_{in} \sin wt + B_{in} \cos wt + C_{in}) - M}{\sum_{i=1}^{N} a_i}$ (harmonic pricing policy),

if $\frac{R}{g} - D < 0$, then $p_i(t) = \frac{\sum_{n=1}^{N} a_i (\hat{A}_{in} e^{zt} + \hat{B}_{in} e^{-zt} + C_{in}) - M}{\sum_{i=1}^{N} a_i}$ for $i \neq n$ and $p_n(t) = \frac{\sum_{i=1}^{N} a_i (\hat{A}_{in} e^{zt} + \hat{B}_{in} e^{-zt} + C_{in}) - M}{\sum_{i=1}^{N} a_i}$ (exponential pricing policy),

where $C_{ij} = \frac{R(P_i - P_j)}{2(R - gD)}$, $w = \sqrt{D \left( \frac{R}{g} - D \right)}$ and $z = \sqrt{D \left( D - \frac{R}{g} \right)}$.

Proof: see appendix.

Lemma 3 proves that when brand loyalties are balanced and the location loyalty has an equal effect on demand for different products, the tense pricing regime does not induce a change in the shape of the general pricing strategy. This implies that if optimal pricing policy for a product was harmonic, then it will remain harmonic under the tense regime, and will even maintain the same fluctuation frequency. If the policy was exponential, then it will continue to maintain the same shape. The major change will be in the pricing amplitudes. Specifically, though the general strategy remains the same, the exact prices will change, sometimes significantly, to maintain the price index at its upper bound. Thus, we sustain our previous discussions that even if the pricing policy was not cyclic (harmonic) and had only one zigzag
over a loose regime, a tense regime can cause the zigzag to repeat. This provides us with the important insight that when the upper bound on the price index is tight enough to be reached from time to time, both tense and loose pricing regimes will alternate consecutively along the planning horizon. Therefore, we may have repeating inter-regime price zigzags even when brand, location and store loyalty coefficients are high and each regime when taken separately does not induce cyclic strategies.

The second case that we study under tense pricing is characterized by two products, \( N=2 \), and similar location effect of different products \( D_n = D \), with no balancing requirements being imposed on brand loyalty (as was the case in Lemma 3). The following lemma presents general pricing strategies for this case.

**Lemma 4.** Let \( A, B, \hat{A} \) and \( \hat{B} \) be unknown integration constants, \( N=2 \), and there exists a time interval, \( \tau \), such that \( \sum_{n=1}^{N=2} a_n p_n(t) = M \) for \( t \in \tau \). Then optimal product prices are determined as follows:

if \( \left( \frac{L_1}{ga_1} - D \right) \frac{1}{a_1} + \left( \frac{L_2}{ga_2} - D \right) \frac{1}{a_2} > 0 \), then \( p_1(t) = A \sin w t + B \cos w t + C \) and

\[
p_2(t) = \frac{M}{a_2} - \left( A \sin w t + B \cos w t + C \right) \frac{a_1}{a_2}
\]

(harmonic pricing policy),

if \( \left( \frac{L_1}{ga_1} - D \right) \frac{1}{a_1} + \left( \frac{L_2}{ga_2} - D \right) \frac{1}{a_2} < 0 \), then \( p_1(t) = \hat{A} e^{zt} + \hat{B} e^{-zt} + C \) and

\[
p_2(t) = \frac{M}{a_2} - \left( \hat{A} e^{zt} + \hat{B} e^{-zt} + C \right) \frac{a_1}{a_2}
\]

(exponential pricing policy),

where

\[
C = \frac{L_1 P_1 - L_2 P_2}{2ga_1^2} - \left( D - \frac{L_2}{ga_2} \right) \frac{M}{a_2} - \frac{L_1}{ga_1} \frac{1}{a_1} + \frac{L_2}{ga_2} \frac{1}{a_2},
\]

\[
w = \sqrt{\left( \frac{L_1}{ga_1} - D \right) \frac{1}{a_1} + \left( \frac{L_2}{ga_2} - D \right) \frac{1}{a_2} \frac{M}{a_2} \frac{a_1a_2}{a_1 + a_2}},
\]

and

\[
z = \sqrt{\left( D - \frac{L_1}{ga_1} \right) \frac{1}{a_1} + \left( D - \frac{L_2}{ga_2} \right) \frac{1}{a_2} \frac{M}{a_2} \frac{a_1a_2}{a_1 + a_2}}.
\]

**Proof:** see appendix.
Lemma 4 provides insights similar to those of Lemma 3. Specifically, the general strategy derived for the loose pricing is optimal for the tense regime, i.e., it is either harmonic or exponential. However, in contrast to Lemma 3, not only may the pricing amplitudes change, but also their shape and harmonic frequency or exponent rate.

Thus, Lemmas 2 – 4 provide us with the following insights:

(i) there are only three types of optimal pricing policies: harmonic, parabolic and exponential in time, irrespective of whether the pricing regime is loose or tense;

(ii) if both loose and tense pricing regimes are implemented over the planning horizon, then the shape of the pricing policy for the same product remains unchanged only if location and brand loyalties are balanced, \( (1-l_1)\frac{b_1}{a_1} = (1-l_2)\frac{b_2}{a_2} = \ldots = (1-l_N)\frac{b_N}{a_N} = R \). This is to say, that only pricing amplitude changes when transforming from one regime to another. If loyalties are not balanced, then the general pricing strategy, including amplitudes and frequencies, may change. Specifically, if optimal pricing for one product is harmonic and for the other exponential, when transforming from the loose to the tense regime optimal pricing strategy for both products must be the same over the tense regime, i.e., either harmonic or exponential.

(iii) harmonic frequencies and exponent rates are determined by individual disloyalty indexes \( I_n, n=1,2,\ldots,N \) and, hence, are different for different products under the loose pricing regime, whereas they are the same (determined by a unique, weighted sum of individual disloyalty indexes, \( \hat{I} \)) for all products under tense pricing. The latter facilitates the determination of optimal pricing, since only the amplitudes need to be adjusted to keep the price index at its upper bound, \( M \).
These insights are used in the next section to find exact pricing policies and their sequences over the entire planning horizon.

5. Detailed Description of Pricing Policies and their Sequencing

In the previous section we derived optimal frequencies of pricing policies for loose and tense pricing regimes occurring over time intervals. Though these findings define general strategies for manipulating prices, they are not applicable unless the exact amplitudes of the pricing policies and their sequencing are identified along the entire planning horizon. To derive such detailed pricing policies, we need to study optimal behavior of the state and co-state variables.

Let us differentiate state (1) and co-state (2) differential equations

\[ \ddot{q}_n(t) = -(1-L_n) b_n \dot{p}_n(t) + dc_n \dot{q}_n(t), \]

\[ \ddot{\psi}_n(t) = -p_n(t) - D_n \dot{\psi}_n(t), \]

which after substituting (1) and (2) takes the following form

\[ \ddot{q}_n(t) - D_n^2 q_n(t) = -L_n \dot{p}_n(t) + D_n L_n (P_n - p_n(t)), \]

\[ \ddot{\psi}_n(t) - D_n^2 \psi_n(t) = -\dot{p}_n(t) - D_n p_n(t). \]

From the second order differential equations (10) and (11), we observe that the optimal quantity demanded, i.e., state variable, \( q_n(t) \), and its corresponding co-state variable, \( \psi_n(t) \), evolve in a very similar manner. Therefore, both are treated in the same lemmas. We start from the analysis for the loose regime.

