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Adomian Decomposition Method for Approximating the Solution of the Parabolic Equations

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Abstract

In this paper, the Adomian decomposition method for solving the linear and nonlinear parabolic equations is implemented with appropriate initial conditions. In comparison with existing techniques, the decomposition method is highly effective in terms of accuracy and rapid convergence. The numerical results obtained by this way have been compared with the exact solution to show the efficiency of the method.

Mathematics Subject Classification: 35K99

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1 Introduction

Over the last 10 years or so many mathematical method that are aimed at solving nonlinear ordinary and partial differential equations have appeared in the research literature [12]. However, most of them require a tedious analysis or a large computer memory to handle this problems. In the beginning of the 1980s, a so-called Adomian decomposition method was introduced by Adomian [1, 2] for solving the nonlinear problems. It is well known that this methods avoids linearization and provides an efficient numerical solution with high accuracy [5, 8, 10, 13].
In this paper, parabolic equations was solved by using Adomian decomposition method. The numerical results are compared with the exact solutions. It is shown that the errors are very small.

2 Analysis of the method

In this section, we demonstrate the main algorithm of Adomian decomposition method on linear and nonlinear parabolic equations with initial condition, namely we consider:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Phi(u) + g(x,t), \quad (x,t) \in [a,b] \times (0,T), \tag{1}$$

with the initial condition

$$u(x,0) = f(x), \tag{2}$$

where Φ is a function of u. We are looking for the solution satisfying Eqs.(1)-(2). The decomposition method consists of approximating the solution of (1)-(2) as an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3}$$

and decomposing Φ as

$$\Phi(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \cdots, u_n), \tag{4}$$

where A_ns are the Adomian polynomials given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\alpha^n}[\Phi(\sum_{k=0}^{\infty} \alpha^k u_k)]_{\alpha=0}, \quad n = 0, 1, 2, \cdots. \tag{5}$$

Applying the decomposition method [7, 9], Eq. (1) can be written as

$$L_t u = L_{xx} u + \Phi(u) + g(x,t), \tag{6}$$

where $L_t = \frac{\partial}{\partial t}$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$.
Assuming the inverse of operatorL_t exists it can be take as

$$L_t^{-1}(.) = \int_0^t (.)dt.$$

Therefore, applying on both sides of Eq. (6) with L_t^{-1} yields

$$u(x,t) = u(x,0) + L_t^{-1}(L_{xx}u) + L_t^{-1}(\Phi(u)) + L_t^{-1}(g(x,t)). \tag{7}$$

Using Eq. (3) and (4) it follows that

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L_t^{-1}(g(x,t)) + L_t^{-1}(\sum_{n=0}^{\infty} u_{nxx}) + L_t^{-1}(\sum_{n=0}^{\infty} A_n). \tag{8}$$

Therefore, one determines the iterates in the following recursive way:

$$\begin{cases} u_0(x,t) = u(x,t) + L_t^{-1}(g(x,t)), \\ u_{n+1}(x,t) = L_t^{-1}(u_{nxx} + A_n), \quad n = 0, 1, 2, \cdots. \end{cases} \tag{9}$$

The convergence of this series has been established, using fixed point theorem [3, 4]. However, in practice, all terms of the series $\sum_{n=0}^{\infty} u_n(x,t)$ can not be determined, so we use an approximation of the solution from the truncated series
$U_M(x,t) = \sum_{n=0}^{M} u_n(x,t)$ with $\lim_{M\longrightarrow\infty} U_M(x,t) = u(x,t)$.

