Fixed Point Solutions of Variational Inequalities for Asymptotically Pseudocontractive Mappings in Banach Spaces

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Abstract. Let $E$ be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, and which possesses uniform normal structure with coefficient $N(E)$, $K$ a nonempty closed convex and bounded subset of $E$. $T : K \rightarrow K$ an asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1, \infty)$, uniformly asymptotically regular and uniformly $L$-Lipschitzian. Let $f : K \rightarrow K$ be a fixed contraction. Let $\{t_n\} \subset (0, 1)$ be such that $\lim_{n \to \infty} t_n = 1$, $y_0 \in K \{t_n\}, \{k_n\}$ satisfy some appropriate conditions, the two iterative process as follows:

$$x_n = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n}T^n x_n,$$

$$y_{n+1} = (1 - \frac{t_n}{k_n})f(y_n) + \frac{t_n}{k_n}T^n y_n.$$ 

for all $n \geq 0$. converge strongly to some fixed point $p$ of $T$, which is the unique solution of variational inequality:

$$\langle (I - f)p, j(p - x^*) \rangle \leq 0 \ \forall x^* \in F(T).$$

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The results presented in this paper extend and improve the corresponding results of Naseer Shahzad and Aniefiok Udomene [Nonlinear Analysis 64(2006)558-567].

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1. Introduction

Let $E$ be a real normed linear space with dual $E^*$. We denote by $J$ the normalized duality mapping from $E$ to $E^*$ defined by

$$J(x) = \{f \in E^*, \langle x, f \rangle = \|x\| \|f\|, \|f\| = \|x\|\}, \forall x \in E,$$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pair. It is well known that if $E^*$ is strictly convex then $J$ is single-valued. In the sequel, we shall denote the single-valued duality mapping by $j$.

Let $K$ be a nonempty closed convex and bounded subset of a real Banach space $E$ and $T : K \to K$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$). It is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim_{n \to \infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all integers $n \geq 0$ and all $x, y \in K$. $T$ is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\}$ with $k_n \geq 1, \lim_{n \to \infty} k_n = 1$ and $j(x - y) \in J(x - y)$ such that the inequality

$$(T^n x - T^n y, j(x - y)) \leq k_n \|x - y\|^2$$

holds for all $x, y \in K$ and for all integers $n \geq 0$. It is trivial to see from the above definitions that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is asymptotically pseudocontractive. The converses do not hold. A mapping $T$ is called uniformly asymptotically regular if $\|T^{n+1} x - T^n x\| \to 0$ as $n \to \infty$ for all $x \in K$. $T$ is called uniformly L-Lipschitzian if there exists $L > 0$ such that $\|T^n x - T^n y\| \leq L \|x - y\|, \forall x, y \in K$ and for all integers $n \geq 0$. We denote by $F(T)$ the set of fixed points of $T$; i.e. $F(T) = \{x \in C : x = T x\}$.


If $\prod_K$ denotes the set of all contractions on $K$, he proved the following theorems.

**Theorem 1.1.** (Xu[12]Theorem4.1). Let $E$ be a uniformly smooth Banach space, $K$ a nonempty closed convex subset of $E$, and $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \prod_K$. Then the path $\{x_t\}$ defined
by
\[ x_t = tf(x_t) + (1 - t)Tx_t, \quad t \in (0, 1) \]
converges strongly to a point in \( F(T) \). If we define \( Q : \prod_K \to F(T) \) by
\[ Q(f) = \lim_{t \to 0} x_t, \]
then \( Q(f) \) solves the variational inequality:
\[ \langle (I - f)Q(f), j(Q(f) - x) \rangle \leq 0, \quad \forall x \in F(T). \]

**Theorem 1.2.** (Xu[12] Theorem 4.2). Let \( E \) be a uniformly smooth Banach space, \( K \) a nonempty closed convex subset of \( E \), and \( T : K \to K \) be a nonexpansive mapping with \( F(T) \neq \emptyset \), and \( f \in \prod_K \). Assume that \( \{\alpha_n\} \subset (0, 1) \) satisfies the following conditions:

(i) \( \lim_{n \to \infty} \alpha_n = 0; \)

(ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty; \)

(iii) either \( \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \) or \( \lim_{n \to \infty} (\alpha_{n+1}/\alpha_n) = 1. \)

Then the sequence \( \{x_n\} \) generated by
\[ x_0 \in K, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2... \]
converges strongly to a fixed point of \( T \).