**LEMMA 5.** Let \( A_n, B_n, E_n, F_n, G_n, K_n, \hat{A}_n, \hat{B}_n, \hat{E}_n, \hat{F}_n, \hat{G}_n, \hat{K}_n \) be unknown integration constants and there exists a time interval, \( \tau \), such that \( \sum_{n=1}^{N} a_n p_n(t) \leq M \) for \( t \in \tau \). Then the optimal demand for the product and its corresponding co-state variable are determined as follows:

\[ q_n(t) = E_n e^{D_n t} + F_n e^{-D_n t} - \frac{(w_n B_n - D_n A_n) L_n}{w_n^2 + D_n^2} \sin w_n t + \frac{(w_n A_n + D_n B_n) L_n}{w_n^2 + D_n^2} \cos w_n t - \frac{(P_n - C_n) L_n}{D_n}, \]

and

\[ \psi_n(t) = G_n e^{D_n t} + K_n e^{-D_n t} - \frac{(w_n B_n - D_n A_n)}{w_n^2 + D_n^2} \sin w_n t + \frac{(w_n A_n + D_n B_n)}{w_n^2 + D_n^2} \cos w_n t + \frac{C_n}{D_n}. \]
if \( \frac{L_n}{g_{a_n}} - D_n < 0 \), then

\[
q_n(t) = \hat{E}_n e^{D_j} + \hat{F}_n e^{-D_j} \left( \frac{z_n + D_n}{z_n^2 - D_n^2} \right) \hat{A}_n L_n + \frac{(z_n - D_n)}{z_n^2 - D_n^2} \hat{B}_n L_n e^{-D_j} - \frac{(P_n - C_n)L_n}{D_n}
\]

and \( \psi_n(t) = \hat{G}_n e^{D_j} + \hat{K}_n e^{-D_j} \left( \frac{z_n + D_n}{z_n^2 - D_n^2} \right) e^{D_j} + \frac{(z_n - D_n)}{z_n^2 - D_n^2} e^{-D_j} + \frac{C_n}{D_n} \).

Proof: see appendix

The tense regime characterized by a general relationship between system parameters and \( N=2 \) is studied in the following lemma.

**Lemma 6.** Let \( A_n, B_n, U_n, V_n, X_n, Y_n, \hat{A}_n, \hat{B}_n, \hat{U}_n, \hat{V}_n, \hat{X} \) and \( \hat{Y} \) be unknown integration constants and there exists a time interval, \( \tau \), such that \( \sum_{n=1}^{N} a_n p_n(t) = M \) for \( t \in \tau \). Then the optimal demand for products and the corresponding co-state variables are determined as follows:

if \( \frac{L_1}{g_{a_1}} - D_1 + \frac{L_2}{g_{a_2}} - D_2 > 0 \), then

\[
q_1(t) = U_1 e^{D_1} + V_1 e^{-D_1} + \frac{(wB - DA)L_1}{w^2 + D^2} \sin wt + \frac{(wA + DB)L_1}{w^2 + D^2} \cos wt - \frac{(P_1 - C_1)L_1}{D_1}
\]

\[
q_2(t) = U_2 e^{D_1} + V_2 e^{-D_1} + \frac{a}{a_2} \left( \frac{(wB - DA)L_2}{w^2 + D^2} \sin wt + \frac{(wA + DB)L_2}{w^2 + D^2} \cos wt - \frac{(P_2 + C_1 - M D_2)}{D_2} \right)
\]

and \( \psi_1(t) = X_1 e^{D_1} + Y_1 e^{-D_1} - \frac{(wB - DA)}{w^2 + D^2} \sin wt + \frac{(wA + DB)}{w^2 + D^2} \cos wt + \frac{C}{D} \),

\[
\psi_2(t) = X_2 e^{D_1} + Y_2 e^{-D_1} + \frac{a}{a_2} \left( \frac{(wA + DB)}{w^2 + D^2} \cos wt - \frac{(C_1 - M D_2)}{D_2} \right)
\]

if \( \frac{L_1}{g_{a_1}} - D_1 + \frac{L_2}{g_{a_2}} - D_2 < 0 \), then

\[
q_1(t) = \hat{U}_1 e^{D_1} + \hat{V}_1 e^{-D_1} - \frac{(z + D)\hat{A}_1 L_1}{z^2 - D^2} e^{D_1} + \frac{(z - D)\hat{B}_1 L_1}{z^2 - D^2} e^{-D_1} - \frac{(P_1 - C_1)L_1}{D_1}
\]
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\[
q_2(t) = \hat{U}_2 e^{D t} + \hat{V}_2 e^{-D t} + \frac{a_1}{a_2} \left( \frac{z + D}{z^2 - D^2} \right) e^{z t} - \frac{a_1}{a_2} \left( \frac{z - D}{z^2 - D^2} e^{-z t} - \frac{P_2 + C}{a_2} \frac{a_1}{a_2} \frac{M}{D} \right) L_2^{\alpha_2}
\]

and
\[
\psi_1(t) = \hat{X}_1 e^{D t} + \hat{Y}_1 e^{-D t} - \frac{(z + D) \hat{A}}{z^2 - D^2} e^{z t} + \frac{(z - D) \hat{B}}{z^2 - D^2} e^{-z t} + C_D
\]

\[
\psi_2(t) = \hat{X}_2 e^{D t} + \hat{Y}_2 e^{-D t} + \frac{a_1}{a_2} \left( \frac{z + D}{z^2 - D^2} \right) e^{z t} - \frac{a_1}{a_2} \left( \frac{z - D}{z^2 - D^2} e^{-z t} - \frac{C}{a_2} \frac{a_1}{a_2} \frac{M}{D} \right)
\]

Proof: see appendix.

Lemmas 5 and 6 provide an important insight with respect to the evolution of the optimal quantity demanded. Specifically, a general form of this state variable as well as of the corresponding co-state variable is very similar for all possible values of the disloyalty index (unlike the case of the decision variable, \( p_n(t) \)) irrespective as to whether loose or tense pricing is exercised. When loyalties are high this form has at most one zigzag characterized by a time point of minimum demand and a steady demand increase afterwards. In the case of low loyalties (high disloyalty index) harmonic fluctuations are added to this trend and demand can fluctuate with multiple zigzags.

Based on Lemmas 1-6, we are now ready to identify exact optimal pricing policies including their amplitudes over the entire planning horizon. We start with the cases characterized by only one pricing regime realized over the entire planning horizon. The following two lemmas consider loose and tense pricing respectively. To facilitate the presentation, we arrange all products into two sets with respect to the disloyalty index, \( I_n \):

\[
S_1 = \{ n | I_n > 0 \}, S_2 = \{ n | I_n < 0 \}.
\]

**Lemma 7.** Let
\[
C_n = \frac{(1 - l_n) b_n P_n}{2(1 - l_n) b_n - g a_n c_n}, w_n = \sqrt{d c_n \left( \frac{(1 - l_n) b_n}{g a_n} - d c_n \right)}\text{ and}
\]

\[
z_n = \sqrt{d c_n \left( d c_n - \frac{(1 - l_n) b_n}{g a_n} \right)}.
\]

If
\[
\sum_{n \in S_1} a_n \left( A_n \sin \omega t + B_n \cos \omega t + C_n \right) + \sum_{n \in S_2} a_n \left( \hat{A}_n e^{\omega t} + \hat{B}_n e^{-\omega t} + C_n \right) \leq M \text{ for } 0 \leq t \leq T,
\]

then
the amplitudes $A_n, B_n, G_n$ and $K_n$, $\hat{A}_n, \hat{B}_n, \hat{G}_n$ and $\hat{K}_n$ of pricing policies defined by Lemma 2 are determined by system

$$q_n(0) = q_n^0, \quad \psi_n(0) = -p_n(0) - D_n\psi_n(0), \quad \psi_n(T) = 0,$$

$$p_n(0) = \frac{q_n(0) - L_n\psi_n(0)}{2ga_n}, \quad \dot{p}_n(0) = \frac{L_nP_n + D_nq_n(0) + L_nD_n\psi_n(0)}{2ga_n}.$$  

(13)

**Proof:** see appendix

To show that system (13) completely identifies unknown amplitudes, consider products $n$ which constitute $S_1$. Then substituting into the system of equations (13) the corresponding optimal solutions (12A) (A: developed in the appendix) and (36A) from Lemmas 2 and 5 respectively, we find for $n \in S_1$

$$2G_nD_n - \frac{(w_nB_n - D_nA_n)}{w_n^2 + D_n^2}w_n = -B_n - 2C_n - \frac{(w_nA_n + D_nB_n)}{w_n^2 + D_n^2}D_n;$$

$$G_n e^{\delta_nT} + K_n e^{-\delta_nT} - \frac{(w_nB_n - D_nA_n)}{w_n^2 + D_n^2} \sin w_nT + \frac{(w_nA_n + D_nB_n)}{w_n^2 + D_n^2} \cos w_nT + \frac{C_n}{D_n} = 0;$$

$$q_n^0 - L_nG_n - L_nK_n = \frac{(w_nA_n + D_nB_n)L_n}{w_n^2 + D_n^2} - \frac{C_n}{D_n}L_n;$$

$$2ga_nAw_n = L_nP_n + D_nq_n^0 + D_nL_nG_n + D_nL_nK_n + \frac{(w_nA_n + D_nB_n)D_nL_n}{w_n^2 + D_n^2} + C_nL_n.$$  

