

# Analysis of a Prey-Predator Fishery Model with Prey Reserve

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## Abstract

In this paper, we consider a prey-predator fishery model with prey dispersal in a two-patch environment, one is assumed to be a free fishing zone and the other is a reserved zone where fishing and other extractive activities are prohibited. The existence of biological and bionomic equilibrium of the system is discussed. The local and global stability analysis has been carried out. An optimal harvesting policy is given using Pontryagin's maximum principle.

**Keywords:** Prey-predator; Global stability; Optimal harvesting

## 1 Introduction

Biological resources are renewable resources. Economic and biological aspects of renewable resources management have been considered by Clark [1]. In recent years, the optimal management of renewable resources, which has a direct relationship to sustainable development, has been studied extensively by many authors [2-7]. The reason is that mankind is facing the problems about shortage of resource at present. Extensive and unregulated harvesting of marine fishes can even lead to the depletion of several fish species. One potential solution to these problems is the creation of marine reserves where fishing and other extractive activities are prohibited. Marine reserve not only protect species inside the reserve area but they can also increase fish abundance

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in adjacent areas. Mathematical model of ecological system, reflecting these problems, has been given in Kar and Swarnakamal [6].

The paper is mainly concerned with the following prey-predator system

$$\begin{cases} \frac{dx_1}{dt} = r_1x_1\left(1 - \frac{x_1}{K_1}\right) - \alpha x_1y - \sigma_1x_1 + \sigma_2x_2 - q_1E_1x_1, \\ \frac{dx_2}{dt} = r_2x_2\left(1 - \frac{x_2}{K_2}\right) + \sigma_1x_1 - \sigma_2x_2, \\ \frac{dy}{dt} = -dy + k\alpha x_1y - q_2E_2y. \end{cases} \quad (1.1)$$

Here,  $x_1(t)$  and  $y(t)$  are biomass densities of prey species and predator species inside the unreserved area which is an open-access fishing zone, respectively, at time  $t$ .  $x_2(t)$  is the biomass density of prey species inside the reserved area where no fishing is permitted at time  $t$ . All the parameters are assumed to be positive.  $r_1$  and  $r_2$  are the intrinsic growth rates of prey species inside the unreserved and reserved areas, respectively.  $d, \alpha$  and  $k$  are the death rate, capturing rate and conversion rate of predators, respectively.  $K_1$  and  $K_2$  are the carrying capacities of prey species in the unreserved and reserved areas, respectively.  $\sigma_1$  and  $\sigma_2$  are migration rates from the unreserved area to the reserved area and the reserved area to the unreserved area, respectively.  $E_1$  and  $E_2$  are the effects applied to harvest the prey species and predator species in the unreserved area.  $q_1$  and  $q_2$  are the catchability coefficients.

Considering the biological background, we only care about the dynamics of system (1.1) in the closed first quadrant  $R_+^3$ .

Here we observe that, if there is no migration of fish population from the reserved area to the unreserved area (i.e.,  $\sigma_2 = 0$ ) and  $r_1 - \sigma_1 - q_1E_1 < 0$ , then  $\dot{x}_1 < 0$ . Similarly, if there is no migration of fish population from the unreserved area to the reserved area (i.e.,  $\sigma_1 = 0$ ) and  $r_2 - \sigma_2 < 0$ , then  $\dot{x}_2 < 0$ . Hence, throughout out analysis, we assume that

$$r_1 - \sigma_1 - q_1E_1 > 0, \quad r_2 - \sigma_2 > 0. \quad (1.2)$$

## 2 Existence of equilibria

Equating the derivatives on the left hand sides to zero and solving the resulting algebraic equations we can find three possible equilibrium  $R_1(0, 0, 0)$ ,  $R_2(\bar{x}_1, \bar{x}_2, 0)$  and  $R_3(x_1^*, x_2^*, y^*)$ .

