The Dual of a Space of Cauchy Transforms

Y. Abu Muhanna and Z. Abdulhadi

Department of Mathematics
American University of Sharjah
P.O. Box 26666, Sharjah, United Arab Emirates
ymuhanna@aus.edu, zhadi@aus.edu

Abstract. Let $F_\alpha$, $\alpha \geq 0$ be the class of Cauchy transforms of order $\alpha$ equipped with the bounded variation norm. The bounded linear functionals on $F_\alpha$ are characterized and an inclusion between duals is given.

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1. Introduction

Let $T$ be the unit circle and $M$ be the set of all complex–valued Borel measures on $T$. For $\alpha > 0$ and $z \in D$, we define the space of weighted Cauchy transforms $F_\alpha$ to be the family of all functions $f(z)$ such that

\[(1.1) \quad f(z) = \int_T K_x^\alpha(z) d\mu(x)\]

where the Cauchy kernel $K_x(z)$ is given by

\[K_x(z) = \frac{1}{1 - \overline{x}z}\]

and where $\mu$ in (1.1) varies over all measures in $M$. The class $F_\alpha$ is a Banach space with respect to the norm

\[(1.2) \quad \|f\|_{F_\alpha} = \inf \|\mu\|_M\]

where the infimum is taken over all Borel measures $\mu$ satisfying (1.1). $\|\mu\|$ denotes the total variation norm of $\mu$. For detailed information about the space $F_\alpha$, see [2, 3, 4, 5].
Let \( W_\alpha \) denote the space of all bounded linear functionals on \( F_\alpha \). Recall that \( L \) is called a bounded linear functional on \( F_\alpha \) if \( L \) is linear and

\[
|L(f)| \leq A\|f\|_{F_\alpha}
\]

for all \( f \in F_\alpha \). Let \( W^*_\alpha \) denote the subspace of \( W_\alpha \) which consists of all bounded functions that preserve weak convergence. This means that if a sequence \( \{f_n\} \) and \( f \) in \( F_\alpha \) with corresponding measures, as in (1.1), \( \{\mu_n\} \) converges weakly to \( \mu \) then \( L(f_n) \to L(f) \) for each \( L \in W^*_\alpha \).

It is known, [3], that \( \|z^n\|_{F_\alpha} \leq cn^{\alpha-1} \), for \( 0 \leq \alpha < 1 \) and that \( \|z^n\|_{F_\alpha} \) is bounded when \( \alpha \geq 1 \).

Let

\[
b_n = L(z^n).
\]

Hence

\[
\limsup_{n \to \infty} |b_n^{1/n}| \leq 1.
\]

So the functions

\[
g(z) = \sum_{0}^{\infty} b_n z^n,
\]

(1.3) \[g_{\alpha}(z) = \sum_{0}^{\infty} A_n(\alpha) b_n z^n\]

are analytic in \( D \). Clearly, \( L(\sum_{0}^{k} a_n z^n) = \sum_{0}^{k} a_k b_k \).

In Theorem 1, we characterize all bounded linear functionals on \( F_\alpha \) and in Theorem 3, we show that \( W_\alpha \subset W^*_\beta \subset W_{\beta} \), where \( 0 \leq \beta < \alpha \).

2. BOUNDED LINEAR FUNCTIONALS ON \( F_\alpha \)

In this section, we shall express a bounded linear functional \( L \) in terms of \( g_\alpha \). As a first step we have:

**Lemma 1.** If \( K^\alpha_x(\rho z) = \sum_{0}^{\infty} A_n(\alpha) \rho^n x^n z^n \), \( \rho < 1 \) then

\[
L(K^\alpha_x(\rho z)) = \sum_{0}^{\infty} A_k(\alpha) b_k \rho^k x^k = g_\alpha(\rho x).
\]

(2.1)
Proof. Let

\[ dσ_ρ(y) = \text{Re} \frac{1}{1 - \frac{yρ}{π}} dt, \]

\[ dσ_{nρ}(y) = \text{Re} \sum_{k=0}^{∞} y^k ρ^k \frac{dt}{π} \]

\[ K_{x_n}^{α}(ρz) = \sum_{n=0}^{∞} A_n(α) ρ^n x^n z^k \]

where \( y = e^{it} \). Then \( K_{x_n}^{α}(ρz) = \int T K_{x}^{α}(z) dσ_{nρ}(y) \) and \( K_{x}^{α}(ρz) = \int T K_{xy}^{α}(z) dσ_ρ(y) \).