(14)

System (14) consists of four algebraic equations which are linear in four unknowns $A_n$, $B_n$, $G_n$ and $K_n$ and, thus, completely determines them. Note, once $A_n$, $B_n$, $G_n$ and $K_n$ are calculated, it is easy to find optimal behavior of the quantity demanded, i.e., identify amplitudes $E_n$ and $F_n$ in equation (34a). This is accomplished by simply plugging boundary conditions

$$q_n(0) = q_n^0 \quad \text{and} \quad \dot{q}_n(0) = L_n(P_n - p_n(0)) + D_nq_n(0)$$

into (34a) and its derivative, thereby obtaining the following two equations in two unknowns $E_n$ and $F_n$.

$$q_n^0 = E_n + F_n + \frac{(w_nA_n + D_nB_n)L_n}{w_n^2 + D_n^2} - \frac{(P_n - C_n)L_n}{D_n};$$

(15)
The other explicit form of system (14) for \( n \in S_0 \) is similarly obtained by substituting the corresponding solutions into (13). This system is relocated to the Appendix (see system A). Lemma 7 details the general pricing strategy derived in the previous section into an exact optimal solution when loose pricing is optimal along the entire planning horizon. The following lemma considers tense pricing over the entire planning horizon.

**Lemma 8.** Let
\[
 C = \frac{L_1 p_{12} - L_2 p_{22} - (D - L_2) M}{2 g a_1 - 2 g a_2 - (D - L_2) g a_2 a_2},
\]

\[
 w = \sqrt{\left( \frac{L_1}{g a_1} - D \right) \frac{1}{a_1} + \left( \frac{L_2}{g a_2} - D \right) \frac{1}{a_2}} \frac{a_1 a_2}{a_1 + a_2}, \quad \text{and} \quad z = \sqrt{\left( \frac{D - L_1}{g a_1} \right) \frac{1}{a_1} + \left( \frac{D - L_2}{g a_2} \right) \frac{1}{a_2}} \frac{a_1 a_2}{a_1 + a_2}.
\]

If \( 2 g M - \sum_{n=1}^N [q_n(t) - L_n \psi_n(t)] \geq 0 \) for \( 0 \leq t \leq T \), then

the amplitudes \( A, B, U_1, U_2, V_1, V_2, X_1, X_2, Y_1, Y_2 \) and \( \hat{A}, \hat{B}, \hat{U}_1, \hat{U}_2, \hat{V}_1, \hat{V}_2, \hat{X}_1, \hat{X}_2, \hat{Y}_1 \) and \( \hat{Y}_2 \) of pricing policies defined by Lemma 4 are determined by System B.

**Proof:** The proof is very similar to that for Lemma 7. Namely, optimal pricing under the tense regime, i.e.,

\[
 2 g M - \sum_{n=1}^N [q_n(t) - L_n \psi_n(t)] \geq 0 \quad \text{and therefore,}
\]

\[
 2 g M - \sum_{n=1}^N \frac{\lambda_n(t)}{a_n} \geq 0
\]

is determined for arbitrary boundary conditions by Lemma 4 without identifying specific amplitudes. The exact values for the amplitudes can be found straightforwardly from equations (1), (7), (18A) - (22A) by setting \( t=0 \) and \( t=T \) as follows,

\[
 q_n(0) = q_n^0, \quad \psi_n(0) = -p_n(0) - D_n \psi_n(0), \quad \psi_n(T) = 0; \quad p_i(0) - p_j(0) = \frac{q_i(0) - L \psi_i(0)}{2 g a_i} - \frac{q_j(0) - L \psi_j(0)}{2 g a_j}; \quad (16)
\]
\[ \dot{p}_i(0) - \dot{p}_j(0) = \frac{L_i P_i + D_i q_i(0) + L_i D_i \psi_i(0)}{2 g a_i} - \frac{L_j P_j + D_j q_j(0) + L_j D_j \psi_j(0)}{2 g a_j}; \]

\[ \dot{q}_n(0) = L_n (P_n - p_n(0)) + D_n q_n(0). \]

To derive an explicit form of system (16), we consider the case of the positive disloyalty coefficient, \( \dot{I} > 0 \). Then substituting the corresponding optimal solutions for \( p_n(t) \), \( q_n(t) \) and \( \psi_n(t) \) from Lemmas 4 and 6 respectively into the system of equations (16), we find a system of equations (System B) which is relocated to the Appendix, and tense regime condition \( \dot{\lambda}(t) = \sum_{n=1}^{N} [a_n(t) - L_n \psi_n(t)] \geq 0 \) transforms to

\[ 2gM - \sum_{n=1}^{N} [a_n(t) - L_n \psi_n(t)] \geq 0 \]

\[ 2gM - \sum_{n=1}^{N} \left[ (U_n - X_n L_n) e^{D_n t} + (V_n - Y_n L_n) e^{-D_n t} - \frac{P L_n}{D_n} \right] \geq 0. \]  

System B consists of 10 algebraic equations linear in 10 unknowns, \( A, B, U_1, U_2, V_1, V_2, X_1, X_2, Y_1 \) and \( Y_2 \). A similar system can be obtained from (16) for the case of the negative disloyalty coefficient, \( \dot{I} < 0 \).

Lemmas 7 and 8 determine exact amplitudes of the optimal solutions for cases of only one pricing regime optimal over the entire planning horizon. That is, the same regime is the first and the last simultaneously. However, the loose/tense pricing regime condition does not necessarily hold over the entire planning horizon. This implies that there can be a sequence of loose and tense pricing regimes that can affect the amplitudes of the pricing curves. We, thus, need to determine the exact amplitudes for the cases where a regime is neither first, nor last, but intermediate with respect to the planning horizon. We use index \( i = 1, 2, \ldots, h \) to number all regimes with respect to the sequence they appear along the planning horizon, i.e., \( h \) pricing regimes switch over at respective time points \( t_1, t_2, \ldots, t_h = T (t_0 = 0) \). To distinguish between optimal state, co-state, and decision variables determined to the accuracy of constant amplitudes for each regime, we adjoin index \( i \) as a superscript to each corresponding variable and its amplitudes. From Lemmas 7 and 8, it immediately follows that the regime switching times, \( t_i \), satisfy the following non-linear equations:
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\[ \sum_{m} a_m (A_m \sin \theta_m t + B_m \cos \theta_m t + C_m) + \sum_{m} a_m (\hat{A}_m e^{\gamma t} + \hat{B}_m e^{-\gamma t} + C_m) = M, \]

if \( i \) is a loose pricing regime; \( 18 \)

\[ 2gM - \sum_{n=1}^{N} [a_n(t) - L_n \psi'(t)] = 0, \]

if \( i \) is a tense pricing regime. \( 19 \)

Using the modified notations, systems \((13)\) and \((16)\) take the following form respectively.

\[ q_n^1(0) = q_n^0, \quad \psi'_n(t_{i-1}) = -p_n^j(t_{i-1}) - D_n \psi'_n(t_{i-1}), \quad \psi_n^h(T) = 0, \]

\[ p_n^j(t_{i-1}) - p_n^j(t_{i-1}) = \frac{q_n^j(t_{i-1}) - L_n \psi'_n(t_{i-1})}{2ga_n}, \quad \psi'_n(t_{i-1}) = \frac{L_nP_n + D_n q_n(t_{i-1}) + L_n D_n \psi'(t_{i-1})}{2ga_n}. \]

And

\[ q_n^1(0) = q_n^0, \quad \psi'_n(t_{i-1}) = -p_n^j(t_{i-1}) - D_n \psi'_n(t_{i-1}), \quad \psi_n^h(T) = 0; \]

\[ p_i^n(t_{i-1}) - p_i^n(t_{i-1}) = \frac{q_i^n(t_{i-1}) - L_i \psi'_i(t_{i-1})}{2ga_i}, \quad \psi'_i(t_{i-1}) = \frac{L_iP_i + D_i q_i(t_{i-1}) + L_i D_i \psi'_i(t_{i-1})}{2ga_i}; \]

\[ \dot{q}_n^i(t_{i-1}) = L_n(P_n - p_n^j(t_{i-1})) + D_n q_n(t_{i-1}). \]

The following lemma shows how to employ systems \((20)\) and \((21)\) in order to obtain an exact optimal pricing for any sequence of regimes.

