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Lower Bounds on the Total Signed Domination Number of Graphs[1]

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Abstract

Let $G = (V, E)$ be a simple graph. A two-valued function $f : V \cup E \rightarrow \{-1, 1\}$ is called a *total signed domination function* of G if $\sum_{y \in N_T[x]} f(y) \geq 1$ for every $x \in V \cup E$, where $N_T[x]$ denote the set of x and the adjacent and incident elements of x. The *total signed domination number* $r_s^T(G)$ of G is defined as $r_s^T(G) = \min\{f(V \cup E)|f$ is a total signed domination function of $G\}$. In this paper, we obtain the lower bounds on the total signed domination number of some classes of graphs. Furthermore, we give the lower bound on the total signed domination number of the general graphs.

Keywords: signed domination function; signed domination number; total signed domination function; total signed domination number

1 Introduction

In this paper, we consider a finite, undirected graph without loops or multiple edges. For terminology and notation not given here, the reader is referred to Bondy and Murty [2]. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *order* of G is given by $n = |V|$ and its *size* by $m = |E|$. For any $x \in V \cup E$, $N_T[x]$ denote the set of x and the adjacent and incident

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elements of x. The *degree* of v in G is denoted by $d(v)$. The *minimum degree* of G is denoted by $\delta(G)$ and the *maximum degree* by $\Delta(G)$.

A function $f : V \rightarrow \{-1, 1\}$ is called a *signed domination function* of G if for any vertex $v \in V$, $f(N[v]) = \sum_{x \in N[v]} f(x) \geq 1$. The *signed domination number* of G is defined as $r_s(G) = \min\{f(V)|f$ is a signed domination function of $G\}$. The signed domination has been studied in [3, 7]. Some results were given as follows:

Lemma 1 (Henning et al. [3]) *Let G be a k-regular graph of order n, then $r_s(G) \geq \frac{n}{k+1}$.*

Lemma 2 (Henning et al. [7]) *Let G be a k-regular graph of order n, and k is odd, then $r_s(G) \geq \frac{2n}{k+1}$.*

A function $f : E \rightarrow \{-1, 1\}$ is called a *signed edge domination function* of G if for any edge $e \in E$, $f(N[e]) = \sum_{x \in N[e]} f(x) \geq 1$. The *signed edge domination number* of G is defined as $r_s'(G) = \min\{f(E)|f$ is a signed edge domination function of $G\}$. The signed edge domination has been studied in [1, 8]. They gave the following results:

Lemma 3 (Xu et al. [1]) *Let G be a graph of order n, then $r_s'(G) \geq 2\lceil(\sqrt{1+8n}-1)/2\rceil - n$.*

Lemma 4 (Xu et al. [1]) *Let G be a graph of order n, $m = |E|$, then $r_s'(G) \geq n - 2m/3$.*

Lemma 5 (Xu et al. [1]) *Let G be a graph of order n, then $r_s'(G) \geq \frac{\delta+2-\Delta}{\delta+2+\Delta} n$.*

In [4], Jin-bu Lu, Lin-zhong Liu gave the definition of total signed domination function as follows:

Definition 1 (Lu et al.[4]) A function $f : V \cup E \rightarrow \{-1, 1\}$ is called a *total signed domination function* of G if $\sum_{y \in N_T[x]} f(y) \geq 1$ for every $x \in V \cup E$, where $N_T[x]$ denote the set of x and the adjacent and incident elements of x. The *total signed domination number* $r_s^T(G)$ of G is defined as $r_s^T(G) = \min\{f(V \cup E)|f$ is a total signed domination function of $G\}$.

Furthermore, they gave the upper bounds on the total signed domination number of some classes of graphs. In this paper, we continue the study of total signed domination in graphs started by them. We obtain the lower bounds on the total signed domination number of some classes of graphs. Furthermore, we give the lower bound on the total signed domination number of the general graphs.

2 The lower bounds on the total signed domination number of some classes of graphs

Theorem 1 *Let C_n be a cycle, then $\lceil \frac{2n}{5} \rceil \leq r_s^T(C_n) \leq \lfloor \frac{2n}{5} \rfloor + 2$, and this bound is sharp.*

Proof: First, we proof $r_s^T(C_n) \geq \lceil \frac{2n}{5} \rceil$.

Let f be a total signed domination function of G. Let $V_+,\ V_-,\ E_+,\ E_-$ denote the sets of those vertices and edges of G which are assigned under f the values +1 and -1. Then $r_s^T(C_n) = |V_+| + |E_+| - |V_-| - |E_-|$.

