Variance Balanced Block Designs with Repeated Blocks

Bronislaw Ceranka and Malgorzata Graczyk

Abstract

Some construction methods of the variance balanced block designs with repeated blocks are given. They are based on the incidence matrices of the balanced incomplete block designs with repeated blocks.

1 Introduction

Block designs with repeated blocks with the equireplications and equiblock sizes are widely used in several fields of research and they are available in the literature, see Foody and Hedayat (1977), Hedayat and Li (1979), Hedayat and Hwang (1984). However from the practical point of view, it may be not possible to construct the design with equisize blocks accommodating the equireplication of each treatment in all the blocks. Here we consider a class of block designs called variance balanced block designs which can be made available in unequal block sizes and for varying replications.

Let us consider $v$ treatments arranged in $b$ blocks in a block design with the incidence matrix $N = [n_{ij}]$, $i = 1, 2, ..., v$, $j = 1, 2, ..., b$, where $n_{ij}$ denotes the number of experimental units in the $j$th block getting the $i$th treatment. When $n_{ij} = 1$ or 0 for all $i$ and $j$, the design is said to be binary. Otherwise it is said to be nonbinary. In this paper we consider binary block design, only.

The following notation is used $r = [r_1, r_2, ..., r_v]'$ is the vector of treatment replications, $k = [k_1, k_2, ..., k_b]'$ is the vector of block sizes, with this $N1_b = r$ and $N'1_v = k$, where $1_a$ is the $a \times 1$ vector of ones.

The information matrix for treatment effects $C$ defined below as

$$C = R - NK^{-1}N',$$  \hspace{1cm} (1.1)
where $R = \text{diag}(r_1, r_2, \ldots, r_v)$, $K = \text{diag}(k_1, k_2, \ldots, k_b)$, is very suitable in determining properties of block design.

For several reasons, in particular from the practical point of view, it is desirable to have repeated blocks in the design. For example, some treatment combinations may be preferable than the others and also the design implementation may cost differently according to the design structure admitting or not repeated blocks. The set of all distinct blocks in a block design is called the support of the design and the cardinality of the support is denoted by $b^*$ and is referred to as the support size of the design.

Though there have been balanced designs in various sens (see Puri and Nigam (1977), Caliński (1977)), we will consider a balanced design of the following type. A block design is said to be balanced if every elementary contrast of treatment is estimated with the same variance (see Rao (1958)). In this sense this design is also called a variance balanced block design.

It is well known that block design is a variance balanced if and only if it has

$$C = \eta \left[ I_v - \frac{1}{v} 1_v 1_v' \right],$$

where $\eta$ is the unique nonzero eigenvalue of the matrix $C$ with the multiplicity $v - 1$, $I_v$ is the $v \times v$ identity matrix. For binary block design

$$\eta = \frac{\sum_{i=1}^v r_i - b}{v - 1}$$

(see Kageyama and Tsuji (1979)).

In particular case when block design is a balanced incomplete block design then $\eta = \frac{vr - b}{v - 1}$.  

2 Construction for $v$ treatments

Let $N_i, i = 1, 2, \ldots, t$, be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters $v$, $b_i$, $r_i$, $k_i$, $\lambda_i$, $b^*_i$. Let $C_i$ be the $C-$matrix of this design defined by $N_i, i = 1, 2, \ldots, t$. Now, we form the matrix $N$ as

$$N = [N_1 \ N_2 \ldots \ N_t].$$

**Theorem 1** Block design with the incidence matrix $N$ of the form (2.1) is the variance balanced block design with repeated blocks with the parameters
v, \quad b = \sum_{i=1}^{t} b_i, \quad r = \sum_{i=1}^{t} r_i, \quad k = [k_1 1'_{b_1} \quad k_2 1'_{b_2} \ldots k_t 1'_{b_t}]', \quad \lambda = \sum_{i=1}^{t} \lambda_i, \quad b^* = \sum_{i=1}^{t} b_i^*.