Very recently, Naseer Shahzad and Aniefiok Udomene [10] extended the results of H.K.Xu [12] from nonexpansive to asymptotically nonexpansive, and they proved the following Theorems:

**Theorem 1.3.** ([10] Theorem 3.1). Let \( E \) be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, \( K \) a nonempty closed convex and bounded subset of \( E \), \( T : K \to K \) an asymptotically nonexpansive mapping with sequence \( \{k_n\} \subset [1, \infty) \), and \( f : K \to K \) a fixed contraction with constant \( \alpha \in [0, 1) \). Let \( \{t_n\} \subset (0, \frac{1-\alpha}{k_n-\alpha}) \) be such that \( \lim_{n \to \infty} t_n = 1 \) and \( \lim_{n \to \infty} \frac{k_n-1}{k_n-t_n} = 0 \). Then,

(i) for each integer \( n \geq 0 \), there is a unique \( x_n \in K \) such that
\[ x_n = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n}Tx_n \tag{1.1} \]
and, if in addition, \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \), then,

(ii) The sequence \( \{x_n\} \) converges strongly to some fixed point \( p \) of \( T \), which is the unique solution of variational inequality:
\[ \langle (I - f)p, j(p - x) \rangle \leq 0 \quad \forall x \in F(T). \]

**Theorem 1.4.** ([10] Theorem 3.3). Let \( E \) be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, \( K \) a nonempty closed convex and bounded subset of \( E \), \( T : K \to K \) an asymptotically nonexpansive mapping with sequence \( \{k_n\} \subset [1, \infty) \), and \( f : K \to K \) a fixed contraction with constant \( \alpha \in [0, 1) \). Let \( \{t_n\} \subset (0, \xi_n) \) be such that \( \lim_{n \to \infty} t_n = 1 \), \( \sum_{n=0}^{\infty} t_n(1 - t_n) = \infty \) and \( \lim_{n \to \infty} \frac{k_n}{k_n-t_n} = 0 \), where \( \xi_n = ... \)
min\(\left\{ \frac{(1-\alpha)k_n}{k_n-\alpha}, \frac{1}{k_n} \right\} \). For an arbitrary \(y_0 \in K\) let the sequence \(\{y_n\}\) be iteratively defined by (1.2)

\[
y_{n+1} = (1 - \frac{t_n}{k_n})f(y_n) + \frac{t_n}{k_n}Ty_n \tag{1.2}
\]

Then,

(i) for each integer \(n \geq 0\), there is a unique \(x_n \in K\) such that the equality (1.1) holds. and, if in addition, \(\lim_{n \to \infty} \|x_n - Tx_n\| = 0, \lim_{n \to \infty} \|y_n - Ty_n\| = 0\). then,

(ii) The sequence \(\{y_n\}\) converges strongly to some fixed point \(p\) of \(T\), which is the unique solution of variational inequality:

\[
\langle (I - f)p, j(p - x^*) \rangle \leq 0 \quad \forall x^* \in F(T).
\]

The main aim of this paper is to obtain fixed point solutions of variational inequalities for an asymptotically pseudocontractive mapping defined on a real reflexive Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure. We proved under the appropriate conditions on \(K\), \(T\) and \(\{t_n\} \subset (0, 1)\) that the sequence iteratively defined by (1.1) and (1.2) converges strongly to some fixed point \(p\) of \(T\), which is the unique solution of variational inequality:

\[
\langle (I - f)p, j(p - x^*) \rangle \leq 0 \quad \forall x^* \in F(T).
\]

Our results extend Theorem 3.1 and Theorem 3.3 of Naseer Shahzad and Aniefiok Udomene[10] to more general class of asymptotically pseudocontractive mappings and to the much more general class of Banach spaces considered here.