Here,  $\bar{x}_2 = \frac{1}{\sigma_2}[(\sigma_1 + q_1 E_1 - r_1)\bar{x}_1 + \frac{r_1}{K_1}\bar{x}_1^2]$  and  $\bar{x}_1$  is the positive solution of the following equation

$$c_{11}x_1^3 + c_{12}x_1^2 + c_{13}x_1 + c_{14} = 0, \tag{2.1}$$

where

$$\begin{aligned} c_{11} &= \frac{r_2 r_1^2}{K_2 K_1^2 \sigma_2^2} > 0, \\ c_{12} &= -\frac{2r_1 r_2 (r_1 - \sigma_1 - q_1 E_1)}{K_1 K_2 \sigma_2^2} < 0, \\ c_{13} &= \frac{r_2 (r_1 - \sigma_1 - q_1 E_1)^2}{K_2 \sigma_2^2} - \frac{r_1 (r_2 - \sigma_2)}{K_1 \sigma_2}, \\ c_{14} &= \frac{(r_2 - \sigma_2)}{\sigma_2} (r_1 - \sigma_1 - q_1 E_1) - \sigma_1. \end{aligned}$$

Eq. (2.1) has a unique positive solution  $x_1 = \bar{x}_1$  if the following inequalities hold:

$$\begin{aligned} \frac{r_2 (r_1 - \sigma_1 - q_1 E_1)^2}{K_2 \sigma_2} &< \frac{r_1 (r_2 - \sigma_2)}{K_1} \\ (r_2 - \sigma_2)(r_1 - \sigma_1 - q_1 E_1) &> \sigma_1 \sigma_2. \end{aligned} \tag{2.2}$$

For  $\bar{x}_2$  to be positive, we must have

$$\bar{x}_1 > \frac{(r_1 - \sigma_1 - q_1 E_1) K_1}{r_1}. \tag{2.3}$$

Again,  $x_1^* = \frac{d+q_2 E_2}{k\alpha}$ ,  $x_2^* = K_2 \{r_2 - \sigma_2 + [(r_2 - \sigma_2)^2 + 4r_2 \sigma_1 x_1^* / K_2]^{1/2}\} / 2r_2$ ,

$$y^* = \frac{(r_1 - \sigma_1 - q_1 E_1)x_1^* - \frac{r_1}{K_1}x_1^{*2} + \sigma_2 x_2^*}{\alpha x_1^*}.$$

For  $y^*$  to be positive, we must have

$$(r_1 - \sigma_1 - q_1 E_1)x_1^* + \sigma_2 x_2^* > \frac{r_1}{K_1}x_1^{*2}. \tag{2.4}$$

### 3 Stability analysis

In the absence of predator species, the model (1.1) becomes

$$\begin{cases} \frac{dx_1}{dt} = r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \sigma_1 x_1 + \sigma_2 x_2 - q_1 E_1 x_1, \\ \frac{dx_2}{dt} = r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) + \sigma_1 x_1 - \sigma_2 x_2, \end{cases} \tag{3.1}$$

which has two equilibria,  $R(0, 0)$  and a positive equilibrium  $\bar{R}(\bar{x}_1, \bar{x}_2)$  if (2.2) and (2.3) hold.

The Jacobian matrix of the system (3.1) is

$$\begin{pmatrix} r_1 - 2\frac{r_1}{K_1}x_1 - \sigma_1 - q_1E_1 & \sigma_2 \\ \sigma_1 & r_2 - 2\frac{r_2}{K_2}x_2 - \sigma_2 \end{pmatrix} \tag{3.2}$$

The characteristic equation of the Jacobian matrix of (3.2) at  $R$  is

$$\lambda^2 - (r_1 - \sigma_1 - q_1E_1 + r_2 - \sigma_2)\lambda + (r_1 - \sigma_1 - q_1E_1)(r_2 - \sigma_2) - \sigma_1\sigma_2 = 0. \tag{3.3}$$

Since  $\lambda_1 + \lambda_2 = r_1 - \sigma_1 - q_1E_1 + r_2 - \sigma_2 > 0$  and  $\lambda_1\lambda_2 = (r_1 - \sigma_1 - q_1E_1)(r_2 - \sigma_2) - \sigma_1\sigma_2 > 0$ .