3 Applications

In this section, we consider the application of the decomposition method to the Eq.(1) with the initial condition (2) by using (5), (9) for two examples.
Example 1. This problem was used by Hopkins and Wait [6] to provide an example of a problem with a nonlinear source term:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-u} + e^{-2u}, \quad (x,t) \in [0,1] \times (0,1], \tag{10}$$

with the initial condition $u(x,0) = \ln(x+2)$. In this example we have $\Phi(u) = e^{-u} + e^{-2u}$, $g(x,t) = 0$ and $f(x) = \ln(x+2)$. Adomian polynomials can be derived as follows

$$\begin{aligned} A_0 &= e^{-u_0} + e^{-2u_0}, \\ A_1 &= u_1(-e^{-u_0} - 2e^{-2u_0}), \\ A_3 &= (-u_2 + \tfrac{1}{2}u_1^2)e^{-u_0} + (-2u_2 + 2u_1^2) - \tfrac{1}{48}u_2u_1^2)e^{-2u_0}, \\ A_4 &= (-\tfrac{1}{12}u_4 + \tfrac{1}{48}u_1u_3 + \tfrac{1}{96}(6u_1u_3 + 4u_2^2) \\ &\quad +(-\tfrac{1}{6}u_4 + \tfrac{1}{24}u_1u_3 + \tfrac{1}{24}(6u_1u_3 + 4u_2^2 - \tfrac{1}{6}u_2u_1^2)e^{-2u_0} \end{aligned} \tag{11}$$

and so on, the rest of the polynomials can be constructed in similar manner. By using (9) we have

$$\begin{aligned} u_0 &= \ln(x+2), \\ u_1 &= \tfrac{t}{x+2}, \\ u_2 &= \tfrac{-t^2}{2(x+2)^2}, \\ u_3 &= \tfrac{t^3}{3(x+2)^3}, \\ &\vdots \\ u_n &= \tfrac{(-1)^{n+1}t^n}{n(x+2)^n}, \end{aligned} \tag{12}$$

and so on. Therefore from (3) we have

$$\begin{aligned} u(x,t) &= \ln(x+2) + \tfrac{t}{x+2} - \tfrac{t^2}{2(x+2)^2} + \tfrac{t^3}{3(x+2)^3} + \cdots + \tfrac{(-1)^{n+1}t^n}{n(x+2)^n} + \cdots \\ &= \ln(x+2) + \ln(\tfrac{t}{x+2} + 1) \\ &= \ln(x+t+2) \end{aligned} \tag{13}$$

Example 2.
This problem was used by Lawson and et. al. [11] as the form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (\pi^2 - 1 - p)u + (pe^{-t} + e^{-pt}) \quad (x,t) \in [0,1] \times (0,1) \tag{14}$$

with the initial condition

$$u(x,0) = 2\sin(\pi x)$$

In this example we have $\Phi(u) = (\pi^2 - 1 - p)u$, $g(x,t) = pe^{-t} + e^{-pt}$ and $f(x) = 2\sin(\pi x)$. By using (9) we have

$$\begin{aligned}
u_0 &= [2 - pe^{-t} - \tfrac{1}{p}e^{-pt} + p + \tfrac{1}{p}]\sin(\pi x),\\
u_1 &= -(1+p)[2t + pe^{-t} + \tfrac{1}{p^2}e^{-pt} + (p + \tfrac{1}{p})t - (p + \tfrac{1}{p^2})]\sin(\pi x),\\
u_2 &= (1+p)^2[t^2 - pe^{-t} - \tfrac{1}{p^3}e^{-pt} + (p + \tfrac{1}{p})\tfrac{t^2}{2!} - (p + \tfrac{1}{p^2})t + (p + \tfrac{1}{p^3})]\sin(\pi x)\\
u_3 &= -(1+p)^3[\tfrac{t^3}{3} + pe^{-t} + \tfrac{1}{p^4}e^{-pt} + (p + \tfrac{1}{p})\tfrac{t^3}{3!}\\
&\quad -(p + \tfrac{1}{p^2})\tfrac{t^2}{2!} + (p + \tfrac{1}{p^3})t - (p + \tfrac{1}{p^4})]\sin(\pi x),\\
&\vdots\\
u_n &= (-1)^n(1+p)^n[2\tfrac{t^n}{n!} + (-1)^{n+1}pe^{-t} + (-1)^{n+1}\tfrac{1}{p^{n+1}}e^{-pt} + (p + \tfrac{1}{p})\tfrac{t^n}{n!}\\
&\quad -(p + \tfrac{1}{p^2})\tfrac{t^{n-1}}{(n-1)!} + \cdots + (-1)^{n-1}(p + \tfrac{1}{p^n})t + (-1)^n(p + \tfrac{1}{p^{n+1}})]\sin(\pi x)
\end{aligned} \tag{15}$$