By the definition of the total signed domination function, for any $y \in V \cup E$, $f(N[y]) \geq 1$. Since C_n is a cycle with $|V| = n$ and $|E| = n$, it follows that

$$\begin{aligned} 2n &\leq \sum_{y \in V \cup E} f(N[y]) \\ &\leq \sum_{y \in V} 5f(y) + \sum_{y \in E} 5f(y) \end{aligned}$$

That is

$$\begin{aligned} \tfrac{2n}{5} &\leq \sum_{y \in V_+} f(y) + \sum_{y \in E_+} f(y) + \sum_{y \in V_-} f(y) + \sum_{y \in E_-} f(y) \\ &\leq |V_+| - |V_-| + |E_+| - |E_-| \end{aligned}$$

Then, $r_s^T(C_n) \geq \frac{2n}{5}$.

Second, we proof $r_s^T(C_n) \leq \lfloor \frac{2n}{5} \rfloor + 2$.

Let $V = \{v_1, v_2, \cdots, v_n\},\ E = \{e_1, e_2, \cdots, e_n\}$. We define a function f' as follows:

We assign the vertices and edges with $\{+1, -1, +1, -1, +1\}$ in order.

Case 1: $n \equiv 0(mod 5)$. For any $x \in V \cup E,\ f'(N[x]) \geq 1$. That is, f' is a total signed domination function of G. $f'(V \cup E) = \frac{2n}{5}$.

Case 2: $n \equiv 1, 2(mod 5)$. We assign v_n the value +1, e_n the value +1. Then, for any $x \in V \cup E,\ f'(N[x]) \geq 1$. That is, f' is a total signed domination function of G. $f'(V \cup E) = \lfloor \frac{2n}{5} \rfloor + 2$.

Case 3: $n \equiv 3, 4(mod 5)$. For any $x \in V \cup E,\ f'(N[x]) \geq 1$. That is, f' is a total signed domination function of G. $f'(V \cup E) = \lfloor \frac{2n}{5} \rfloor + 1$.

By Case 1, Case 2 and Case 3, $f'(V \cup E) \leq \lfloor \frac{2n}{5} \rfloor + 2$. So $r_s^T(C_n) \leq \lfloor \frac{2n}{5} \rfloor + 2$.

That the bound is sharp may be seen as follows: $r_s^T(C_5) = 2 = \frac{2 \times 5}{5} = \lceil \frac{2n}{5} \rceil$, $r_s^T(C_6) = 4 = \lfloor \frac{2 \times 6}{5} \rfloor + 2 = \lfloor \frac{2n}{5} \rfloor + 2$, and this upper bound is better than the upper bound in [4].■

Theorem 2 *Let G be a k-regular graph, then* $r_s^T(G) \geq \frac{2n+nk}{2(2k+1)}$*, and this bound is sharp.*

Proof: Let f be a total signed domination function of G. Let $V_+,\ V_-,\ E_+,\ E_-$ denote the sets of those vertices and edges in G which are assigned under f the values +1 and -1. Then $r_s^T(G) = |V_+| + |E_+| - |V_-| - |E_-|$.

By the definition of the total signed domination function, for any $y \in V \cup E$, $f(N[y]) \geq 1$. Since G is a k-regular graph with $|V| = n$ and $|E| = \frac{nk}{2}$, it follows that

$$\begin{aligned} n + \tfrac{nk}{2} &\leq \sum_{y \in V \cup E} f(N[y]) \\ &\leq \sum_{y \in V \cup E} (2k+1) f(y) \end{aligned}$$

That is

$$\begin{aligned} \tfrac{2n+nk}{2(2k+1)} &\leq \sum_{y \in V \cup E} f(y) \\ &\leq |V_+| - |V_-| + |E_+| - |E_-| \end{aligned}$$

Then, $r_s^T(G) \geq \frac{2n+nk}{2(2k+1)}$.

That the bound is sharp may be seen as follows: $r_s^T(C_5) = 2 = \frac{10+10}{2\times 5} = \frac{2n+nk}{2(2k+1)}$.■

Theorem 3 *Let $K_{m,n}$ be a complete bipartite graph. Then*

$$r_s^T(K_{m,n}) \geq \begin{cases} \frac{2n+n^2}{2n+1}, & if \quad m = n; \\ \frac{3mn-2n^2+m+n+m^2n-mn^2}{2n+1}, & if \quad m \neq n. \end{cases}$$

Proof: Let f be a total signed domination function of G. Let$V_+,\ V_-,\ E_+,\ E_-$ denote the sets of those vertices and edges in G which are assigned under f the values +1 and -1. Let $|X| = m,\ |Y| = n$, then $r_s^T(K_{m,n}) = |V_+| + |E_+| - |V_-| - |E_-| = 2|V_+| + 2|E_+| - m - n - mn$.