**Lemma 9.** Let optimal solution for \( p_n(t), q_n(t) \) and \( \psi_n(t) \) be determined by \( h \) pricing regimes switching over at time points \( t_i \) which satisfy \((18)/(19)\) if \( i \) is a loose/tense pricing regime i.e., \( p_n(t), q_n(t) \) and \( \psi_n(t) \), \( t_{i-1} \leq t \leq t_i \) be known for \( i=1,2,...,h \) \( (t_0=0) \) and \( n=1,2,...,N \) to an accuracy of constant amplitudes. These amplitudes satisfy the following conditions:

- if \( i=1 \) is a loose/tense pricing regime then boundary condition \( \psi_n^h(T) = 0 \) in \((13)/(16)\) is replaced with \( \psi'_n(t_1) = \psi'_n(t_1) \);

- if \( i \neq 1 \) and \( i \neq h \) is a loose/tense pricing regime then boundary conditions \( q_n^i(0) = q_n^0 \) and \( \psi_n^h(T) = 0 \) in \((13)/(16)\) are replaced with \( q_n^{i-1}(t_{i-1}) = q_n^i(t_{i-1}) \) and \( \psi_n(t_i) = \psi_n^{i+1}(t_i) \) respectively;

- if \( i=h \) is a loose/tense pricing regime then boundary conditions \( q_n^h(0) = q_n^0 \) in \((13)/(16)\) is replaced with \( q_n^{h-1}(t_{h-1}) = q_n^h(t_{h-1}) \).
Proof: see appendix

Note, that according to Lemmas 1–9, when the planning horizon is relatively long, loose pricing eventually causes the price index to rise to its upper bound, which affects consumer loyalties. Therefore, to restore and maintain those consumer loyalties, the tense pricing regime must follow the loose pricing regime. Once the loyalties are ensured for a period of time we can get back to loose pricing again and so forth. This methodology incurs significant computations to locate all consecutive regimes for products having a long life cycle. Most modern products, however, have a quite limited life, which implies that only a very limited number of regimes will be revealed over the planning horizon. Moreover, a short life cycle indicates that even if consumer loyalties weaken due to a loose upper bound, $M$, of the price index, they can be restored by introducing a new product instead of a new pricing regime. Such a product can be very similar to the existing one and characterized by a new, preferably lower price. This is a well known practical alternative, which yields a reduction in the overall price index, decreases customer “fatigue” to the product, and eliminates the need for tense pricing irrespective as to how high $M$ is. Therefore, for most products, the calculations can be reduced to that of loose pricing, which will be re-determined each time a new product modification is created. Thus, on top of the theoretical findings and important insights discussed above, the approach is easy and practical to implement for most products. We illustrate this with a numerical example in the next section. It is important to note, that the example exhibits a common practice of setting low introductory prices for new products or new product modifications.

6. Numerical Example

We consider two products having substantially different characteristics over planning horizon $T=105$ time units, location loyalty $d=0.2$, very loose upper bound $M$ and moderate price index effect $g=4450$. One product, $n=1$, is an expensive brand name which is popular and has a significant effect on consumers. In contrast, the other product, $n=2$, is relatively cheap, and has a low effect on consumers. The data are summarized in Table 1.
Table 1: Problem parameters for the example

<table>
<thead>
<tr>
<th>N</th>
<th>$c_n$</th>
<th>$P_n$</th>
<th>$D_n$</th>
<th>$b_n$</th>
<th>$l_n$</th>
<th>$L_n$</th>
<th>$a_n$</th>
<th>$q_n^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.07</td>
<td>120.0</td>
<td>0.014</td>
<td>509.0</td>
<td>0.6</td>
<td>203.6</td>
<td>0.6</td>
<td>20000</td>
</tr>
<tr>
<td>2</td>
<td>0.005</td>
<td>7.0</td>
<td>0.001</td>
<td>18.75</td>
<td>0.2</td>
<td>15.0</td>
<td>0.4</td>
<td>4000</td>
</tr>
</tbody>
</table>

We first apply Lemmas 2 and 7 to solve problem (1)-(3) by loose pricing over the entire planning horizon. Since bound $M$ is loose the feasibility condition will always be met, i.e., the calculated solution is optimal.

We begin by calculating the disloyalty index for both products.

\[ I_1 = dc_1 \left( \frac{(1-l_1)b_1}{ga_1} - dc_1 \right) = 0.00087025 \] and
\[ I_2 = dc_2 \left( \frac{(1-l_2)b_2}{ga_2} - dc_2 \right) = 0.00000729. \]

Since this index is positive, then \( \frac{(1-l_n)b_n}{ga_n} - dc_n > 0 \), for both \( n=1 \) and \( n=2 \) and according to Lemma 2 the optimal pricing is of harmonic type:

\[ p_n(t) = A_n \sin w_n t + B_n \cos w_n t + C_n \quad \text{for} \ n=1 \text{ and } n=2 \quad \text{and} \ 0 \leq t \leq 105. \]

To determine the parameters of these harmonics, consider first product \( n=1 \). According to Lemma 3, optimal fluctuation frequency and centricity parameters are

\[ w_1 = \sqrt{dc_1 \left( \frac{(1-l_1)b_1}{ga_1} - dc_1 \right)} = 0.0295 \quad \text{and} \quad C_1 = \frac{(1-l_1)b_1 P_1}{2(1-l_1)b_1 - gd_1 c_1} = 73.5; \quad (22) \]

respectively. To find the optimal amplitudes we turn to Lemma 7. That is, we plug problem parameters from Table 1 and (22) into system (14) of four equations:

\[ 2G_1 \cdot 0.014 - \left( \frac{0.0295B_1 - 0.014A_1}{0.0295^2 + 0.014^2} \right) 0.0295 = -B_1 - 2 \cdot 73.5 - \frac{(0.0295A_1 + 0.014B_1) \cdot 0.014}{0.0295^2 + 0.014^2}; \]

\[ G_1 e^{0.014105 \cdot K_s e^{-0.014105}} - \left( \frac{0.0295B_1 - 0.014A_1}{0.0295^2 + 0.014^2} \right) \sin(0.0295 \cdot 105) + \]
\[ \frac{(0.0295A_1 + 0.014B_1)}{0.0295^2 + 0.014^2} \cos(0.0295 \cdot 105) + \frac{73.5}{0.014} = 0; \]

\[ (B_1 + 73.5) \cdot 2.4450 \cdot 0.6 = 20000 - 203.6G_1 - 203.6K_1 - \frac{(0.0295A_1 + 0.014B_1) \cdot 203.6}{0.0295^2 + 0.014^2} - \frac{73.5}{0.014} 203.6; \]
Solving this system we obtain
\[ A_1 = 2431.41, B_1 = -4866.4, G_1 = -8669.66 \text{ and } K_1 = 125806. \]
Thus, the optimal loose pricing is
\[ p(t) = 2431.41\sin(0.0295t) - 4866.4\cos(0.0295t) + 73.5, \quad 0 \leq t \leq 105. \]
To find the demand evolution, we solve System A which yields
\[ E_1 = -0.037 \text{ and } F_1 = 0.034. \]
Thus, \[ q(t) = -0.037e^{0.014t} + 0.034e^{-0.014t} + 3.34 \cdot 10^3 \sin(0.0295t) - 6.963 \cdot 10^5 \cos(0.0295t) - 676345. \]
The dynamics of the obtained optimal prices and quantity demanded for the first product are depicted in Figure 1. From this figure one can observe that the optimal price starts from negative values, indicating an introductory price which is below the marginal cost for products of type \( n=1 \). In addition, these products can be given away for free as a promotional gimmick. These gimmicks are widely used and have become fairly standard marketing devices in many industries. An example that comes immediately to mind is that of wine tasting. Free tasting is common for high quality wines, and often free sample bottles (albeit, rather small bottles) are simply given away to encourage potential customers to try (and hopefully get hooked on) the product. Once the introductory period is completed, the price increases harmonically and substantially in order to project an image of a high quality product, and to regain profits forfeited during the advertising period. Note, that starting from the advertising period the demand for the product is continuously increasing. The rate of increase, however, declines as the price rises. Eventually the demand regresses back towards its initial levels. However, by the end of the planning horizon the demand, \( q(105) = 42543 \), is still well above the initial level \( q(0) = 20000 \).