Proof. The matrix \( C \) of the block design (2.1) is

\[
C = rI_v - \sum_{i=1}^{t} \frac{1}{\lambda_i} N_i N_i' = \sum_{i=1}^{t} \left( \left( r_i - \frac{1}{\lambda_i}(r_i - \lambda_i) \right) I_v - \frac{\lambda_i}{\lambda_i} I_v I_v' \right)
\]

\[
= \sum_{i=1}^{t} C_i = \sum_{i=1}^{t} \eta_i \left[ I_v - \frac{1}{v} I_v I_v' \right] = \eta \left[ I_v - \frac{1}{v} I_v I_v' \right],
\]

where \( \eta = \sum_{i=1}^{t} \eta_i, \quad \eta_i \) is the unique nonzero eigenvalue of the matrix \( C_i, \quad i = 1, 2, \ldots, t. \) So, the theorem is proved.

**Example 1** Let us consider the balanced incomplete block design with the parameters \( v = 7, \quad b_1 = 21, \quad r_1 = 6, \quad k_1 = 2, \quad \lambda_1 = 1, \quad b_1^* = 21 \) with the incidence matrix \( N_1 \) given through the blocks \((1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (3, 4), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 7)\) and the balanced incomplete block design with the parameters \( v = 7, \quad b_2 = 14, \quad r_2 = 6, \quad k_2 = 3, \quad \lambda_2 = 2, \quad b_2^* = 7 \) with the incidence matrix \( N_2 \) given through the blocks \((1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 7), (1, 5, 6), (2, 6, 7), (1, 3, 7)\), each block is repeated two times. Based on the matrices \( N_1 \) and \( N_2 \) for \( t = 2 \) we form the incidence matrix \( N \) in the form (2.1) of the variance balanced block design with repeated blocks with the parameters \( v = 7, \quad b = 35, \quad r = 12, \quad k = [2 \cdot 1'_{21} \quad 3 \cdot 1'_{14}]' \), \( \lambda = 3, \quad b^* = 28. \) Hence the matrix \( C \) is given as \( C = \frac{40}{v} \left[ I_7 - \frac{1}{v} I_7 I_7' \right]. \)

**Example 2** Let us consider the balanced incomplete block design with the parameters \( v = 9, \quad b_1 = 24, \quad r_1 = 8, \quad k_1 = 3, \quad \lambda_1 = 2, \quad b_1^* = 12 \) with the incidence matrix \( N_1 \) given through the blocks \((1, 2, 6), (1, 3, 7), (1, 4, 8), (1, 5, 9), (2, 3, 8), (2, 4, 9), (2, 5, 7), (3, 4, 5), (3, 6, 9), (4, 6, 7), (5, 6, 8), (7, 8, 9), \) each block is repeated two times. And we consider the balanced incomplete block design with the parameters \( v = 9, \quad b_2 = 36, \quad r_2 = 16, \quad k_2 = 4, \quad \lambda_2 = 6, \quad b_2^* = 18 \) with the incidence matrix \( N_2 \) given through the blocks \((1, 2, 3, 4), (1, 2, 4, 9), (1, 2, 5, 7), (1, 3, 6, 8), (1, 3, 8, 9), (1, 4, 6, 7), (1, 5, 6, 9), (1, 5, 7, 8), (2, 3, 5, 6), (2, 3, 6, 7), (2, 4, 5, 8), (2, 6, 8, 9), (2, 7, 8, 9), (3, 4, 5, 8), (3, 4, 7, 9), (3, 5, 7, 9), (4, 5, 6, 9), (4, 6, 7, 8), \) each block is repeated two times. Based on the matrices \( N_1 \) and \( N_2 \) for \( t = 2 \) we form the incidence matrix \( N \) in (2.1) of the variance balanced block design with repeated blocks with the parameters \( v = 9, \quad b = 60, \quad r = 24, \quad k = [3 \cdot 1'_{24} \quad 4 \cdot 1'_{36}]', \quad \lambda = 8, \quad b^* = 30. \) Hence we have
\[ C = \frac{39}{7} [I_9 - \frac{1}{9} 1_9 1_9^T] . \]

In particular case when \( t = 1 \) we have the following corollaries.