Hence  $R(0, 0)$  is unstable.

Similarly, the characteristic equation of the Jacobian matrix of (3.2) at  $\bar{R}$  is

$$\lambda^2 + \left(\frac{r_1}{K_1}\bar{x}_1 + \frac{r_2}{K_2}\bar{x}_2 + \frac{\sigma_1\bar{x}_1}{\bar{x}_2} + \frac{\sigma_2\bar{x}_2}{\bar{x}_1}\right)\lambda + \frac{r_2\bar{x}_2}{K_2}\left(\frac{r_1}{K_1}\bar{x}_1 + \frac{\sigma_2\bar{x}_2}{\bar{x}_1}\right) + \frac{r_1\sigma_1\bar{x}_1^2}{K_1\bar{x}_2}. \tag{3.4}$$

Since  $\lambda_1 + \lambda_2 = -\left(\frac{r_1}{K_1}\bar{x}_1 + \frac{r_2}{K_2}\bar{x}_2 + \frac{\sigma_1\bar{x}_1}{\bar{x}_2} + \frac{\sigma_2\bar{x}_2}{\bar{x}_1}\right) < 0$ , and  $\lambda_1\lambda_2 = \frac{r_2\bar{x}_2}{K_2}\left(\frac{r_1}{K_1}\bar{x}_1 + \frac{\sigma_2\bar{x}_2}{\bar{x}_1}\right) + \frac{r_1\sigma_1\bar{x}_1^2}{K_1\bar{x}_2} > 0$ .

Thus  $\bar{R}(\bar{x}_1, \bar{x}_2)$  is locally asymptotically stable.

Let us now suppose that system (1.1) has a unique positive equilibrium  $R_3(x_1^*, x_2^*, y^*)$ . The Jacobian matrix of (1.1) at  $R_3$  is

$$\begin{pmatrix} r_1 - 2\frac{r_1}{K_1}x_1^* - \alpha y^* - \sigma_1 - q_1E_1 & \sigma_2 & -\alpha x_1^* \\ \sigma_1 & r_2 - 2\frac{r_2}{K_2}x_2^* - \sigma_2 & 0 \\ k\alpha y^* & 0 & -d + k\alpha x_1^* - q_2E_2 \end{pmatrix} \tag{3.5}$$

The characteristic equation of the Jacobian matrix of (1.1) at  $R_3$  is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{3.6}$$

where

$$\begin{aligned} a_1 &= \frac{r_1}{K_1}x_1^* + \frac{r_2}{K_2}x_2^* + \frac{\sigma_2x_2^*}{x_1^*} + \frac{\sigma_1x_1^*}{x_2^*}, \\ a_2 &= \left(\frac{r_1}{K_1}x_1^* + \frac{\sigma_2x_2^*}{x_1^*}\right)\left(\frac{r_2}{K_2}x_2^* + \frac{\sigma_1x_1^*}{x_2^*}\right) + \sigma_1\sigma_2 + k\alpha^2x_1^*y^*, \\ a_3 &= k\alpha^2x_1^*y^*\left(\frac{r_2}{K_2}x_2^* + \frac{\sigma_1x_1^*}{x_2^*}\right). \end{aligned}$$