Assuming now that product \( n=1 \) undergoes a modification by the end of the planning horizon, we now recalculate the optimal solution for the next equal length planning horizon of \( T=105 \) time units based on actual terminal demand for its prototype, \( q(105) = 42543 \). Solving system (15) now yields the following constants
\[ A_1 = 2771.37, B_1 = -4942.23, G_1 = -8780.38, K_1 = 127905.7, \]
This new result is depicted in Figure 2 and presents a second zigzag in pricing from $t=105$ to $t=210$. The evolution over the second planning period is very similar to that of the first, which implies that in order to restore customer interest and to arouse this interest for a modification of the expensive product, we would once again need to undertake a promotion campaign. Note that the quantity demanded by the end of the second planning horizon $q_1(210)=42299$ is almost equal to its initial level $q_1(105)=42543$. That is to say, we will obtain the same result if we recalculate the optimal solution again for the third planning horizon, i.e., each product modification that does not effect the product’s main characteristics from Table 1 will simply result in a very similar price zigzag.

We now repeat the same calculations for product $n=2$ from $t=0$ to $t=105$. Plugging numbers for $n=2$ from Table 1 into the same system of equation, we find

$$p_2(t) = 11.584\sin(0.0027t) - 3.8\cos(0.0027t) + 3.971, \quad 0 \leq t \leq 105.$$ 

$$q_2(t) = 0.000135e^{0.001t} - 0.000065e^{-0.001t} + 39055.7\sin(0.0027t) + 49431.2\cos(0.0027t) - 45431.2.$$ 

Note that this time both price and quantity gradually increase over the first planning horizon. This happens again if we recalculate the optimal solution for the second planning horizon from $t=105$ to $t=210$, with the initial demand for the second horizon equal to the demand we obtained at the end of the first horizon, $q_2(105)=13014$. Both solutions are depicted in Figure 2. It turns out that loose pricing for product $n=2$ results in a zigzag policy very different from that for product $n=1$. As the effect of product $n=2$ on customers is weak, the optimal solution simply starts from a low price $p_2(0)=0.17$ rather than with a price which is below the marginal cost. This price then slowly increases towards its basic level along with the demand for this product type, $p_2(105)=3.6$ and $q_2(105)=13014$. At the second planning horizon, the initial $p_2(105+0)=1.8$ and terminal $p_2(210)=5.5$ prices are higher than the corresponding prices for the first planning horizon $p_2(0)=0.17$ and $p_2(105)=3.6$, but the initial price at the second horizon is lower than the price at the end of the first planning horizon. Thus, each new zigzag initially lowers the product price to increase the quantity demanded. This price then exceeds the level arrived at by the previous zigzag policy. Quantity demanded for product $n=2$ monotonously increases to $q_2(210)=20086$ instead of “zigzagging” as was the case with $n=1$.

Thus, unlike the pricing zigzags of the first product which are almost identical at each planning horizon, and mostly well above the basic price $P_1=120$, pricing zigzags for the second
product differ for different planning horizons, increase gradually in value, and remain below
the basic price $P_2=7$.

7. Conclusions
In this paper we suggest a general model for pricing of products over a continuous-time
planning horizon under brand, location, and store loyalty. The objective is to find an
appropriate trade-off between net profit goals and the overall price level. With the aid of the
maximum principle we prove that there can be only two optimal pricing regimes: loose and
tense pricing which follow each other consecutively along the planning horizon. In contrast to
loose pricing, tense pricing is characterized by a constant price index set at an upper bound to
maintain consumer loyalty.

We derive a disloyalty index which determines the form of both pricing regimes. This
index compares the ratio between brand “disloyalty”, and store loyalty weighted with system
constants and the location loyalty. In general, the lower the brand, location, and store loyalties
the greater the disloyalty index. If this index is positive the optimal tense and loose pricing has
harmonic shape, otherwise it is of exponential form. The major insight we gain is that zigzag
pricing is advantageous if the store is willing to compromise its net profit goals with the goal of
maintaining a reasonable price index. Furthermore, harmonic cyclic pricing is not optimal if
the store does not compromise at all, or vise versa if it compromises too much, so that the
loyalty index becomes negative and price index considerations come to dominate net profit
considerations.

We show that if the upper bound of the price index is high, i.e., the loose pricing is
exercised over a long planning horizon, and the disloyalty index is positive, then the optimal
cyclic pricing is characterized by a frequency proportional to the disloyalty index. Moreover, if
a product has a relatively short life cycle, so that the pricing trajectory is too short for even a
single cycle to be realized, and the consumer loyalties are maintained by re-determining the
optimal solution each time product modification is available (rather than by imposing tense
pricing), the optimal pricing policy will still have multiple zigzags.

The rationale for the derived pricing zigzags is that starting with a price reduction of a
product encourages customers to visit the store, and increases product sales. The stronger the
effect that a particular product has on customers with respect to their location and brand
loyalties, the greater the initial price reduction that is justified. This may generate a future demand large enough to enable a much desired price increase. The price increase however eventually reduces the quantity demanded of the product, and as a result, future price reductions become inevitable. This cycle repeats itself for the purpose of renewing and maintaining the brand and location loyalties, which otherwise may diminish over time.
Figure 1. Optimal evolution of price and quantity demand of product $n=1$

Figure 2. Optimal evolution of price and quantity demand of product $n=2$
References


Appendix

**Proof of Lemma 1**: We use the Lagrange multipliers method to maximize (8) subject to (2). Namely, by differentiating the Lagrangian,

\[
\Lambda(t) = \sum_{n=1}^{N} \left[ p_n(t)q_n(t) - ga_n p_n^2(t) \right] - \sum_{n=1}^{N} \psi_n(t) L_n p_n(t) + \lambda \left[ \sum_{n=1}^{N} a_n p_n(t) - M \right]
\]

with respect to \( p_n(t) \) we find,

\[
p_n(t) = \frac{q_n(t) - L_n \psi_n(t) + \lambda(t)a_n}{2ga_n},
\]

where the Lagrange multiplier, \( \lambda(t) \geq 0 \), satisfies the following complementary slackness condition

\[
\lambda(t) \left[ \sum_{n=1}^{N} a_n p_n(t) - M \right] = 0.
\]

Thus, if \( \sum_{n=1}^{N} a_n p_n(t) < M \), then \( \lambda(t) = 0 \) and \( p_n(t) = \frac{q_n(t) - L_n \psi_n(t)}{2ga_n} \) as stated in the lemma. Otherwise,

\[
\sum_{n=1}^{N} a_n p_n(t) = M.
\]

Substituting (2A) into (4A) we immediately determine \( \lambda(t) \) as stated in the lemma.