Hence \( \| K_{x_n}^{α}(ρz) - K_{x}^{α}(ρz) \|_{F_α} \leq \int \text{Re} \sum_{n+1}^{∞} y^k ρ^k \frac{dt}{π} \to 0 \), as \( n \to ∞ \). Hence, as \( L \) is continuous, the result follows.

Lemma 2. Let \( f \) be as in (1.1), then

\[ L(f) = \int T L(K_{x}^{α}(z)) dμ(x). \]

Proof. Let \( dλ_n = \sum_j^{n} μ_j x_j \to dμ \) in the total variation norm and let \( f_n = \int T K_{x}^{α}(z) dλ_n(x) \). Then \( f_n \to f \) in \( F_α \) and, as \( dλ_n \) is a finite sum,

\[ L(f_n) = \int T L(K_{x}^{α}(z)) dλ_n(x) \to L(f). \]

Since \( L(K_{x}^{α}(z)) \) is bounded,

\[ \left| \int T L(K_{x}^{α}(z)) dλ_n(x) - \int T L(K_{x}^{α}(z)) dμ(x) \right| \leq C \| μ - λ_n \| \]

and hence \( L(f) = \int T L(K_{x}^{α}(z)) dμ(x). \)

Lemma 3. \( g_α(z) = \sum_0^{∞} A_n(α) b_n z^n \) is a bounded function in \( D \).

Proof. By (2.1) and (2.2), \( L(K_{x}^{α}(ρz)) = \sum_0^{∞} A_k(α) b_k ρ^k x^k = \int T L(K_{xy}^{α}(z)) dσ_ρ(y) \).

Since \( K_{x}^{α}(z) \) is uniformly bounded by 1 in \( F_α \), \( L(K_{x}^{α}(ρz)) \) is also uniformly bounded in \( C \). Hence \( g_α(z) \) is a bounded function.

The first three lemmas lead to the following proposition:
Proposition 4. \( g_\alpha(\rho \overline{x}) = \lim_{\rho \to 1} g_\alpha(\rho x) = L(K_\alpha^\alpha(z)), \) for all \( x \) with \( |x| = 1. \)

Proof. Lemma 1 imply that \( g_\alpha(\rho x) = \int_T L(K_\alpha^\alpha(z)) d\sigma_\rho(y). \) This, the fact that \( g_\alpha(z) \) is bounded and \( d\sigma_\rho(y) = \frac{\text{Re} \frac{1}{1 - y\rho}}{\pi} \) imply that \( L(K_\alpha^\alpha(z)) = g_\alpha(\overline{x}). \)

A consequence of Proposition 1 is:

Corollary 5. Let \( f \) be as in (1.1), \( f(z) = \sum_0^\infty a_k z^k \) and \( L \) a bounded linear functional on \( F_\alpha. \) Then

\[
(2.3) \quad L(f) = \int_T g_\alpha(\overline{x}) d\mu(x) = \lim_{\rho \to 1} \sum_0^\infty A_k(\alpha) b_k a_k \rho^k,
\]

where \( g_\alpha \in H^\infty, \) defined as in (1.3), has radial limits at all \( x \in T. \)

In the converse direction of Corollary 1, we have

Lemma 6. Let \( g \) and \( g_\alpha \) be related as in (1.3). If \( g_\alpha \in H^\infty \) with radial limits at all \( x \in T \) then \( g \) generates a bounded linear functional on \( F_\alpha. \)

Proof. Define \( L(K_\alpha^\alpha(z)) = g_\alpha(\overline{x}) \) and if \( f \) as in (1), \( L(f) = \int_T g_\alpha(\overline{x}) d\mu(x). \)

Clearly \( L \) is linear and as \( g_\alpha \) is bounded \( |L(f)| \leq \|g_\alpha\|_{H^\infty} \|f\|_{F_\alpha}. \)

In conclusion, Corollary 1 and Lemma 4 lead to the following theorem which characterizes the dual of \( F_\alpha: \)

Theorem 7. The dual of \( F_\alpha = W_\alpha = \{ g : g_\alpha \in H^\infty \text{ with radial limits at all } x \in T \} \)