According to Routh-Hurwitz criteria, the necessary and sufficient conditions for local stability of equilibrium point  $R_3$  are

$$a_1 > 0, a_3 > 0 \text{ and } a_1a_2 - a_3 > 0.$$

It is evident that  $a_1 > 0, a_3 > 0$ . Thus, the stability of  $R_3$  is determined by the sign of  $a_1a_2 - a_3$ . By direct calculations, we obtain

$$a_1a_2 - a_3 = \left(\frac{r_1}{K_1}x_1^* + \frac{r_2}{K_2}x_2^* + \frac{\sigma_2x_2^*}{x_1^*} + \frac{\sigma_1x_1^*}{x_2^*}\right)\left[\left(\frac{r_1}{K_1}x_1^* + \frac{\sigma_2x_2^*}{x_1^*}\right)\left(\frac{r_2}{K_2}x_2^* + \frac{\sigma_1x_1^*}{x_2^*}\right) + \sigma_1\sigma_2\right] + k\alpha^2x_1^*y^*\left(\frac{r_1}{K_1}x_1^* + \frac{\sigma_2x_2^*}{x_1^*}\right) > 0.$$

Hence  $R_3(x_1^*, x_2^*, y^*)$  is locally asymptotically stable.

**Theorem 1.** The equilibrium point  $\bar{R}$  is globally asymptotically stable.

**Proof.** Let us consider the following Lyapunov function:

$$V(x_1, x_2) = \omega_1(x_1 - \bar{x}_1 - \bar{x}_1 \ln \frac{x_1}{\bar{x}_1}) + \omega_2(x_2 - \bar{x}_2 - \bar{x}_2 \frac{x_2}{\bar{x}_2}),$$

where  $\omega_1, \omega_2$  are positive constants, to be chosen later on.

Differentiating  $V$  with respect to time  $t$ , we get

$$\frac{dV}{dt} = \omega_1 \frac{(x_1 - \bar{x}_1)}{x_1} \frac{dx_1}{dt} + \omega_2 \frac{(x_2 - \bar{x}_2)}{x_2} \frac{dx_2}{dt}.$$

Choosing  $\frac{\omega_2}{\omega_1} = \frac{\bar{x}_2 \sigma_2}{\bar{x}_1 \sigma_1}$ , a little algebraic manipulation yields

$$\frac{dV}{dt} = -\frac{r_1\omega_2\sigma_1\bar{x}_1}{K_1\sigma_2\bar{x}_2}(x_1 - \bar{x}_1)^2 - \frac{r_2}{K_2}\omega_2(x_2 - \bar{x}_2)^2 - \frac{\omega_2\sigma_2}{x_1x_2\bar{x}_2}(\bar{x}_1x_2 - x_1\bar{x}_2)^2 < 0.$$

Therefore,  $\bar{R}(\bar{x}_1, \bar{x}_2)$  is globally asymptotically stable.

**Theorem 2.**  $R_3(x_1^*, x_2^*, y^*)$  is globally asymptotically stable.

**Proof.** Let us choose the Lyapunov function

$$V(x_1, x_2, y) = \omega_1(x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*})$$

$$+\omega_2(x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*}) + \omega_3(y - y^* - y^* \ln \frac{y}{y^*}),$$

where  $\omega_1, \omega_2$  and  $\omega_3$  are positive constants, to be chosen later on.

Differentiating  $V$  with respect to time  $t$ , we get

$$\frac{dV}{dt} = \omega_1 \frac{(x_1 - x_1^*)}{x_1} \frac{dx_1}{dt} + \omega_2 \frac{(x_2 - x_2^*)}{x_2} \frac{dx_2}{dt} + \omega_3 \frac{(y - y^*)}{y} \frac{dy}{dt}.$$

Choosing  $\frac{\omega_1}{\omega_3} = k, \frac{\omega_1}{\omega_2} = \frac{x_1^* \sigma_1}{x_2^* \sigma_2}$ , a little algebraic manipulation yields

$$\frac{dV}{dt} = -\frac{r_1}{K_1} \omega_1 (x_1 - x_1^*)^2 - \frac{r_2 \omega_1 \sigma_2 x_2^*}{K_2 \sigma_1 x_1^*} (x_2 - x_2^*)^2 - \frac{\omega_1 \sigma_2 (x_2 x_1^* - x_1 x_2^*)^2}{x_1 x_1^* x_2} < 0.$$

Therefore,  $R_3(x_1^*, x_2^*, y^*)$  is globally asymptotically stable.