2. Preliminaries

Throughout this paper, we denote by \(\mathbb{N}\) and \(\mathbb{R}_+\) the sets of positive integers and nonnegative real numbers, respectively. Recall that a self-mapping \(f : K \to K\) is a fixed contraction on \(K\) if there exists a constant \(\alpha \in (0, 1)\) such that

\[
\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad x, y \in K.
\]

Let \(E\) be a real normed linear space with \(\dim E \geq 2\). The norm of \(E\) is said to be uniformly Gâteaux differentiable if for each \(y \in S := \{x \in E : \|x\| = 1\}\), the limit

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

is attained uniformly for \(x \in S\). The modulus of convexity of \(E\) is the function \(\delta_E : (0, 2] \to [0, 1]\) defined by

\[
\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1; \epsilon = \|x - y\| \right\}.
\]
Asymptotically pseudocontractive mappings

$E$ is uniformly convex if and only if $\delta_E(\epsilon) > 0, \forall \epsilon \in (0, 2]$. Typical examples of spaces which are uniformly convex are the Lebesgue $L_p$, the sequence $l_p$, and the Sobolev $W^m_p$ spaces for $1 < p < \infty$.

Let $K$ be a nonempty bounded closed convex subset of a real Banach space $E$ and let $d(K) := \sup\{\|x - y\| : x, y \in K\}$ be the diameter of $K$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$, the Chebyshev radius of $K$ relative to itself. The normal structure coefficient of $E$ is defined (e.g.,[3]) as the number

$$N(E) := \inf\{d(K)/r(K) : K \text{ is a bounded closed convex subset of } E \text{ with } d(K) > 0\}.$$ 

A space $E$ such that $N(E) > 1$ is said to have uniformly normal structure. It is known that a space with a uniformly normal structure is reflexive and that all uniformly convex Banach spaces have uniformly normal structure(e.g.,[1]).

Recall that (see e.g.,[11]) a Banach limit $LIM$ is a bounded linear functional on $l^\infty$ such that

$$\|LIM\| = 1, \lim inf t_n \leq LIM_n t_n \leq \lim sup t_n,$$

and $LIM t_n = LIM t_{n+1}$ for all $t_n \in l^\infty$.

In the sequel, we shall make use of the following lemmas.

**Lemma 2.1.** ([5]) Suppose $E$ is a Banach space with uniformly normal structure. $K$ is a nonempty closed convex subset of $E$, and $T : K \to K$ is uniformly $L$-Lipschitzian with $L < N(E)^{1/2}$. Suppose also there exists a nonempty closed convex subset $C$ of $K$ with the following property:

$$\text{(P)}: \text{ } x \in C \text{ implies } \omega_w(x) \subset C,$$

where $\omega_w(x)$ is the weak $w$-limit set of $T$ at $x$, i.e., the set $\{y \in E : y = \text{weak} - \lim j T^n y, \text{for some } n_j \to \infty\}$. Then $T$ has a fixed point in $C$.

**Lemma 2.2.** Let $E$ be a real normed linear space. Then the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \forall x, y \in E, \forall j(x + y) \in J(x + y).$$

**Lemma 2.3.** ([Xu/7,12,13]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \gamma_n)a_n + \beta_n \gamma_n, \text{ } n \geq 0,$$

where,

1. $\{\gamma_n\} \subset (0, 1), \sum_{n=0}^\infty \gamma_n = \infty;$
2. $\lim sup_{n \to \infty} \beta_n \leq 0; (n \geq 0).$

Then, $a_n \to 0$ as $n \to \infty$. 

3. Main Results

**Theorem 3.1.** Let $E$ be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, and which possesses uniform normal structure with coefficient $N(E)$, $K$ a nonempty closed convex and bounded subset of $E$. $T : K \to K$ an asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1, \infty)$, uniformly asymptotically regular and uniformly $L$-Lipschitzian. Let $f : K \to K$ be a fixed contraction. Let $\{t_n\} \subset (0, \frac{(1-\alpha)k_n}{k_n-\alpha})$ be such that $\lim_{n \to \infty} t_n = 1$ and

$$\lim_{n \to \infty} \frac{k_n-1}{k_n-t_n} = 0, \quad \text{and} \quad L < N(E)^{1/2}$$

(i) for each integer $n \geq 0$, there is a unique $x_n \in K$ such that

$$x_n = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n}T^n x_n$$

and $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$;