\( W_\alpha^* = \{ g : g_\alpha \text{ is continuous on } D \}. \)

Remark 1. The dual of \( F_1 = W_1 = \{ g : g \in H^\infty \text{ with radial limits at all } x \in T \}. \)

Remark 2. The dual of \( F_2 = W_2 = \{ g : g \text{ is analytic on } D \} = \text{The dual of } F_\alpha, \)

for all \( \alpha > 2. \) This is the same as the dual of the space of all analytic functions

equipped with the topology of uniform convergence on compact subsets of \( D. \).
We start this section with the statement of the well known Abel’s Theorem:

**Theorem 8. (Abel’s Theorem)** If \( \sum_{1}^{\infty} c_n \) is convergent, with \( \left| \sum_{1}^{\infty} c_n \right| \leq M \) and \( \{d_n\} \) is a decreasing positive sequence converging to 0 then \( \left| \sum_{s+1}^{u} c_n d_n \right| \leq 2Md_{s+1} \) and hence \( \sum_{1}^{\infty} c_n d_n \) converges.

As a consequence of Abel’s Theorem we have:

**Lemma 9.** Let \( g \) and \( g_\alpha \) be as in (1.3) with \( \alpha > 1 \). If \( g_\alpha \in H^\infty \) then \( g \) is continuous on \( \overline{D} \).

**Proof.** Let \( d_n = \frac{1}{A_n(\alpha)} \), where \( \alpha > 1 \). Then \( d_n \) is decreasing to 0. Let \( c_n = A_n(\alpha)b_nz^n \). Then apply Abel’s Theorem to get

\[
\left| \sum_{s+1}^{u} c_n d_n \right| = \left| \sum_{s+1}^{u} b_n z^n \right| \leq 2 \|g_\alpha\|_\infty d_{s+1} \to 0.
\]

and consequently

\[
\left| \sum_{s+1}^{u} b_n e^{int} \right| \leq 2 \|g_\alpha\|_\infty d_{s+1} \to 0.
\]

Hence \( \sum_{1}^{\infty} b_n z^n \) is uniformly convergent on \( \overline{D} \). Therefore, it is continuous. \( \blacksquare \)

Here is another important consequence of Abel’s Theorem.

**Lemma 10.** If \( g_\alpha \in H^\infty \) for any \( \alpha \geq 0 \) then \( g_\beta \) is continuous for any \( \beta < \alpha \).

**Proof.** Let \( d_n = \frac{A_n(\beta)}{A_n(\alpha)} \) and \( c_n = A_n(\alpha)b_nz^n \). Then it can be shown that \( d_n \) decreases to 0. Hence, by Abel’s Theorem,

\[
\left| \sum_{s+1}^{u} c_n d_n \right| = \left| \sum_{s+1}^{u} A_n(\beta)b_n z^n \right| \leq 2 \|g_\alpha\|_\infty d_{s+1} \to 0
\]

and that

\[
\left| \sum_{s+1}^{u} A_n(\beta)b_n e^{int} \right| \leq 2 \|g_\alpha\|_\infty d_{s+1} \to 0
\]

Hence \( \sum_{1}^{\infty} A_n(\beta)b_n z^n \) is uniformly convergent on \( \overline{D} \). Hence, it is continuous. \( \blacksquare \)
As a consequence of the last lemma, we have:

**Theorem 11.** $W_\alpha \subset W'_{\beta} \subset W_\beta$, where $0 \leq \beta < \alpha$.

Finally we have the following property of $W_\alpha, \alpha > 0$.

**Theorem 12.** If $g \in W_\alpha$ and $g_\alpha \in F_\alpha$ then $\sum_{n=0}^{\infty} A_n(\alpha) |b_n|^2 < \infty$.

**Proof.** Let $g, g_\alpha$ be as in (1.3) and that $L$ is the corresponding linear functional. In specific, $L(z^n) = b_n$ and $g_\alpha = \sum_{n=1}^{\infty} A_n(\alpha) b_n r^{i\theta} \in F_\alpha$ is bounded. Hence $g_\alpha(z) = \sum_{n=1}^{\infty} \frac{1}{n} b_n z^n \in F_\alpha$ is bounded. Apply $L$ to $g_\alpha(z)$ to conclude the result. 

**Remark 3.** The result implies, for $\alpha = 0$, that the area of the image of $g_0$ is finite.

**References**


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