**Proof of Lemma 2**: According to Lemma 1, given \( \sum_{n=1}^{N} a_n p_n(t) \leq M \) for \( t \in \tau \), the optimal pricing for products is

\[
p_n(t) = \frac{q_n(t) - L_n \psi_n(t)}{2ga_n}, \quad n=1,2,\ldots,N.
\]

By differentiating equation (6A) over the interval \( \tau \), we find

\[
\dot{p}_n(t) = \frac{\dot{q}_n(t) - L_n \dot{\psi}_n(t)}{2ga_n} \quad \text{for} \quad t \in \tau, \quad n=1,2,\ldots,N.
\]

Taking into account equations (1) and (7), we obtain from (7A):

\[
\dot{p}_n(t) = \frac{L_n(P_n - p_n(t)) + D_n q_n(t) + L_n p_n(t) + L_n D_n \psi_n(t)}{2ga_n} = \frac{L_n P_n + D_n q_n(t) + L_n D_n \psi_n(t)}{2ga_n}.
\]
Next, multiplying equation (6A) by $D_n$ and adding it to equation (8A), we have

$$D_n p_n(t) + \dot{p}_n(t) = \frac{L_n P_n + 2D_n q_n(t)}{2ga_n}. \quad (9A)$$

By differentiating equation (9A) over the interval $\tau$ and substituting equation (1), we obtain:

$$D_n \ddot{p}_n(t) + \dddot{p}_n(t) = \frac{2D_n L_n (P_n - p_n(t)) + 2D_n^2 q_n(t)}{2ga_n}. \quad (10A)$$

Finally, multiplying equation (9A) by $(-D_n)$ and adding it to equation (10A), we find a second order differential equation in prices $p_n(t)$ which involves only constant parameters of the model and is independent of both state $q_n(t)$ and co-state $\psi_n(t)$ variables.

$$\ddot{p}_n(t) + D_n \left( \frac{L_n}{ga_n} - D_n \right) p_n(t) = \frac{D_n L_n P_n}{2ga_n}. \quad (11A)$$

The solution to this equation depends on whether the relationship $\frac{L_n}{ga_n} - D_n$ is positive or negative. If $\frac{L_n}{ga_n} - D_n > 0$, then the optimal price is a linear combination of two harmonic functions:

$$p_n(t) = A_n \sin w_n t + B_n \cos w_n t + C_n, \quad (12A)$$

where amplitudes $A_n$ and $B_n$ are the integration constants,

$$C_n = \frac{L_n P_n}{2(L_n - ga_n D_n)} > 0 \quad (13A)$$

and the frequency of the price fluctuations is

$$w_n = \sqrt{D_n \left( \frac{L_n}{ga_n} - D_n \right)} \quad (14A)$$

On the other hand, if $\frac{L_n}{ga_n} - D_n < 0$, then the optimal price is a linear combination of two diverging exponents:

$$p_n(t) = \hat{A}_n e^{\gamma t} + \hat{B}_n e^{-\gamma t} + C_n, \quad (15A)$$

where $\hat{A}_n, \hat{B}_n$ are the integration constants, $C_n$ becomes negative but is still defined by (15A), and
Optimal pricing with loyalty

\[ z_n = \sqrt{D_n \left( D_n - \frac{L_n}{g_n} \right)} . \]  

(16A)

The last condition of the lemma is due to \( \frac{L_n}{g_n} - D_n = 0 \). In such a case,

\[ p_n(t) = \frac{D_n L_n P_n}{4g_n} t^2 + A_n t + B_n . \]  

(17A)

The proof is completed by returning to our original parameters with the aid of (4) and (5) in all derived conditions as stated in the lemma.

\[ \blacksquare \]

**PROOF of LEMMA 3**: According to Lemma 1, given \( \sum_{n=1}^{N} a_n p_n(t) = M \) for \( t \in \tau \), the optimal pricing for products is determined by equation (2A), where

\[ \lambda(t) = \frac{2gM - \sum_{n=1}^{N} [q_n(t) - L_n \psi_n(t)]}{\sum_{n=1}^{N} a_n} \geq 0 . \]  

(18A)

By substituting (18A) into (2A) and differentiating the resulting equation over interval \( \tau \), we find

\[ p_n(t) = \frac{\dot{q}_n(t) - L_n \dot{\psi}_n(t)}{2ga_n} - \frac{\sum_{i=1}^{N} [\dot{q}_i(t) - L_i \dot{\psi}_i(t)]}{2g \sum_{n=1}^{N} a_i} \quad \text{for} \quad t \in \tau , \quad n=1,2,\ldots,N . \]  

(19A)

Taking into account equations (1) and (7), we obtain from (19A):

\[ \dot{p}_n(t) = \frac{L_n P_n + D_n q_n(t) + L_n D_n \psi_n(t)}{2ga_n} \cdot \frac{\sum_{i=1}^{N} [L_i P_i + D_j q_j(t) + L_j D_j \psi_j(t)]}{2g \sum_{i=1}^{N} a_i} . \]  

(20A)

Using (2A) and (20A) for two different products \( i \) and \( j \), we find

\[ p_i(t) - p_j(t) = \frac{q_i(t) - L_i \psi_i(t)}{2ga_i} - \frac{q_j(t) - L_j \psi_j(t)}{2ga_j} ; \]  

(21A)

\[ \dot{p}_i(t) - \dot{p}_j(t) = \frac{L_i P_i + D_j q_j(t) + L_i D_j \psi_j(t)}{2ga_i} \cdot \frac{L_j P_j + D_j q_j(t) + L_j D_j \psi_j(t)}{2ga_j} . \]  

(22A)
Next, denoting $X_{ij}(t) = p_i(t) - p_j(t)$, multiplying equation (21A) by $D$, taking into account that $D_n=D$ for $n=1,2,\ldots,N$ and adding it to equation (22A), we have

$$DX_{ij}(t) + \dot{X}_{ij}(t) = \frac{L_i P_i + 2Dq_i(t)}{2ga_j} - \frac{L_j P_j + 2Dq_j(t)}{2ga_j}.$$  \hfill (23A)

By differentiating equation (23A) over the interval $\tau$ and substituting equation (1), we obtain:

$$DX_{ij}(t) + \dot{X}_{ij}(t) = \frac{2DL_i(P_i(t) - p_i(t)) + 2D^2q_i(t)}{2ga_i} - \frac{2DL_j(P_j(t) - p_j(t)) + 2D^2q_j(t)}{2ga_j}.$$  \hfill (24A)

Finally, multiplying equation (23A) by $(-D)$, adding it to equation (24A) and taking into account that $\frac{L_n}{a_n} = R$, we find a second order differential equation in price difference $X_{ij}(t) = p_i(t) - p_j(t)$ which involves only constant parameters of the model and is independent of both state $q_n(t)$ and co-state $\psi_n(t)$ variables.

$$\ddot{X}_{ij}(t) + D\left(\frac{R}{g} - D\right)X_{ij}(t) = \frac{DR}{2g}(P_i(t) - P_j(t)).$$ \hfill (25A)

Similar to Lemma 3, solution to this equation depends on the sign of $\frac{L_n}{a_n} - D_n = \frac{R}{g} - D$, i.e., disloyalty coefficient, $I_n=I$ (as it is independent now of $n$). If it is positive, equal to zero, or negative the price difference has a periodic, parabolic and exponential shape respectively. The only difference is that parameter $w_n = w$ does not depend on product index $n$, while integration constants $A_n, B_n$ and coefficient $C_n$ now depend on two indexes $i$ and $j$:

$$C_{ij} = \frac{R(P_i(t) - P_j(t))}{2(R - Dg)}.$$ \hfill (26A)

To find explicit equations for pricing policies rather than for their differences, we recall the tense regime condition, $\sum_{n=1}^{N} a_n p_n(t) = M$. Specifically, if $I_n>0$, then we can select a product, $n$, to express prices of all the other products, $i$, via this product:

$$p_i(t) = A_n \sin wt + B_n \cos wt + C_m - p_n(t), \quad i \neq n.$$ \hfill (27A)

Then substituting equation (36) into the tense regime condition we obtain

$$\sum_{i=1}^{N} a_i(A_n \sin wt + B_n \cos wt + C_m) - p_n(t)\sum_{i=1}^{N} a_i = M.$$ \hfill (28A)
from where the optimal price for product $n$ is readily obtained as stated in the lemma. This price is then substituted into equations (27A) to find the optimal pricing for the other products. Similarly, one can obtain optimal pricing for the cases of $I<0$.