(ii) if in addition,

$$\|x_n - T^m x\|^2 \leq \langle x_n - T^m x, j(x_n - x) \rangle, \quad (3.1), \forall m, n \geq 1, \forall x \in K$$

then, The sequence $\{x_n\}$ converges strongly to some fixed point $p$ of $T$, which is the unique solution of variational inequality:

$$\langle (I - f)p, j(p - x^*) \rangle \leq 0 \quad \forall x^* \in F(T).$$

**Proof.** (i) Clearly, by the conditions on $\{t_n\}$, for each integer $n \geq 0$, we have that $\alpha_n := 1 - t_n/k_n \in (0, 1)$ and the mapping $T_n(x) := \alpha_n f(x) + (1 - \alpha_n)T^n x$ is continuous and strongly pseudocontractive. Indeed,

$$\langle T_n x - T_n y, j(x - y) \rangle$$

$$= \langle (1 - \alpha_n)(T(t_n)^n x - T(t_n)^n y) + \alpha_n (f(x) - f(y)), j(x - y) \rangle$$

$$\leq (1 - \alpha_n) k_n \|x - y\|^2 + \alpha_n \|x - y\|^2$$

$$= ((1 - \alpha_n) k_n + \alpha_n \|x - y\|^2$$

Since $(1 - \alpha_n) k_n + \alpha_n \alpha \in (0, 1)$ for all $n \geq 0$.

Therefore, by Theorem 5 of [8], $T_n$ has unique fixed point (say) $x_n \in K$. This means that the equation

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)^n x_n$$

has a unique solution for each integer $n \geq 0$.

Moreover, since $K$ is bounded we have that

$$\|x_n - T^n x_n\| = \alpha_n \|f(x_n) - T^n x_n\| \to 0 \text{ as } n \to \infty. \quad (3.2)$$

Thus

$$\|x_n - Tx_n\| = \|\alpha_n (f(x_n) - Tx_n) + (1 - \alpha_n)(T^n x_n - Tx_n)\|$$

$$\leq \alpha_n \|f(x_n) - Tx_n\| + (1 - \alpha_n) \|T^n x_n - T^n x_n + T^{n+1} x_n - Tx_n\|$$

$$\leq \alpha_n \|f(x_n) - Tx_n\| + (1 - \alpha_n) \|T^n x_n - T^{n+1} x_n\|$$

$$+ (1 - \alpha_n) L \|T^n x_n - x_n\|$$
Therefore, from (3.2) and the uniformly asymptotic regularity of $T$ we get that
\[ \|x_n - T x_n\| \to 0 \text{ as } n \to \infty. \] (3.3)

(ii) Define the mapping $\phi : K \to \mathbb{R}_+$ by $\phi(x) := \text{LIM}_n \|x_n - x\|^2 \text{ for all } x \in K$.

Furthermore, since $E$ is reflexive and $\phi$ is continuous, convex and $\phi(z) \to \infty$ as $\|z\| \to \infty$, $\phi$ attains its infimum over $K$ (see, e.g., [2,5,11]). Hence $C := \{x \in K : \phi(x) = \min_{z \in K} \phi(z)\}$ is nonempty. It is also closed and convex. Now we show that $C$ has property (P). Let $y_0 \in C$ and let $x := w - \text{lim}_{m \to \infty} T^{m} y_0$. For any $t > 0$, belong to the weak $w$-limit set $\omega_w(x^*)$ of $T$ at $y_0$. Then from the weak-lower-semi continuity of $\phi$, and the fact that $\lim_{n \to \infty} \|x_n - T^n x_n\| = 0$, using condition (3.1), we have the following estimates:

\[
\phi(x) \leq \lim_{j \to \infty} \inf \phi((T)^m y_0) \leq \lim_{m \to \infty} \sup \phi(T^m y_0) \leq \lim_{m \to \infty} \text{sup} (\text{LIM}_n \|x_n - T^m y_0\|^2) 
\]

\[
\leq \lim_{m \to \infty} \text{sup} (\text{LIM}_n \langle x_n - T^m y_0, j(x_n - y_0) \rangle) 
\]

\[
= \lim_{m \to \infty} \text{sup} (\text{LIM}_n \langle x_n - T x_n + (T x_n - T^2 x_n) + \ldots + (T^m x_n - T^m y_0), j(x_n - y_0) \rangle) 
\]

\[
\leq \lim_{m \to \infty} \text{sup} (\text{LIM}_n \|x_n - T x_n\| + \|T x_n - T^2 x_n\| + \ldots + \|T^{m-1} x_n - T^m x_n\| d + \text{LIM}_n k_m \|x_n - y_0\|^2) 
\]