By the definition of the total signed domination function, for any $y \in V \cup E$, $f(N[y]) \geq 1$. Since $K_{m,n}$ is a complete bipartite graph with$|V| = m + n$ and $|E| = mn$, it follows that

$$\begin{aligned} m+n+mn &\leq \textstyle\sum_{y\in V\cup E} f(N[y]) \\ &\leq \textstyle\sum_{y\in X}(2n+1)f(y) + \sum_{y\in Y}(2m+1)f(y) + \sum_{y\in E}(m+n+1)f(y) \\ &\leq (2n+1)(|X_+| - |X_-|) + (2m+1)(|Y_+| - |Y_-|) + (m+n+1)(|E_+| - |E_-|) \end{aligned}$$

That is

$$6mn + 2m + 2n + m^2n + mn^2 \leq 2(2n+1)|X_+| + 2(2m+1)|Y_+| + 2(m+n+1)|E_+|$$

Case 1: $m = n$. Then

$$\begin{aligned} 6n^2 + 4n + 2n^3 &\leq 2(2n+1)|X_+| + 2(2n+1)|Y_+| + 2(2n+1)|E_+| \\ \frac{6n^2+4n+2n^3}{2n+1} &\leq 2|V_+| + 2|E_+| \end{aligned}$$

So $r_s^T(K_{m,n}) \geq \frac{6n^2+4n+2n^3}{2n+1} - m - n - mn = \frac{6n^2+4n+2n^3}{2n+1} - 2n - n^2 = \frac{n^2+2n}{2n+1}$.

Case 2: $m \neq n$. We may assume $m < n$. Then

$$\begin{aligned} 6mn + 2m + 2n + m^2n + mn^2 &\leq 2(2n+1)|X_+| + 2(2n+1)|Y_+| + 2(2n+1)|E_+| \\ \frac{6mn+2m+2n+m^2n+mn^2}{2n+1} &\leq 2|V_+| + 2|E_+| \end{aligned}$$

So $r_s^T(K_{m,n}) \geq \frac{6mn+2m+2n+m^2n+mn^2}{2n+1} - m - n - mn = \frac{3mn-2n^2+m+n+m^2n-mn^2}{2n+1}$.

By Case 1 and Case 2, the proof is completed .■

3 The lower bounds on the total signed domination number of general graphs

Lemma 6 (Bondy [2]) $\sum_{v\in V} d(v) = 2m$.

Theorem 4 *Let $G = (V, E)$ be a simple graph, then $r_s^T(G) \geq \frac{(m+n)(4m+n-2n\Delta)}{n(2\Delta+1)}$.*

Proof: Let f be a total signed domination function of G. Let $V_+,\ V_-,\ E_+,\ E_-$

denote the sets of those vertices and edges in G which are assigned under f the values +1 and -1. Then $r_s^T(G) = |V_+| + |E_+| - |V_-| - |E_-| = 2|V_+| + 2|E_+| - m - n$.

By the definition of the total signed domination function, for any $y \in V \cup E$, $f(N[y]) \geq 1$, v_i, v_j is the vertex incident to edge y. Then

$$\begin{aligned} n+m &\leq \sum_{y\in V\cup E} f(N[y]) \\ &\leq \sum_{y\in V} f(N[y]) + \sum_{y\in E} f(N[y]) \\ &\leq \sum_{y\in V}[2d(y)+1]f(y) + \sum_{y\in E}[d(v_i)+d(v_j)+1]f(y) \\ &\leq (\sum_{y\in V_+} - \sum_{y\in V_-})[2d(y)+1] + (\sum_{y\in E_+} - \sum_{y\in E_-})[d(v_i)+d(v_j)+1] \\ &\leq (2\sum_{y\in V_+} - \sum_{y\in V})[2d(y)+1] + (2\sum_{y\in E_+} - \sum_{y\in E})[d(v_i)+d(v_j)+1] \\ &\leq 2\sum_{y\in V_+}[2d(y)+1] - (4m+n) + 2\sum_{y\in E_+}[d(v_i)+d(v_j)+1] - \sum_{y\in V} d^2(y) - m \end{aligned}$$

That is

$$n+m+4m+n+m+\sum_{y\in V} d^2(y) \leq 2\sum_{y\in V_+}[2d(y)+1] + 2\sum_{y\in E_+}[d(v_i)+d(v_j)+1]$$

Since $(\sum_{y\in V} d(y))^2/n \leq \sum_{y\in V} d^2(y)$, it follows that

$$\begin{aligned} 6m+2n+(\sum_{y\in V} d(y))^2/n &\leq 2\sum_{y\in V_+}[2d(y)+1] + 2\sum_{y\in E_+}[d(v_i)+d(v_j)+1] \\ &\leq 2(2\Delta+1)|V_+| + 2(2\Delta+1)|E_+| \end{aligned}$$

That is

$$\frac{6m+2n+4m^2/n}{2\Delta+1} \leq 2|V_+| + 2|E_+|$$

Then
$r_s^T(G) = 2|V_+| + 2|E_+| - m - n \geq \frac{6m+2n+4m^2/n}{2\Delta+1} - m - n = \frac{(m+n)(4m+n-2n\Delta)}{n(2\Delta+1)}$.■

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