

The Aharonov-Bohm Effect: Mathematical Aspects of the Quantum Flow

Luis Fernando Mello

Instituto de Ciências Exatas, Universidade Federal de Itajubá
CEP 37.500-903, Itajubá, MG, Brazil
lfmelo@unifei.edu.br

Yuri Cândido Ribeiro

Instituto de Sistemas Elétricos e Energia, Universidade Federal de Itajubá
CEP 37.500-903, Itajubá, MG, Brazil
yuricandido@yahoo.com.br

Abstract

This paper addresses the scattering of a beam of charged particles by an infinitely long magnetic string in the context of the hydrodynamical approach to quantum mechanics. The scattering is qualitatively analyzed by two approaches. In the first approach, the quantum flow is studied via a one-parameter family of complex potentials. In the second approach, the qualitative theory of planar differential equations is used to obtain a one-parameter family of Hamiltonian functions which determine the phase portraits of the systems.

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1 Introduction

Consider a long solenoid of small transverse cross section endowed with a magnetic flux. The limit configuration when the cross section becomes vanishingly small while the magnetic flux enclosed is kept constant is called a *magnetic string*. The magnetic field vanishes everywhere except inside the magnetic string and, as there is no Lorentz force, charged particles around the string are not affected by it, according to the classical mechanics point of view. Nevertheless, there is a nontrivial quantum scattering due to the Aharonov-Bohm effect. See Ref. [3] and [4].

The nontrivial scattering mentioned above can be studied from the following family of vector fields (the probability current density) (see [4])

$$\mathcal{J}(x, y, \delta) = \frac{\hbar k}{M} \left(-1 + \frac{\delta}{k} \frac{y}{x^2 + y^2}, -\frac{\delta}{k} \frac{x}{x^2 + y^2} \right), \quad (1)$$

or equivalently by the following family of planar ordinary differential equations

$$\begin{cases} x' = \frac{\hbar k}{M} \left(-1 + \frac{\delta}{k} \frac{y}{x^2 + y^2} \right) \\ y' = -\frac{\hbar k}{M} \left(\frac{\delta}{k} \frac{x}{x^2 + y^2} \right) \end{cases} \quad (2)$$

where M is the mass of the particle, $0 \leq k < \infty$ is associated to the energy $\hbar^2 k^2 / 2M$ for a stationary state and $0 \leq \delta \leq \frac{1}{2}$ is the flux parameter (see section 2).

In [3] the numerical solution of the system (2) is plotted, showing the main features of the quantum flow near the magnetic string. When $\delta = 0$ the streamlines are parallel to the x -axis which is identical to the classical flow (see Fig. 1). However, when $\delta > 0$ the topology of the streamlines changes near the origin, where open lines become cycles. As Fig. 4 shows, the quantum and the classical flow differ radically from each other.

This paper provides a new analysis of the scattering problem near the magnetic string, using complex potentials and the qualitative theory of planar differential equations. In Section 2 we introduce some notation and definitions. The family of velocity fields (1) is analyzed via complex potentials in Section 3, showing that the quantum flow of (1) is determined by the streamlines of a family of complex potentials. In Section 4, the system (2) is studied using the

qualitative theory of planar differential equations. We show that the family of differential equations (2) is a family of planar Hamiltonian systems whose phase portraits are obtained by the level curves of the Hamiltonian functions. Conclusions are presented in Section 5, and a review on complex potentials is given in the Appendix.

2 The magnetic string

Let us assume an infinity magnetic string carrying magnetic flux Φ coinciding with the z axis. The vector potential is then given by

$$\mathbf{A} = \frac{\Phi}{2\pi r} e_\theta, \quad (3)$$

where $e_\theta = (-\sin\theta, \cos\theta)$, r and θ are the polar coordinates in the plane xy . The flux parameter δ is defined by

$$\delta = \frac{e\Phi}{2\pi\hbar c}. \quad (4)$$

The evolution of the state Ψ of a particle of charge e and mass M interacting with an electromagnetic field of scalar potential φ and vector potential \mathbf{A} is given by the Schrödinger equation

$$i\hbar \frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \quad (5)$$

where the Hamiltonian operator \hat{H} is

$$\hat{H} = \frac{1}{2M} \left[-i\hbar\nabla - \frac{e}{c}\mathbf{A} \right]^2 + e\varphi. \quad (6)$$