## 4 Bionomic equilibrium

Let

$c_1$  =fishing cost per unit effort for prey species,

$c_2$  =fishing cost per unit effort for predator species,

$p_1$  =price per unit biomass of the prey,

$p_2$  =price per unit biomass of the predator.

Therefore, the economic rent(net revenue)at any time is given by

$$\begin{aligned} \Pi &= (p_1 q_1 x_1 - c_1) E_1 + (p_2 q_2 y - c_2) E_2 \\ &= \Pi_1 + \Pi_2 \end{aligned}$$

where  $\Pi_1 = (p_1 q_1 x_1 - c_1) E_1, \Pi_2 = (p_2 q_2 y - c_2) E_2$  i.e.,  $\Pi_1$  and  $\Pi_2$  represent the net revenues for the prey and predator species, respectively.

The bionomic equilibrium  $(x_{1\infty}, x_{2\infty}, y_{\infty}, E_{1\infty}, E_{2\infty})$  is given by the following simultaneous equations

$$r_1 x_1 (1 - \frac{x_1}{K_1}) - \alpha x_1 y - \sigma_1 x_1 + \sigma_2 x_2 - q_1 E_1 x_1 = 0, \tag{4.1}$$

$$r_2 x_2 (1 - \frac{x_2}{K_2}) + \sigma_1 x_1 - \sigma_2 x_2 = 0, \tag{4.2}$$

$$-d + k \alpha x_1 - q_2 E_2 = 0, \tag{4.3}$$

$$\Pi = (p_1 q_1 x_1 - c_1) E_1 + (p_2 q_2 y - c_2) E_2 = 0. \tag{4.4}$$

In order to determine the bionomic equilibrium, we now consider the following cases:

Case 1: If  $c_2 > p_2q_2y$ , i.e.the cost is greater than the revenue for the predator, then the predator fishing will be stopped ( $E_2 = 0$ ). Only the prey fishing remains operational (i.e. $c_1 < p_1q_1x_1$ ).

We then have  $x_{1\infty} = \frac{c_1}{p_1q_1}$ ,  $x_{2\infty} = \frac{K_2}{2r_2} \{r_2 - \sigma_2 + [(r_2 - \sigma_2)^2 + 4\sigma_1 \frac{r_2c_1}{K_2p_1q_1}]^{1/2}\}$ . Since  $c_1 < p_1q_1x_1 < p_1q_1K_1$ . Hence  $1 - \frac{c_1}{p_1q_1K_1} > 0$ .  $(y_\infty, E_{1\infty})$  will be any point on the line

$$\begin{aligned} & \sigma_1 + \alpha y + q_1 E_1 \\ &= r_1 \left(1 - \frac{c_1}{p_1q_1K_1}\right) + \frac{\sigma_2 p_1 q_1 K_2}{2r_2 c_1} \{r_2 - \sigma_2 + [(r_2 - \sigma_2)^2 + 4\sigma_1 \frac{r_2 c_1}{K_2 p_1 q_1}]^{1/2}\} \end{aligned}$$

in the first quadrant of the  $yE_1$ - plane.

Case 2: If  $c_1 > p_1q_1x_1$ , i.e.the cost is greater than the revenue in the prey fishing, then prey fishing will be closed ( $E_1 = 0$ ). Only predator fishing remains operational (i.e. $c_2 < p_2q_2y$ ).