**Corollary 1** Block design with the incidence matrix \( N \) of the form

\[ N = [N_1 \quad I_v] \]  \hspace{1cm} (2.2)

is the variance balanced block design with repeated blocks with the parameters \( v, \quad b = b_1 + v, \quad r = r_1 + 1, \quad k = [k_1 1_{b_1}^T 1_v^T], \quad \lambda = \lambda_1, \quad b^* = b_1^* + v. \]

**Example 3** Let us consider the balanced incomplete block design with the parameters \( v = 7, \quad b_1 = 14, \quad r_1 = 6, \quad k_1 = 3, \quad \lambda_1 = 2, \quad b_1^* = 7 \) with the incidence matrix \( N_1 \) given through the blocks \( (1,2,4), \quad (2,3,5), \quad (3,4,6), \quad (4,5,7), \quad (1,5,6), \quad (2,6,7), \quad (1,3,7) \), each block is repeated two times. Based on the matrix \( N_1 \) we form the incidence matrix \( N \) in the form (2.2) of the variance balanced block design with repeated blocks with the parameters \( v = 7, \quad b = 21, \quad r = 7, \quad k = [3 \cdot 1_{14}^T 1_7^T], \quad \lambda = 2, \quad b^* = 14. \) Hence we have \( C = \frac{14}{3} [I_7 - \frac{1}{7} 1_7 1_7^T] . \)

**Corollary 2** Block design with the incidence matrix \( N \) of the form

\[ N = [N_1 \quad I_v] \]  \hspace{1cm} (2.3)

is the variance balanced block design with repeated blocks with the parameters \( v, \quad b = b_1 + 1, \quad r = r_1 + 1, \quad k = [k_1 1_{b_1}^T v^T], \quad \lambda = \lambda_1 + 1, \quad b^* = b_1^* + 1. \)

**Example 4** Let us consider the balanced incomplete block design with the parameters \( v = 9, \quad b_1 = 24, \quad r_1 = 8, \quad k_1 = 3, \quad \lambda_1 = 2, \quad b_1^* = 12 \) with the incidence matrix \( N_1 \) given through the blocks \( (1,2,6), \quad (1,3,7), \quad (1,4,8), \quad (1,5,9), \quad (2,3,8), \quad (2,4,9), \quad (2,5,7), \quad (3,4,5), \quad (3,6,9), \quad (4,6,7), \quad (5,6,8), \quad (7,8,9) \), each block is repeated two times. Based on the matrices \( N_1 \) we form the incidence matrix \( N \) in the form (2.3) of the variance balanced block design with repeated blocks with the parameters \( v = 9, \quad b = 25, \quad r = 9, \quad k = [3 \cdot 1_{24}^T 9^T], \quad \lambda = 3, \quad b^* = 13. \) Hence we have \( C = 7 [I_9 - \frac{1}{9} 1_9 1_9^T] . \)
3 Construction for \( v + 1 \) treatments

Let \( N_i, i = 1, 2, \) be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters \( v, b_i, r_i, k_i, \lambda_i, b_i^*. \) Now, we form the matrix \( N \) as

\[
N = \begin{bmatrix}
N_1 \otimes I'_i & N_2 \otimes I'_u \\
I'_{b_i} \otimes I'_i & 0_{b_2} \otimes I'_u
\end{bmatrix}.
\]  

(3.1)