\[
\leq \lim_{m \to \infty} \text{sup} (\text{LIM}_n \|x_n - T x_n\| + L \|x_n - T x_n\| + \ldots + L \|x_n - T x_n\| d + \text{LIM}_n k_m \|x_n - y_0\|^2) 
\]

\[
\leq \text{LIM}_n \|x_n - y_0\|^2 = \phi(y_0) = \min_{z \in K} \phi(z). 
\]

where $d = \text{diam} K$. Thus we have $x \in C$, i.e., $\omega_w(y_0) \subseteq C$ and hence $C$ has property (P). Since $E$ is uniformly convex and has uniformly normal structure and since $L < N(E)^{1/2}$, it follows from Lemma 2.1 that $T$ has a fixed point (say) $P \in F(T) \cap C$. and so $F(T) \neq \emptyset$. Now for any $x \in F(T)$ the asymptotically pseudocontractivity of $T$ gives the following estimates:

\[
\langle x_n - T^n x_n, j(x_n - x^*) \rangle = \langle x_n - x^*, j(x_n - x) \rangle + \langle x^* - T^n x_n, j(x_n - x^*) \rangle 
\]

\[
\geq \|x_n - x^*\|^2 - k_n \|x_n - x^*\|^2 
\]

\[
\geq -(k_n - 1)d^2 
\]

(3.4)

Moreover, from (1.1) we have that
\[ x_n - T^n x_n = \frac{1}{t_n}(k_n - t_n)(f(x_n) - x_n) \] (3.5)

and from (3.4) and (3.5) we get that
\[
\text{LIM}_n \langle x_n - f(x_n), j(x_n - x^*) \rangle \leq \text{LIM}_n t_n \frac{k_n - 1}{k_n - t_n} d^2 \to 0 \text{ as } n \to \infty \text{ (by hypothesis)}. 
\] (3.6)

In particular,
\[ \text{LIM}_n \langle x_n - f(x_n), j(x_n - p) \rangle \leq 0. 
\] (3.7)
Let \( s \in (0, 1] \). Then, by Lemma 2.2 we get that
\[
\| x_n - p - s(f(x_n) - p) \|^2 \\
\leq \| x_n - p \|^2 + 2\langle -s(f(x_n) - p), j(x_n - p - s(f(x_n) - p)) \rangle \\
= \| x_n - p \|^2 - 2s\langle f(x_n) - p, j(x_n - p) \rangle \\
- 2s\langle f(x_n) - p, j(x_n - p - s(f(x_n) - p)) - j(x_n - p) \rangle.
\]
Let \( \varepsilon > 0 \) be arbitrary. Then since \( j \) is norm-to-weak* uniformly continuous on bounded subsets of \( E \), there exists \( \delta > 0 \) such that for all \( s \in (0, \delta) \) we have
\[
LIM_n \langle f(x_n) - p, j(x_n - p) \rangle \\
\leq \frac{1}{2s}[LIM_n \| x_n - p \|^2 - LIM_n \| x_n - p - s(f(x_n) - p) \|^2] + \varepsilon < \varepsilon,
\]
since \( p \in C \), and is a minimizer of \( \phi \) over \( K \). Now, since \( \varepsilon \) is arbitrary this implies that
\[
LIM_n \langle f(x_n) - p, j(x_n - p) \rangle \leq 0 \tag{3.8}
\]
Combining inequalities (3.7) and (3.8) we get
\[
LIM_n \langle x_n - p, j(x_n - p) \rangle = LIM_n \| x_n - p \|^2 \leq 0.
\]
Therefore, there is a subsequence \( \{ x_{n_j} \} \) of \( \{ x_n \} \) which converges strongly to \( p \).

Thus, it follows from inequality (3.6) and norm to weak* uniformly continuity of \( j \) that
\[
\langle (I - f)p, j(p - x^*) \rangle = \lim \langle x_{n_j} - f(x_{n_j}), j(x_{n_j} - x^*) \rangle \leq 0.
\]
Now, suppose there exists another subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) which converges strongly to \( q \). Then since \( \lim \| x_n - T x_n \| = 0 \) for each \( t \in \mathbb{R}^+ \), we have that \( q \) is a fixed point of \( T \).