We then have  $y_\infty = \frac{c_2}{p_2q_2}$ . Substituting  $y_\infty$  into (4.1) we get  $x_{2\infty} = \frac{1}{\sigma_2} [\frac{\alpha c_2}{p_2q_2} x_{1\infty} + \sigma_1 x_{1\infty} - r_1 x_{1\infty} (1 - \frac{x_{1\infty}}{K_1})]$ ,  $x_{1\infty}$  is the positive solution of the following equation

$$c_{21}x_1^3 + c_{22}x_1^2 + c_{23}x_1 + c_{24} = 0, \tag{4.5}$$

where

$$\begin{aligned} c_{21} &= \frac{r_2 r_1^2}{K_2 K_1^2 \sigma_2^2} > 0, \\ c_{22} &= -\frac{2r_1 r_2 (r_1 - \sigma_1 - \frac{\alpha c_2}{p_2 q_2})}{K_1 K_2 \sigma_2^2}, \\ c_{23} &= \frac{r_2 (r_1 - \sigma_1 - \frac{\alpha c_2}{p_2 q_2})^2}{K_2 \sigma_2^2} - \frac{r_1 (r_2 - \sigma_2)}{K_1 \sigma_2}, \\ c_{24} &= \frac{(r_2 - \sigma_2)}{\sigma_2} (r_1 - \sigma_1 - \frac{\alpha c_2}{p_2 q_2}) - \sigma_1. \end{aligned}$$

Now, if

$$(1) \quad r_1 - \sigma_1 - \frac{\alpha c_2}{p_2 q_2} < 0, \quad \frac{r_2 (r_1 - \sigma_1 - \frac{\alpha c_2}{p_2 q_2})^2}{K_2 \sigma_2^2} < \frac{r_1 (r_2 - \sigma_2)}{K_1 \sigma_2},$$

or

$$(2) \quad r_1 - \sigma_1 - \frac{\alpha c_2}{p_2 q_2} > 0, \quad \frac{r_2 (r_1 - \sigma_1 - \frac{\alpha c_2}{p_2 q_2})^2}{K_2 \sigma_2^2} < \frac{r_1 (r_2 - \sigma_2)}{K_1 \sigma_2}, \quad (r_2 - \sigma_2) (r_1 - \sigma_1 - \frac{\alpha c_2}{p_2 q_2}) >$$

$\sigma_1 \sigma_2$ .

then Eq (4.5) has a unique positive solution  $x_1 = x_{1\infty}$ . For  $x_{2\infty}$  to be positive, we must have

$$x_{1\infty} > K_1 - \frac{\sigma_1 K_1}{r_1} - \frac{\alpha c_2 K_1}{p_2 q_2 r_1}.$$

Substituting  $x_{1\infty}$  into (4.3) we get

$$E_{2\infty} = \frac{k\alpha x_{1\infty} - d}{q_2}.$$

$E_{2\infty} > 0$ , provided  $k\alpha x_{1\infty} > d$ .

Case 3: If  $c_1 > p_1 q_1 x_1, c_2 > p_2 q_2 y$ , then the cost is greater than revenues for both the species and the whole fishery will be closed.

Case 4: If  $c_1 < p_1 q_1 x_1, c_2 < p_2 q_2 y$ , then the revenues for both the species being positive, then the whole fishing will be in operation.

In this case,  $x_{1\infty} = c_1/p_1 q_1$  and  $y_\infty = c_2/p_2 q_2$ .

Now substituting  $x_{1\infty}$  and  $y_\infty$  into (4.1), (4.2) and (4.3), we get

$$x_{2\infty} = \frac{K_2}{2r_2} \{r_2 - \sigma_2 + [(r_2 - \sigma_2)^2 + 4\sigma_1 \frac{r_2 c_1}{K_2 p_1 q_1}]^{1/2}\}, \tag{4.6}$$

$$E_{1\infty} = \frac{r_1}{q_1} \left(1 - \frac{c_1}{p_1 q_1 K_1}\right) + \frac{p_1 \sigma_2 x_{2\infty}}{c_1} - \frac{\sigma_1}{q_1} - \frac{\alpha c_2}{q_1 p_2 q_2}, \tag{4.7}$$

and

$$E_{2\infty} = \frac{k\alpha c_1}{p_1 q_1 q_2} - \frac{d}{q_2}. \tag{4.8}$$

Now,

$$E_{1\infty} > 0, \text{ if } \frac{r_1}{q_1} \left(1 - \frac{c_1}{p_1 q_1 K_1}\right) + \frac{p_1 \sigma_2 x_{2\infty}}{c_1} > \frac{\sigma_1}{q_1} + \frac{\alpha c_2}{q_1 p_2 q_2}, \tag{4.9}$$

$$E_{2\infty} > 0, \text{ if } \frac{k\alpha c_1}{p_1 q_1} > d. \tag{4.10}$$

Thus the nontrivial bionomic equilibrium point  $(x_{1\infty}, x_{2\infty}, E_{1\infty}, E_{2\infty})$  exists if conditions (4.9) and (4.10) hold.