and so on. Therefore from (3) we have

$$\begin{aligned}
u(x,t) &= [2 - pe^{-t} - \tfrac{1}{p}e^{-pt} + p + \tfrac{1}{p}]\sin(\pi x)\\
&\quad -(1+p)[2t + pe^{-t} + \tfrac{1}{p^2}e^{-pt} + (p + \tfrac{1}{p})t - (p + \tfrac{1}{p^2})]\sin(\pi x)\\
&\quad +(1+p)^2[t^2 - pe^{-t} - \tfrac{1}{p^3}e^{-pt} + (p + \tfrac{1}{p})\tfrac{t^2}{2!} - (p + \tfrac{1}{p^2})t + (p + \tfrac{1}{p^3})]\sin(\pi x)\\
&\quad -(1+p)^3[\tfrac{t^3}{3} + pe^{-t} + \tfrac{1}{p^4}e^{-pt} + (p + \tfrac{1}{p})\tfrac{t^3}{3!} - (p + \tfrac{1}{p^2})\tfrac{t^2}{2!} + (p + \tfrac{1}{p^3})t\\
&\quad -(p + \tfrac{1}{p^4})]\sin(\pi x) + \cdots + +(-1)^n(1+p)^n[2\tfrac{t^n}{n!} + (-1)^{n+1}pe^{-t}\\
&\quad +(-1)^{n+1}\tfrac{1}{p^{n+1}}e^{-pt} + (p + \tfrac{1}{p})\tfrac{t^n}{n!} - (p + \tfrac{1}{p^2})\tfrac{t^{n-1}}{(n-1)!} + \cdots\\
&\quad +(-1)^{n-2}(p + \tfrac{1}{p^{n-1}})\tfrac{t^2}{2!} + (-1)^{n-1}(p + \tfrac{1}{p^n})t + (-1)^n(p + \tfrac{1}{p^{n+1}})]\sin(\pi x)\\
&\quad + \cdots = (e^{-t} + e^{-pt})\sin\pi x
\end{aligned} \tag{16}$$

4 Numerical implementation of ADM

In order to verify numerically whether the proposed methodology lead to accurate solutions, we will evaluate the ADM solutions using the M-terms approximation for some examples of the parabolic equations solved in the previous section. To show the efficiency of the present method for our problem in comparison with the exact solution we report absolute errors which is defined by

$$|u(x_i,t_k) - U_M(x_i,t_k)|$$

where u is the exact solution and $U_M = \sum_{n=0}^{M} u_n(x,t)$. For $M = 5, 10, 20$ we achieved a very good approximation with the actual solution of M-terms only of the decomposition series derived above. However, many terms can be calculated in order to achieve a high level accuracy of the decomposition method with help of *Matlab*.

Example 1. Table 1 shows absolute error for test problem 1 for various values of x, t and M. As this Table shows errors are very small.

Example 2. Table 2 shows absolute error for test problem 2 for various values of x, t, p and M. As this Table shows errors are very small.

Table1. Absolute error for various values of x , t and M for test problem 1.