The hydrodynamical approach (see [4]) is obtained by writing the wave function in the form

$$\Psi = \rho e^{i\chi},$$

and separating the real and imaginary parts of the Schrödinger equation. The real part leads to

$$M \left(\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B} - \nabla V, \quad (7)$$

where

$$\mathbf{v} = \frac{1}{M} \left(\hbar\nabla\chi - \frac{e}{c}\mathbf{A} \right), \quad (8)$$

and

$$V = -\frac{\hbar}{2M} \frac{\Delta\rho}{\rho}, \quad (9)$$

is the *quantum potential*. The imaginary part yields

$$\frac{\partial\rho^2}{\partial t} + \nabla \cdot (\rho^2 \mathbf{v}) = 0, \quad (10)$$

where ρ^2 is the *probability density* and $\mathcal{J} = \rho^2 \mathbf{v}$ is the *probability current*.

In the scattering problem near the magnetic string, Eq. (7) and (10) have the forms

$$M \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla V, \quad (11)$$

$$\nabla \cdot \mathcal{J} = 0, \quad (12)$$

since $\mathbf{E} = 0$ and $\mathbf{B} = 0$, whereas the probability current associated with the Aharonov-Bohm wave function is given by Eq. (1) (see [4]).

3 Complex potentials

We begin with the family of complex analytic functions

$$F(z, \delta) = \frac{\hbar k}{M} \left(-z + i \frac{\delta}{k} \log z \right), \quad (13)$$

where $z = x + iy$. It follows that

$$F'(z, \delta) = \frac{\hbar k}{M} \left(-1 + i \frac{\delta}{k} \frac{1}{z} \right), \quad (14)$$

which can be written as

$$F'(z, \delta) = \frac{\hbar k}{M} \left[\left(-1 + \frac{\delta}{k} \frac{y}{x^2 + y^2} \right) - i \left(-\frac{\delta}{k} \frac{x}{x^2 + y^2} \right) \right]. \quad (15)$$

From Eqs. (A.38) and (15) it follows that

$$u(x, y, \delta) = \frac{\hbar k}{M} \left(-1 + \frac{\delta}{k} \frac{y}{x^2 + y^2} \right) \quad (16)$$

and

$$v(x, y, \delta) = -\frac{\hbar k}{M} \left(\frac{\delta}{k} \frac{x}{x^2 + y^2} \right), \quad (17)$$

are the components of probability current (1). Therefore we have proved the following lemma.

Lemma 3.1 *The family of complex analytic functions (13) is the family of complex potentials of the probability current (1).*

The complex potential F in Eq. (13) is the sum of the “classical” complex potential F_1 and the “quantum” complex potential F_2 ,

$$F_1(z) = -\frac{\hbar k}{M} z \tag{18}$$

and

$$F_2(z, \delta) = i \frac{\hbar \delta}{M} \log z. \tag{19}$$

The stream function of the complex potential $F_1(z) = -(\hbar k/M) z = (\hbar k/M) (-x - iy)$ is $\psi_1(x, y) = -(\hbar k/M) y$. Thus the streamlines are the horizontal lines $y = c_0$ and the velocity at any point is $F_1'(z) = -\hbar k/M$. The flow is uniform to the left and is called *parallel*. When $\delta = 0$ we have $F = F_1$, and the flow is illustrated in Fig. 1.

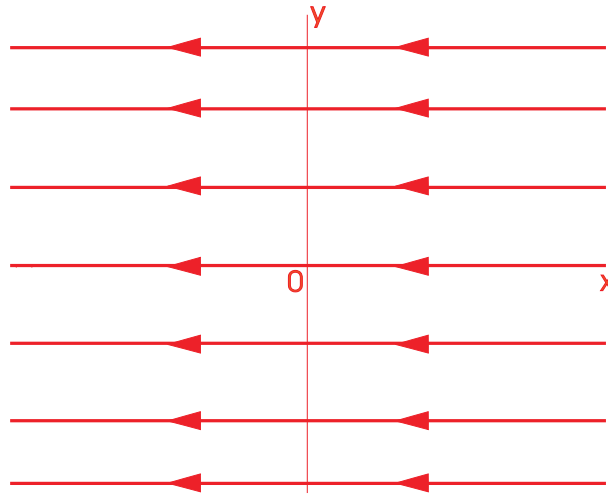


Figure 1: Flow of (1) when $\delta = 0$.