**PROOF of LEMMA 4:** From $\sum_{n=1}^{N} a_n p_n(t) = M$ for $t \in \tau$, $N=2$, we first express $p_2(t)$:

$$p_2(t) = \frac{M - a_1 p_1(t)}{a_2}. \quad (29A)$$

Differentiating equation (29A) twice over $t \in \tau$, we obtain

$$\dot{p}_2(t) = -\frac{a_1}{a_2} \dot{p}_1(t), \quad (30A)$$

$$\ddot{p}_2(t) = -\frac{a_1}{a_2} \ddot{p}_1(t). \quad (31A)$$

Next, multiplying equation (23A) by $(-D)$, adding it to equation (24A) and returning to the original notations $X_j(t) = p_j(t) - p_j(t)$, we find

$$-D^2(p_1(t) - p_2(t)) + (\dot{p}_1(t) - \ddot{p}_2(t)) = \frac{2DL_1(P_1 - p_1(t)) - DL_2P_1}{2ga_1} - \frac{2DL_2(P_2 - p_2(t)) - DL_2P_2}{2ga_2}. \quad (32A)$$

Substituting (29A) and (31A) into the last equation we obtain

$$\ddot{p}_1(t) + \left(\frac{L_1}{ga_1} - D\right) + \left(\frac{L_2}{ga_2} - D\right) \frac{a_1}{a_2} D\frac{a_2}{a_1 + a_2} p_1(t) = \frac{a_2}{a_1 + a_2} D\frac{L_1}{2ga_1} \frac{L_2}{2ga_2} (D - \frac{L_2}{ga_2}) \frac{M}{a_2}. \quad (32A)$$

Similar to Lemmas 2 and 3, the solution to this equation is either a sum of harmonics or of exponents. Different from the previous lemmas, the type of the solution is determined by a weighted sum of the disloyalty indexes, which enables coordinating the pricing of different products

$$\hat{I} = \left(\frac{L_1}{ga_1} - D\right) \frac{1}{a_1} + \left(\frac{L_2}{ga_2} - D\right) \frac{1}{a_2} D\frac{a_1 a_2}{a_1 + a_2}, \quad (32A)$$

as stated in Lemma 4. Based on this index and (29A) the prices stated in the lemma are immediately determined.
PROOF of LEMMA 5: Consider first the case of positive disloyalty index. Then according to Lemma 3, we have \( \frac{L_n}{g a_n} - D_n > 0 \), and \( p_n(t) = A_n \sin w_n t + B_n \cos w_n t + C_n \). Substituting this price and its derivative into (10), we obtain
\[
\ddot{q}_n(t) - D^2 q_n(t) = \left( w_n B_n - D_n A_n \right) L_n \sin w_n t - \left( w_n A_n + D_n B_n \right) L_n \cos w_n t + \left( P_n - C_n \right) D_n L_n. 
\tag{33A}
\]
Solving this equation yields:
\[
q_n(t) = E_n e^{D_1 t} + F_n e^{-D_1 t} - \left( \frac{w_n B_n - D_n A_n}{w_n^2 + D_n^2} \right) \sin w_n t + \left( \frac{w_n A_n + D_n B_n}{w_n^2 + D_n^2} \right) \cos w_n t - \frac{P_n - C_n}{D_n} L_n.
\tag{34A}
\]
Similarly, substituting the optimal price (15A) into (11), we have
\[
\dot{\psi}_n(t) - D^2 \psi_n(t) = \left( w_n B_n - D_n A_n \right) \sin w_n t - \left( w_n A_n + D_n B_n \right) \cos w_n t - C_n D_n.
\tag{35A}
\]
Solving this equation yields:
\[
\psi_n(t) = G_n e^{D_2 t} + K_n e^{-D_2 t} - \left( \frac{w_n B_n - D_n A_n}{w_n^2 + D_n^2} \right) \sin w_n t + \left( \frac{w_n A_n + D_n B_n}{w_n^2 + D_n^2} \right) \cos w_n t + \frac{C_n}{D_n}.
\tag{36A}
\]
Let us now \( \frac{L_n}{g a_n} - D_n < 0 \). Then equation (33A) and (35A) take the following form
\[
\ddot{q}_n(t) - D^2 q_n(t) = - \left( z_n + D_n \right) \hat{A}_n L_n e^{z_1 t} + \left( z_n - D_n \right) \hat{B}_n L_n e^{z_1 t} + \left( P_n - C_n \right) D_n L_n,
\tag{37A}
\]
\[
\dot{\psi}_n(t) - D^2 \psi_n(t) = - \left( z_n + D_n \right) \hat{A}_n e^{z_1 t} + \left( z_n - D_n \right) \hat{B}_n e^{z_1 t} - C_n D_n L_n.
\tag{38A}
\]
The solutions to these equations are
\[
q_n(t) = \hat{E}_n e^{D_1 t} + \hat{F}_n e^{-D_1 t} - \left( \frac{z_n + D_n}{z_n^2 - D_n^2} \right) \hat{A}_n L_n e^{z_1 t} + \left( \frac{z_n - D_n}{z_n^2 - D_n^2} \right) \hat{B}_n L_n e^{z_1 t} - \frac{P_n - C_n}{D_n} L_n,
\tag{39A}
\]
\[
\psi_n(t) = \hat{G}_n e^{D_2 t} + \hat{K}_n e^{-D_2 t} - \left( \frac{z_n + D_n}{z_n^2 - D_n^2} \right) \hat{A}_n e^{z_1 t} + \left( \frac{z_n - D_n}{z_n^2 - D_n^2} \right) \hat{B}_n e^{z_1 t} + \frac{C_n}{D_n},
\tag{40A}
\]
respectively as stated in the lemma.

PROOF of LEMMA 6: Consider first the case of positive weighted disloyalty index,
\[
\left( \frac{L_n}{g a_1} - D_n \right) \frac{1}{a_1} + \left( \frac{L_n}{g a_2} - D_n \right) \frac{1}{a_2} > 0.
\]
Then according to Lemma 4, \( p_1(t) = A \sin wt + B \cos wt + C \).
and \( p_2(t) = \frac{M}{a_2} - \left( A \sin wt + B \cos wt + C \right) \frac{a_1}{a_2}. \) Substituting this price and its derivative into (10), we obtain,

\[
\dot{q}_1(t) - D^2 q_1(t) = \left( wB - DA \right) L_1 \sin wt - \left( wA + DB \right) L_1 \cos wt + \left( P_1 - C \right) D L_1,
\]

(41A)

\[
\ddot{q}_2(t) - D^2 q_2(t) = -\frac{a_1}{a_2} \left( wB - DA \right) L_2 \sin wt + \frac{a_1}{a_2} \left( wA + DB \right) L_2 \cos wt + \left( P_2 + C \frac{a_1}{a_2} - \frac{M}{a_2} \right) D L_2.
\]

(42A)

The solutions to these equations are:

\[
q_1(t) = U_1 e^{\nu t} + V_1 e^{-\nu t} - \frac{\left( wB - DA \right) L_1}{w^2 + D^2} \sin wt + \frac{\left( wA + DB \right) L_1}{w^2 + D^2} \cos wt - \frac{\left( P_1 - C \right) L_1}{D},
\]

(43A)

\[
q_2(t) = U_2 e^{\nu t} + V_2 e^{-\nu t} + \frac{\left( wB - DA \right) L_2}{w^2 + D^2} \sin wt - \frac{\left( wA + DB \right) L_2}{w^2 + D^2} \cos wt - \frac{\left( P_2 + C \frac{a_1}{a_2} - \frac{M}{a_2} \right) L_2}{D}.
\]

(44A)

Next, substituting the optimal price (15A) into (11), we have

\[
\dddot{q}_1(t) - D^3 \psi_1(t) = \left( wB - DA \right) \sin wt - \left( wA + DB \right) \cos wt - CD.
\]

(45A)

\[
\dddot{q}_2(t) - D^3 \psi_2(t) = -\frac{a_1}{a_2} \left( wB - DA \right) \sin wt + \frac{a_1}{a_2} \left( wA + DB \right) \cos wt + \left( C \frac{a_1}{a_2} - \frac{M}{a_2} \right) D.
\]

(46A)

The solutions to these equations are:

\[
\psi_1(t) = X_1 e^{\nu t} + Y_1 e^{-\nu t} - \frac{\left( wB - DA \right)}{w^2 + D^2} \sin wt + \frac{\left( wA + DB \right)}{w^2 + D^2} \cos wt + \frac{C}{D},
\]