Similarly, we also can show
\[
\langle (I - f)q, j(q - x^*) \rangle = \lim \langle x_{n_k} - f(x_{n_k}), j(x_{n_k} - x^*) \rangle \leq 0.
\]
Replace \( x^* \) with \( q \) to obtain
\[
\langle (I - f)p, j(p - q) \rangle \leq 0.
\]
Replace \( x^* \) with \( p \) to obtain
\[
\langle (I - f)q, j(q - p) \rangle \leq 0.
\]
Adding the above two inequalities, we get
\[
(1 - \alpha) \| q - p \|^2 \leq \langle (I - f)q - (I - f)p, j(q - p) \rangle \leq 0.
\]
Thus, \( q = p \). Therefore, \( \{ x_n \} \) converges strongly to \( p \). The proof is completed.

\[ \square \]

**Theorem 3.2.** Let \( E \) be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, and which possesses uniform normal structure with coefficient \( N(E) \), \( K \) a nonempty closed convex and bounded subset of \( E \). \( T : K \rightarrow K \) an asymptotically pseudocontractive mapping with sequence \( \{ k_n \} \subset [1, \infty) \), uniformly asymptotically regular and uniformly \( L \)-Lipschitzian. Let \( f : K \rightarrow K \)
be a fixed contraction. Let \( \{t_n\} \subset (0, \xi_n) \), \((2\alpha)^{1/2} < L \neq 1, \alpha \in (0, \frac{1}{2})\) be such that \( \lim_{n \to \infty} t_n = 1 \) and \( \lim_{n \to \infty} \frac{k_n - 1}{k_n - t_n} = 0 \), and \( L < N(E)^{1/2} \), where \( \xi_n = \min \left\{ \frac{(1-\alpha)k_n}{k_n - \alpha}, \frac{k_n(1-2\alpha)}{L^2 - 2\alpha} \right\} \). Suppose that \( \|x_n - T^m x\|^2 \leq \langle x_n - T^m x, j(x_n - x) \rangle \), and \( \lim_{n \to \infty} \|y_n - Ty_n\| = 0 \). For an arbitrary \( y_0 \in K \). Then the iterative sequence \( \{y_n\} \) defined by (1.2) i.e.,

\[
y_{n+1} = (1 - \frac{t_n}{k_n}) f(y_n) + \frac{t_n}{k_n} T^m y_n
\]

converges strongly to some fixed point \( p \) of \( T \), which is the unique solution of variational inequality:

\[
\langle (I - f)p, j(p - x^*) \rangle \leq 0 \quad \forall x^* \in F(T).
\]

**Proof.** Let \( n \geq m \). Then from (1.1),

\[
x_m - y_n = (1 - \frac{t_m}{k_m})(f(x_m) - y_n) + \frac{t_m}{k_m}(T^m x_m - y_n).
\]

Setting \( \alpha_n := 1 - \frac{t_m}{k_m} \), we have the following estimate:

\[
\|x_m - y_n\|^2 = \langle x_m - y_n, j(x_m - y_n) \rangle
\]

\[
= \langle \alpha_m (f(x_m) - y_n) + (1 - \alpha_m)(T^m x_m - y_n), j(x_m - y_n) \rangle
\]

\[
= (1 - \alpha_m)\langle T^m x_m - T^m y_n + T^m y_n - y_n, j(x_m - y_n) \rangle
\]

\[
+ \alpha_m \langle f(x_m) - x_m + x_m - y_n, j(x_m - y_n) \rangle
\]

\[
\leq [\alpha_m + k_m(1 - \alpha_m)]\|x_m - y_n\|^2 + (1 - \alpha_m)\|T^m y_n - y_n\|\|x_m - y_n\|
\]

\[
+ \alpha_m \langle f(x_m) - x_m, j(x_m - y_n) \rangle
\]

Since \( K \) is bounded, for some constant \( M > 0 \), it follows that

\[
\lim_{n \to \infty} \sup_{m \to \infty} \langle f(x_m) - x_m, j(y_n - x_m) \rangle \leq \frac{t_m}{k_m} \frac{k_m - 1}{k_m - t_m} M
\]

\[
+ \lim_{n \to \infty} \sup_{m \to \infty} \frac{1 - \alpha_m}{\alpha_m} \|T^m y_n - y_n\| M,
\]

so that

\[
\lim_{n \to \infty} \sup_{m \to \infty} \langle f(x_m) - x_m, j(y_n - x_m) \rangle \leq \lim_{n \to \infty} \sup_{m \to \infty} \frac{t_m}{k_m} \frac{k_m - 1}{k_m - t_m} M.
\]