**Theorem 2** Block design with the incidence matrix \( N \) of the form (3.1) is the variance balanced block design with repeated blocks with the parameters \( v+1, b = tb_1+ub_2, \quad r = [(tr_1 + ur_2)1'_v, \quad tb_1]', \quad k = [(k_1 + 1)1'_{b_1}, \quad k_21'_{b_2}], \quad b^* = tb_1^* + ub_2^* \) if and only if the constants \( t \) and \( u \) satisfy the equality

\[
t (r_1 - \lambda_1) k_2 = u \lambda_2 (k_1 + 1).
\]  

(3.2)

Proof. For the block design with the matrix \( N \) given in (3.1) we have

\[
C = \begin{bmatrix}
\left[ t \left( r_1 - \frac{r_1 - \lambda_1}{k_1+1} \right) + u \left( r_2 - \frac{r_2 - \lambda_2}{k_2} \right) \right] I_v - \left[ \frac{t \lambda_1}{k_1+1} + \frac{u \lambda_2}{k_2} \right] 1'_v 1'_v - \frac{tr_1}{k_1+1} 1'_v \\
-\frac{tr_1}{k_1+1} 1'_v & \frac{tb_1 k_1}{k_1+1}
\end{bmatrix}.
\]  

(3.3)

Comparing the diagonal elements of the matrix \( C \) of the form (3.3) we have

\[
\frac{t (r_1(k_1 - 1) + r_1)}{k_1+1} + \frac{ur_2(k_2 - 1)}{k_2} = \frac{tb_1 k_1}{k_1+1}
\]

\[
\frac{t (\lambda_1(v - 1) + r_1 - vr_1)}{k_1+1} = -\frac{u \lambda_2(v - 1)}{k_2}
\]

\[
t(r_1 - \lambda_1) k_2 = u \lambda_2 (k_1 + 1).
\]

Comparing the offdiagonal elements of the matrix \( C \) of the form (3.3) we have

\[
\frac{t \lambda_1}{k_1+1} + \frac{u \lambda_2}{k_2} = \frac{tr_1}{k_1+1}
\]

\[
t(r_1 - \lambda_1) k_2 = u \lambda_2 (k_1 + 1).
\]

If the condition (3.2) holds then the matrix \( C = \eta \left[ I_{v+1} - \frac{1}{v+1} 1_{v+1} 1'_{v+1} \right] \), where \( \eta = \frac{tr_1(v+1)}{k_1+1} \). So, the Theorem is proved.

In particular case when \( t = u = 1 \) we have
Corollary 3 Block design with the incidence matrix $N$ of the form

$$
N = \begin{bmatrix}
N_1 & N_2 \\
1_{b_1} & 0_{b_2}
\end{bmatrix}
$$

is the variance balanced block design with repeated blocks with the parameters $v+1$, $b = b_1 + b_2$, $r = [(r_1 + r_2)1_v' b_1]'$, $k = [(k_1 + 1)1_{b_1}' k_21_{b_2}']'$, $b^r = b_1^r + b_2^r$ if and only if

$$(r_1 - \lambda_1)k_2 = \lambda_2(k_1 + 1).$$

Example 5 Let us consider the balanced incomplete block design with the parameters $v = 7$, $b_1 = 21$, $r_1 = 6$, $k_1 = 2$, $\lambda_1 = 1$, $b_1^r = 21$ with the incidence matrix $N_1$ given through the blocks $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, 6)$, $(1, 7)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(2, 6)$, $(2, 7)$, $(3, 4)$, $(3, 5)$, $(3, 6)$, $(3, 7)$, $(4, 5)$, $(4, 6)$, $(4, 7)$, $(5, 6)$, $(5, 7)$, $(6, 7)$ and the balanced incomplete block design with the parameters $v = 7$, $b_2 = 35$, $r_2 = 15$, $k_2 = 3$, $\lambda_2 = 5$, $b_2^r = 7$ with the incidence matrix $N_2$ given through the blocks $(1, 2, 4)$, $(2, 3, 5)$, $(3, 4, 6)$, $(4, 5, 7)$, $(1, 5, 6)$, $(2, 6, 7)$, $(1, 3, 7)$, each block is repeated five times. Based on the matrices $N_1$ and $N_2$ for $t = u = 1$ we construct the incidence matrix $N$ in the form (3.4) of the variance balanced block design with repeated blocks with the parameters $v = 8$, $b = 56$, $r = 21$, $k = 3$, $b^r = 28$. Hence we have