In polar coordinates the complex potential F_2 has the form

$$F_2(r, \theta, \delta) = i \frac{\hbar \delta}{M} \log(re^{i\theta}) = -\frac{\hbar \delta}{M} \theta + i \frac{\hbar \delta}{M} \log r. \tag{20}$$

The stream function is

$$\psi_2(r, \theta, \delta) = \frac{\hbar \delta}{M} \log r \tag{21}$$

and the streamlines are given by $\psi_2(r, \theta, \delta) = (\hbar \delta/M) \log r = c_0$, which are circles centered at the origin as represented in Fig. 2. This flow is a *pure rotation* with a *vortex* at the origin.

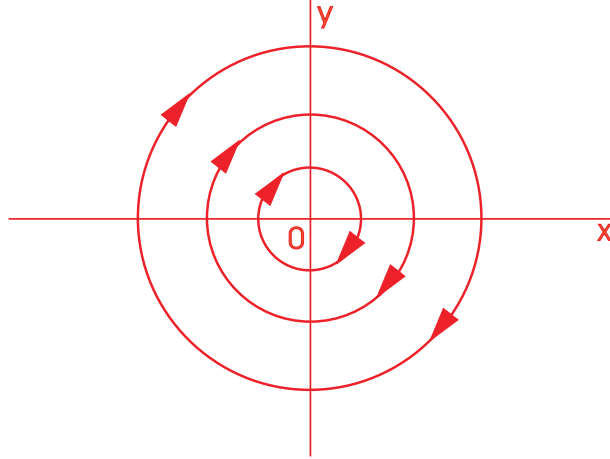


Figure 2: Streamlines of quantum complex potential F_2 .

This analysis shows that the flow of the complex potential F in Eq. (13) is the superposition of the parallel and the pure rotation flows. A new stagnation point appears to solve the compatibility problem of the two flows. It follows from Eq. (14)

$$F'(z, \delta) = -\frac{\hbar k}{M} \left(1 - \frac{z_0}{z}\right),$$

where $z_0 = i(\delta/k)$. Thus $F'(z, \delta) = 0$ if and only if $z = z_0$.

The Taylor expansion of $F'(z, \delta)$ at z_0 is given by

$$F'(z, \delta) = -\frac{\hbar k}{M} \frac{z - z_0}{z_0} + O(|z|^2) = i \frac{\hbar k^2}{\delta M} (z - z_0) + O(|z|^2). \quad (22)$$

Therefore near of the stagnation point z_0 the complex potential can be written as

$$F_3(z, \delta) = i \frac{\hbar k^2}{2\delta M} (z - z_0)^2. \quad (23)$$

Take $z = z - z_0$ for the study of the complex potential F_3 . Thus

$$F_3(z, \delta) = i \frac{\hbar k^2}{2\delta M} (x + i y)^2, \quad (24)$$

and the stream function is given by

$$\psi_3(x, y, \delta) = \frac{\hbar k^2}{2M\delta} (x^2 - y^2). \quad (25)$$

The streamlines are the level curves

$$\frac{\hbar k^2}{2M\delta} (x^2 - y^2) = c_0.$$

By a translation, the flow near $z_0 = i(\delta/k)$ is determined, as illustrated in Fig. 3. We have proved the following lemma.

Lemma 3.2 *The flow of the complex potential F in Eq. (13) is the superposition of the parallel and pure rotation flows, as represented in Fig. 1 and 2 respectively. The complex potential F has an stagnation point at $z_0 = i(\delta/k)$ for all $\delta > 0$. The flow of F near of z_0 is represented in Fig. 3.*

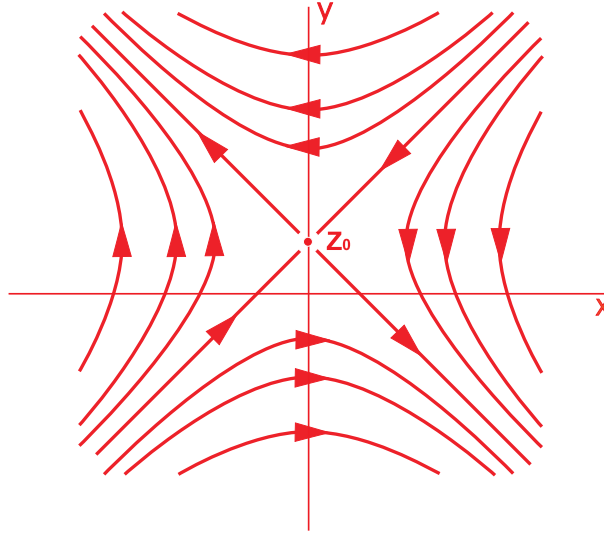


Figure 3: Streamlines of the complex potential F_3 .