(47A)

\[
\psi_2(t) = X_2 e^{\nu t} + Y_2 e^{-\nu t} + \frac{\left( wB - DA \right)}{w^2 + D^2} \sin wt - \frac{\left( wA + DB \right)}{w^2 + D^2} \cos wt - \frac{\left( C \frac{a_1}{a_2} - \frac{M}{a_2} \right)}{D}.
\]

(48A)

Similarly, the solutions for the case of \( \left( \frac{L_1}{ga_1} - D \right) \frac{1}{a_1} + \left( \frac{L_2}{ga_2} - D \right) \frac{1}{a_2} < 0 \) are determined as stated in this lemma.
PROOF of LEMMA 7: The loose regime is realized over the entire planning horizon if \[
\sum_{n=1}^{N} a_n p_n(t) \leq M \quad \text{for} \quad 0 \leq t \leq T.
\]
Substituting the prices found in Lemma 2 for this regime with respect to the sets (12), this condition takes the following form:

\[
\sum_{n \in S_i} a_n \left( A_n \sin \omega_n t + B_n \cos \omega_n t + C_n \right) + \sum_{n \in S_i} a_n \left( \hat{A}_n e^{\omega_n t} + \hat{B}_n e^{-\omega_n t} + C_n \right) \leq M,
\]
as stated in the lemma. The pricing policies are determined by Lemma 2 to an accuracy of the constant amplitudes for the arbitrary boundary conditions imposed on the state, co-state and decision variables, as well as their derivatives. This implies, that these policies are optimal over the entire planning horizon, i.e., the maximum principle (6)-(8) is satisfied, if the exact values for the amplitudes take into account all existing boundary conditions. Recall that according to equation (1), (7), (6A), (8A) and (18A) we have by setting \( t=0 \) and \( t=T \):

\[
q_n(0) = q_n^0, \quad \psi_n'(0) = -p_n(0) - D_n \psi_n'(0), \quad \psi_n(T) = 0,
\]

\[
p_n(0) = \frac{q_n(0) - L_n \psi_n(0)}{2g a_n}, \quad \dot{p}_n(0) = \frac{L_n p_n + D_n q_n(0) + L_n D_n \psi_n(0)}{2g a_n}.
\]
as stated in the lemma.

**System A:** for \( n \in S_2 \)

\[
2 \hat{G}_n D_n - \left( \frac{z_n + D_n}{D_n} \right) \hat{A}_n z_n = \frac{z_n - D_n}{D_n} \hat{B}_n z_n = -\hat{A}_n - \hat{B}_n - 2C_n + \left( \frac{z_n + D_n}{D_n} \right) \hat{A}_n D_n - \frac{z_n - D_n}{D_n} \hat{B}_n D_n;
\]

\[
\hat{G}_n e^{D_n T} + \hat{K}_n e^{-D_n T} = \frac{z_n + D_n}{D_n} \hat{A}_n e^{T} + \frac{z_n - D_n}{D_n} \hat{B}_n e^{-T} + C_n; \quad \frac{z_n - D_n}{D_n} = 0;
\]

\[
2 g a_n \left( \hat{A}_n + \hat{B}_n + C_n \right) = q_n^0 - L_n \hat{G}_n - L_n \hat{K}_n + \left( \frac{z_n + D_n}{D_n} \right) \hat{A}_n L_n - \frac{z_n - D_n}{D_n} \hat{B}_n L_n - \frac{C_n}{D_n} L_n;
\]

\[
2 g a_n \left( \hat{A}_n \dot{z}_n - \hat{B}_n \dot{z}_n \right) = L_n p_n + D_n q_n^0 + L_n \hat{G}_n D_n + L_n \dot{\hat{K}}_n D_n - \frac{z_n + D_n}{D_n} \hat{A}_n L_n D_n + \frac{z_n - D_n}{D_n} \hat{B}_n L_n D_n + C_n L_n.
\]

**System B:** for \( n \in S_1 \)

\[
2 X_1 D - \left( \frac{(wB - DA)w}{w^2 + D^2} \right) = -B - 2C - \left( \frac{(wA + DB)D}{w^2 + D^2} \right);
\]
\[ 2X_2 D + \frac{a}{a_2} \left( \frac{wB - DA}{w^2 + D^2} \right) = -\frac{2M}{a_2} + \left( B + 2C \right) \frac{a_1}{a_2} + \frac{a}{a_2} \left( \frac{wA + DB}{w^2 + D^2} \right); \]

\[ X_1 e^{\lambda t} + Y_1 e^{-\lambda t} - \frac{(wB - DA)}{w^2 + D^2} \sin wT + \frac{(wA + DB)}{w^2 + D^2} \cos wT + \frac{C}{D} = 0; \]

\[ X_2 e^{\lambda t} + Y_2 e^{-\lambda t} + \frac{a}{a_2} \left( \frac{wB - DA}{w^2 + D^2} \right) \sin wT - \frac{a_1}{a_2} \left( \frac{wA + DB}{w^2 + D^2} \right) \cos wT - \frac{C}{D} \left( \frac{a}{a_2} - \frac{M}{a_2} \right) = 0; \]

\[ (B + C)(1 + \frac{a_1}{a_2}) - \frac{M}{a_2} = \frac{q_1^0}{2ga_1} - \left( X_1 + Y_1 + \frac{(wA + DB)}{w^2 + D^2} + \frac{C}{D} \right) \frac{L_1}{2ga_2}. \]

\[ -\frac{q_2^0}{2ga_2} + \left( X_2 + Y_2 - \frac{a_1}{a_2} \left( \frac{wA + DB}{w^2 + D^2} \right) - \left( \frac{C}{a_2} - \frac{M}{a_2} \right) \frac{1}{D} \right) \frac{L_2}{2ga_2}; \]

\[ Aw(1 + \frac{a_1}{a_2}) = \frac{L_1 P_1 + Dq_1^0}{2ga_1} + \left( X_1 + Y_1 + \frac{(wA + DB)}{w^2 + D^2} + \frac{C}{D} \right) \frac{L_1 D}{2ga_2}. \]

\[ L_2 P_2 + Dq_2^0 - \left( X_2 + Y_2 - \frac{a_1}{a_2} \left( \frac{wA + DB}{w^2 + D^2} \right) - \left( \frac{C}{a_2} - \frac{M}{a_2} \right) \frac{1}{D} \right) \frac{L_2 D}{2ga_2}; \]

\[ q_1^0 = U_1 + V_1 + \frac{(wA + DB) L_1}{w^2 + D^2} - \frac{(P_1 - C) L_1}{D}; \]

\[ q_2^0 = U_2 + V_2 - \frac{a_1}{a_2} \left( \frac{(wA + DB) L_2}{w^2 + D^2} \right) - \left( \frac{P_2 + C}{a_2} - \frac{M}{a_2} \right) \frac{L_2}{D}; \]

\[ -2V_1 D - \frac{(wB - DA) L_1 w}{w^2 + D^2} = L_1 (-B - 2C) + \frac{(wA + DB) L_1 D}{w^2 + D^2}; \]

\[ -2V_2 D + \frac{a_1}{a_2} \left( \frac{(wB - DA) L_2 w}{w^2 + D^2} \right) = L_2 B \frac{a_1}{a_2} - \frac{a_1}{a_2} \left( \frac{(wA + DB) L_2 D}{w^2 + D^2} \right). \]

**Proof of Lemma 9:** The proof is straightforward. According to the maximum principle, the state and co-state variables must be continuous. Therefore, given optimal solutions for each separate pricing regime, in order for the overall solution to be optimal, i.e., in order to merge different regimes, say \( i-1, i \) and \( i+1 \), we need the continuity, \( q_1^{i-1}(t_{i-1}) = q_1^i(t_{i+1}) \) and
\( \psi^i_n(t) = \psi^{i+1}_n(t) \) as stated in the lemma for the case of intermediate regimes. When \( i-1=0 \), the demand condition simply reduces to the given initial requirement of our problem \( q^1_n(0) = q^0_n \), while \( i=h \) corresponds to the given terminal co-state boundary condition \( \psi^h_n(T) = 0 \). ■

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