$$
C = 16 \left[I_8 - \frac{1}{8}1_81_8'\right].
$$

Example 6 Let us consider the balanced incomplete block design with the parameters $v = 9$, $b_1 = 24$, $r_1 = 8$, $k_1 = 3$, $\lambda_1 = 2$, $b_1^r = 12$ with the incidence matrix $N_1$ given through the blocks $(1, 2, 6)$, $(1, 3, 7)$, $(1, 4, 8)$, $(1, 5, 9)$, $(2, 3, 8)$, $(2, 4, 9)$, $(2, 5, 7)$, $(3, 4, 5)$, $(3, 6, 9)$, $(4, 6, 7)$, $(5, 6, 8)$, $(7, 8, 9)$, each block is repeated two times. And we consider the balanced incomplete block design with the parameters $v = 9$, $b_2 = 36$, $r_2 = 16$, $k_2 = 4$, $\lambda_2 = 6$, $b_2^r = 18$ with the incidence matrix $N_2$ given through the blocks $(1, 2, 3, 4)$, $(1, 2, 4, 9)$, $(1, 2, 5, 7)$, $(1, 3, 6, 8)$, $(1, 3, 8, 9)$, $(1, 4, 6, 7)$, $(1, 5, 6, 9)$, $(1, 5, 7, 8)$, $(2, 3, 5, 6)$, $(2, 3, 6, 7)$, $(2, 4, 5, 8)$, $(2, 6, 8, 9)$, $(2, 7, 8, 9)$, $(3, 4, 5, 8)$, $(3, 4, 7, 9)$, $(3, 5, 7, 9)$, $(4, 5, 6, 9)$, $(4, 6, 7, 8)$, each block is repeated two times. Based on the matrices $N_1$ and $N_2$ for $t = u = 1$ we construct the incidence matrix $N$ in the form (3.4) of the variance balanced block design with repeated blocks with the parameters $v = 10$, $b = 60$, $r = 24$, $k = 4$, $b^r = 30$. Hence we have

$$
C = 20 \left[I_{10} - \frac{1}{10}1_{10}1_{10}'\right].
$$
If $N_1 = I_v$ then

**Corollary 4** Block design with the incidence matrix $N$ of the form

$$N = \begin{bmatrix} I_v & N_2 \\ 1_v' & 0_{b_2}' \end{bmatrix}$$

(3.5)

is the variance balanced block design with repeated blocks with the parameters $v+1, b = v+b_2, \ r = [(r_2 + 1)1_v' v]'$, $k = [21_v' k_2 1_{b_2}'], b^*_2 = b^*_2 + v$ if and only if $k_2 = 2\lambda_2$.

**Example 7** Let us consider the balanced incomplete block design with the parameters $v = 13, b_2 = 26, r_2 = 8, k_2 = 4, \lambda_2 = 2, b^*_1 = 13$ with the incidence matrix $N_2$ given through the blocks $(1,2,4,10), (2,3,5,11), (3,4,6,12), (4,5,7,13), (1,5,6,8), (2,6,7,9), (3,7,8,10), (4,8,9,11), (5,9,10,12), (6,10,11,13), (1,7,11,12), (2,8,12,13), (1,3,9,13)$, each block is repeated two times. Based on the matrix $N_2$ we construct the incidence matrix $N$ in the form (3.5) of the variance balanced block design with repeated blocks with the parameters $v = 14, b = 39, \ r = [91_{13}' 13]', k = [21_{13}' 41_{26}'], b^* = 26$. Hence we have $C = 7[I_{14} - \frac{1}{14}1_{14}1_{14}'].$

**References**


Received: April 27, 2007