One sees from Eq. (14) that

$$\lim_{|z| \rightarrow \infty} F'(z, \delta) = -\frac{\hbar k}{M} = F'_1(z).$$

Thus as the distance from the origin increases the quantum flow approaches that of a plane wave, as expected.

It should be noted that the stream function is given by

$$\psi(x, y, \delta) = \psi_1(x, y) + \psi_2(x, y, \delta) = \frac{\hbar k}{M} \left(-y + \frac{\delta}{k} \log \left(\sqrt{x^2 + y^2} \right) \right). \quad (26)$$

Thus $\psi(-x, y, \delta) = \psi(x, y, \delta)$ and therefore the quantum flow is symmetric with respect to y -axis.

The separatrices of the stagnation point $z_0 = i(\delta/k)$, for $\delta > 0$, represented in Fig. 4 by dashed lines, are subsets of the level curve

$$L(\delta) = \left\{ (x, y) \in \mathbb{R}^2 \mid \psi(x, y, \delta) = \frac{\hbar \delta}{M} \left(\log \frac{\delta}{k} - 1 \right), \delta > 0 \right\}.$$

There is a homoclinic streamline of the stagnation point z_0 bounding a neighborhood filled with cycles surrounding the vortex $(0,0)$. This neighborhood shrinks to $(0,0)$ as $\delta \rightarrow 0^+$.

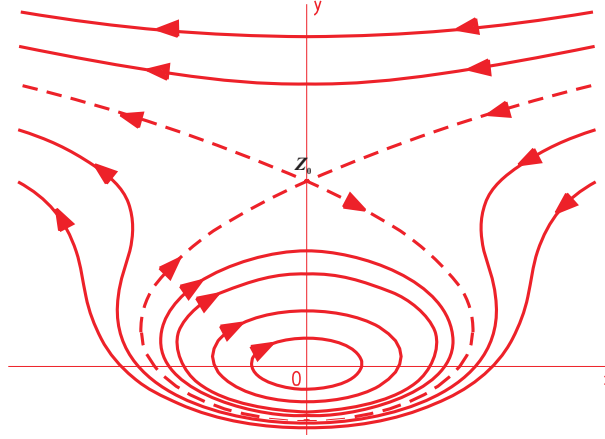


Figure 4: Quantum flow of (1) near the magnetic string ($\delta > 0$).

The following theorem is an immediate consequence of our analysis.

Theorem 3.3 *The family of complex analytic functions $F(z, \delta)$ in (13) is the family of quantum complex potentials of the quantum flow of (1). The quantum flow of (1) are represented in Fig. 1 and Fig. 4 for $\delta = 0$ and $\delta > 0$, respectively.*

The circulation of \mathcal{J} is given by

$$\Gamma_{\mathcal{J}} = \oint_C \mathcal{J} \cdot t \, ds = \oint_C F'(z) \, dz = \oint_C \left[\frac{\hbar k}{M} \left(-1 + i \frac{\delta}{k} \frac{1}{z} \right) \right] dz = -2\pi \frac{\hbar \delta}{M}, \quad (27)$$

where C is a closed curve encircling the origin. From Eqs. (4) and (27), this circulation can be rewritten as

$$\Gamma_{\mathcal{J}} = -\frac{e}{cM} \Phi. \quad (28)$$

Therefore from the mathematical point of view the quantum flow associated to the Aharonov-Bohm effect is a parallel flow at infinity with a vortex at the origin, where the circulation around the vortex is due to nonzero magnetic flux Φ .

4 Planar Hamiltonian differential equations

The planar differential equations of the form

$$\begin{cases} x' = \frac{\partial H}{\partial y}(x, y) \\ y' = -\frac{\partial H}{\partial x}(x, y) \end{cases} \quad (29)$$

are called *Hamiltonian differential equations*, where H is a C^2 real function called a *Hamiltonian function*. The Hamiltonian differential equations generally arise in mechanical systems without friction, and the total energy of the system can often be taken as the Hamiltonian function. See [1] and references therein for more details about mathematical aspects of Hamiltonian differential equations.