By Theorem 3.1 \( x_m \to p \in F(T) \), which is the unique solution of variational inequality:

\[
\langle (I - f)p, j(p - x^*) \rangle \leq 0 \quad \forall x^* \in F(T).
\]

Since \( j \) is norm to weak* uniformly continuous on bounded sets in Banach space \( E \) with uniformly Gâteaux differentiable norm, Then taking limit as \( m \to \infty \), we obtain that

\[
\lim_{n \to \infty} \sup_{m \to \infty} \langle f(p) - p, j(y_n - p) \rangle \leq 0
\]

(3.9)
Now from the iterative process (1.2), we estimate as follows:
\[
\|y_{n+1} - p\|^2 = \| (1 - \alpha_n) (T^n y_n - p) + \alpha_n (f(y_n) - p) \|^2 \\
\leq (1 - \alpha_n)^2 \| T^n y_n - p \|^2 + 2 \alpha_n \langle f(y_n) - p, j(y_{n+1} - p) \rangle \\
\leq (1 - \alpha_n)^2 L^2 \| y_n - p \|^2 + 2 \alpha_n \| y_n - p \| \| y_{n+1} - p \| \\
+ 2 \alpha_n \langle f(p) - p, j(y_{n+1} - p) \rangle \\
\leq (1 - \alpha_n)^2 L^2 \| y_n - p \|^2 + 2 \alpha_n \| y_n - p \| \| y_{n+1} - p \| \\
2 \alpha_n \langle f(p) - p, j(y_{n+1} - p) \rangle \\
\leq (1 - \alpha_n)^2 L^2 \| y_n - p \|^2 + \alpha_n (\| y_n - p \|^2 + \| y_{n+1} - p \|^2) \\
2 \alpha_n \langle f(p) - p, j(y_{n+1} - p) \rangle
\]
So that
\[
\|y_{n+1} - p\|^2 \leq \frac{(1 - \alpha_n)^2 L^2 + \alpha_n}{1 - \alpha_n} \| y_n - p \|^2 + \frac{2 \alpha_n}{1 - \alpha_n} \langle f(p) - p, j(y_{n+1} - p) \rangle \\
= (1 - \frac{1 - (1 - \alpha_n) L^2 - 2 \alpha_n}{1 - \alpha_n}) \| y_n - p \|^2 + \frac{2 \alpha_n}{1 - \alpha_n} \langle f(p) - p, j(y_{n+1} - p) \rangle \\
\leq (1 - \frac{1 - (1 - \alpha_n) L^2 - 2 \alpha_n}{1 - \alpha_n}) \| y_n - p \|^2 + \frac{2 \alpha_n}{1 - \alpha_n} \langle f(p) - p, j(y_{n+1} - p) \rangle
\]
Hence,
\[
\|y_{n+1} - p\|^2 \leq (1 - \gamma_n) \| y_n - p \|^2 + \beta_n \gamma_n \tag{3.10}
\]
where \( \gamma_n = \frac{1 - (1 - \alpha_n) L^2 - 2 \alpha_n}{1 - \alpha_n} \) and \( \beta_n = \frac{2 \alpha_n}{1 - \alpha_n} \langle f(p) - p, j(y_{n+1} - p) \rangle \).
By the condition on \( L \) and \( t_n \) and inequality (3.9), we obtain that
\[
\gamma_n \subset (0, 1), \quad \sum_{n=0}^{\infty} \gamma_n = \infty, \quad \limsup_{n \to \infty} \beta_n \leq 0
\]
Now we apply Lemma2.3 to (3.10), we have that
\[
\lim_{n \to \infty} \| y_n - p \|^2 = 0.
\]
The proof is completed. \( \square \)

Remark 3.3. Theorem3.1 and Theorem3.2 generalize and improve Theorem3.1 and Theorem3.3 of Naseer Shahzad and Aniefiok Udomene[10] in the following ways:

1. The asymptotically nonexpansive mappings is extended to asymptotically pseudocontractive mappings.
2. \( \lim_{n \to \infty} \|x_n - T x_n\| = 0 \) is the result of Theorem3.1 and Theorem3.2 in this paper, but it is assumption in Theorem3.1 and Theorem3.3 of Naseer Shahzad and Aniefiok Udomene[10].

References


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