x/t	0.2	0.4	0.6	0.8	1
$M = 5$					
0.2	$8.7288E-8$	$5.2113E-6$	$5.5632E-5$	$2.9414E-4$	0.0011
0.4	$5.2100E-8$	$3.1268E-6$	$3.3532E-5$	$1.7801E-4$	$6.4360E-4$
0.6	$3.2396E-8$	$1.9530E-6$	$2.1027E-5$	$1.1202E-4$	$4.0630E-4$
0.8	$2.0859E-8$	$1.2624E-6$	$1.3640E-5$	$7.2890E-5$	$2.6512E-4$
1.0	$1.3842E-8$	$8.4059E-7$	$9.1099E-6$	$4.8819E-5$	$1.7801E-4$
$M = 10$					
0.2	$2.9388E-13$	$5.5942E-10$	$4.5169E-8$	$1.0028E-6$	$1.0989E-5$
0.4	$1.1369E-13$	$2.1740E-10$	$1.7638E-8$	$3.9323E-7$	$4.3255E-6$
0.6	$4.6851E-14$	$9.1057E-11$	$7.4176E-9$	$1.6601E-7$	$1.8321E-6$
0.8	$2.1094E-14$	$4.0656E-11$	$3.3245E-9$	$7.4639E-8$	$8.2616E-7$
1.0	$9.9920E-15$	$1.9182E-11$	$1.5736E-9$	$3.5433E-8$	$3.9323E-7$
$M = 20$					
0.2	$2.2204E-16$	$3.3307E-16$	$5.2625E-14$	$2.1011E-11$	$2.1403E-9$
0.4	$1.1102E-16$	$6.6613E-16$	$9.1038E-15$	$3.4535E-12$	$3.5318E-10$
0.6	$4.4409E-16$	$2.2204E-16$	$1.5543E-15$	$6.5525E-13$	$6.7235E-11$
0.8	$2.2204E-16$	$2.2204E-16$	$2.2204E-16$	$1.4078E-13$	$1.4458E-11$
1.0	$2.2204E-16$	$6.6613E-16$	$6.6613E-16$	$3.4195E-14$	$3.4537E-12$

Table2. Absolute error for various values of x , t, p and M for test problem 2.

x/t	0.2	0.4	0.6	0.8	1
$M=20, p=1$					
0.2	$4.9449E-13$	$3.5554E-12$	$5.4122E-10$	$2.3279E-11$	$2.4691E-10$
0.4	$8.0003E-13$	$5.7551E-12$	$8.7571E-10$	$3.7667E-11$	$3.9951E-10$
0.6	$8.0025E-13$	$5.7527E-12$	$8.7571E-10$	$3.7667E-11$	$3.9951E-10$
0.8	$4.9438E-13$	$3.5553E-12$	$5.4122E-10$	$2.3280E-11$	$2.4691E-10$
1.0	$1.0304E-28$	$7.4077E-28$	$1.1276E-25$	$4.8503E-27$	$1.1444E-26$
$M=10, p=2$					
0.2	$1.0967E-10$	$2.0841E-7$	$1.7609E-5$	$4.0750E-4$	0.0046
0.4	$1.7745E-10$	$3.3721E-7$	$2.8492E-5$	$6.5934E-4$	0.0075
0.6	$1.7745E-10$	$3.3721E-7$	$2.8492E-5$	$6.5934E-4$	0.0075
0.8	$1.0967E-10$	$2.0841E-7$	$1.7609E-5$	$4.0750E-4$	0.0046
1.0	$2.2850E-26$	$4.3421E-23$	$3.6689E-21$	$8.8901E-21$	$9.6653E-19$
$M=20, p=3$					
0.2	$3.8205E-4$	$5.1061E-4$	$6.6805E-5$	$2.8638E-4$	$3.1596E-4$
0.4	$6.1817E-4$	$8.2618E-4$	$1.0809E-4$	$4.6338E-4$	$5.1124E-4$
0.6	$6.1817E-4$	$8.2618E-4$	$1.0809E-4$	$4.6338E-4$	$5.1124E-4$
0.8	$3.8205E-4$	$5.1061E-4$	$6.6805E-5$	$2.8638E-4$	$3.1596E-4$
1.0	$7.9600E-20$	$1.0639E-19$	$1.3919E-20$	$5.9668E-20$	$6.5830E-20$

5 Conclusions

In this paper, we have proposed an efficient method for solving system of parabolic equations, with high convergence and small error. As seen in Tables 1-2, errors are very small and they are very better than the results of another papers cited in this article.

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