It is clear from the special form of Eq. (29) that the Hamiltonian H is a first integral (conservation of energy). Thus the phase portrait of Eq. (29) is obtained by the level curves of H .

The differential equations (2) are Hamiltonian differential equations with Hamiltonian

$$H(x, y, \delta) = \frac{\hbar k}{M} \left(\frac{\delta}{k} \log \left(\sqrt{x^2 + y^2} \right) - y \right). \quad (30)$$

Observing Eqs. (26) and (30), one sees that $H(x, y, \delta) = \psi(x, y, \delta)$. We have the following analog of the Theorem 3.3.

Theorem 4.1 *The planar differential equations (2) are Hamiltonian differential equations with the Hamiltonian H given by (30). The phase portraits of the differential equations (2) are illustrated in Fig. 1 and Fig. 4 for $\delta = 0$ and $\delta > 0$, respectively.*

Therefore, in the context of qualitative theory of differential equations, (1) is a Hamiltonian vector field where the stream function plays the role of the Hamiltonian.

5 Concluding remarks

In sections 3 and 4 we have analyzed, by two different points of view, the mathematical aspects of the quantum flow associated with the Aharonov-Bohm

effect. In section 3 we use the complex potentials theory to obtain the features of the velocity fields (1). The main goal was to obtain the family of complex potentials (13).

In section 4, we used the qualitative theory of planar differential equations to obtain the phase portraits of the planar differential equations (2), showing that the differential equations (2) are planar Hamiltonian differential equations.

Appendix: A review on complex potentials

Complex potentials play an important role in hydrodynamics [2]. Here the motion of the fluid is assumed to be two-dimensional, i.e., the same in all planes parallel to the xy -plane, and is independent of time. Thus, it is sufficient to consider the motion in the xy -plane. We consider only irrotational flows, and we also assume that the fluid is incompressible and free from viscosity.

Let $q(x, y) = (u(x, y), v(x, y))$ be the velocity of a particle of the fluid at any point (x, y) . Here the functions $u(x, y)$ and $v(x, y)$ are of class C^1 . From the above hypotheses

$$\oint_C q \cdot n \, ds = 0,$$

where C is a positively oriented simple closed contour lying in a simply connected domain, n is the normal vector to C and ds is the arc length of C . From Gauss Theorem

$$\nabla \cdot q(x, y) = \frac{\partial u}{\partial x}(x, y) + \frac{\partial v}{\partial y}(x, y) = 0. \quad (\text{A.31})$$

As the flow is irrotational one finds

$$\oint_C q \cdot t \, ds = 0, \quad (\text{A.32})$$

where t is the tangent vector to C . From Green Theorem

$$\nabla \times q(x, y) = \frac{\partial v}{\partial x}(x, y) - \frac{\partial u}{\partial y}(x, y) = 0. \quad (\text{A.33})$$

As Eqs. (A.31) and (A.33) are the Cauchy-Riemann equations for the functions u and $-v$,

$$u(x, y) - iv(x, y) \quad (\text{A.34})$$

is a complex analytic function.

It follows from Eq. (A.32) that $u dx + v dy$ is an exact differential. Therefore there is a function $\phi(x, y)$, the *velocity potential*, such that

$$q(x, y) = \nabla\phi(x, y). \quad (\text{A.35})$$

From (A.35) and (A.31) it follows that

$$\Delta\phi(x, y) = \frac{\partial^2\phi}{\partial x^2}(x, y) + \frac{\partial^2\phi}{\partial y^2}(x, y) = 0. \quad (\text{A.36})$$

In other words, ϕ is a harmonic function. Let $\psi(x, y)$ be a harmonic conjugate of $\phi(x, y)$. Thus

$$F = \phi + i\psi \quad (\text{A.37})$$

is a complex analytic function, called *complex potential* of the flow.

As the level curves of ψ and ϕ are orthogonal at points where the velocity vector is not the zero vector, the velocity vector is tangent to the level curves of ψ . Thus the function ψ characterizes the flow in a region. This function ψ is called *stream function* and their level curves are called the *streamlines* of the flow. A point z where $F'(z) = 0$ is called an *stagnation point*.

From Eq. (A.37) and (A.34)

$$F' = u - iv = \bar{q}. \quad (\text{A.38})$$

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