## 5 Optimal harvesting policy

In this section, our objective is to maximize the present value  $J$  of a continuous time stream of revenues given by

$$J = \int_0^\infty e^{-\delta t} \{(p_1 q_1 x_1 - c_1)E_1(t) + (p_2 q_2 y - c_2)E_2(t)\} dt, \tag{5.1}$$

where  $\delta$  denotes the instantaneous annual rate of discount. We intend to maximize (5.1) subject to the state equations (1.1) by invoking Pontryagin's maximal principle (Clark [1]). The control variable  $E_i(t)(i = 1, 2)$  are subjected to the constraints

$$0 \leq E_i(t) \leq (E_i)_{max}.$$

The Hamiltonian for the problem is given by

$$\begin{aligned} H = & e^{-\delta t}[(p_1q_1x_1 - c_1)E_1 + (p_2q_2y - c_2)E_2] + \lambda_1[r_1x_1(1 - x_1/K_1) - \alpha x_1y \\ & - \sigma_1x_1 + \sigma_2x_2 - q_1E_1x_1] + \lambda_2[r_2x_2(1 - r_2/K_2) + \sigma_1x_1 - \sigma_2x_2] \\ & + \lambda_3(-dy + k\alpha x_1y - q_2E_2y), \end{aligned}$$

where  $\lambda_1, \lambda_2$  and  $\lambda_3$  are the adjoint variables (Clark [1]).

The control variables  $E_1$  and  $E_2$  appear linearly in the Hamiltonian function  $H$ .

Assuming that the control constraints are not binding i.e., the optimal solution does not occur at  $(E_i)_{max}$  or  $(E_i)_{max}$ , we have singular control (Clark [1]).

According to Pontryagin's maximum principle

$$\frac{\partial H}{\partial E_1} = 0; \quad \frac{\partial H}{\partial E_2} = 0; \quad \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1}; \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2}; \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial y}.$$

Substitution and simplification yields

$$\frac{dH}{dE_1} = 0 \Rightarrow \lambda_1 = e^{-\delta t}(p_1 - \frac{c_1}{q_1x_1}), \tag{5.2}$$

$$\frac{dH}{dE_2} = 0 \Rightarrow \lambda_3 = e^{-\delta t}(p_2 - \frac{c_2}{q_2y}), \tag{5.3}$$

$$\frac{d\lambda_1}{dt} = - [e^{-\delta t}p_1q_1E_1 + \lambda_1(r_1 - 2\frac{r_1}{K_1}x_1 - \alpha y - \sigma_1 - q_1E_1) + \lambda_2\sigma_1 + \lambda_3k\alpha y], \tag{5.4}$$

$$\frac{d\lambda_2}{dt} = - [\lambda_1\sigma_2 + \lambda_2(r_2 - 2\frac{r_2}{K_2}x_2 - \sigma_2)], \tag{5.5}$$

$$\frac{d\lambda_3}{dt} = - [e^{-\delta t}p_2q_2E_2 - \lambda_1\alpha x_1 + \lambda_3(-d + k\alpha x_1 - q_2E_2)]. \tag{5.6}$$

Now, Substituting  $\lambda_1$  and  $\lambda_3$  into (5.6) and using equilibrium equations we get

$$y^* = \frac{\delta c_2q_1}{q_1p_2q_2(\delta + d - k\alpha x_1^*) + (p_1q_1x_1^* - c_1)\alpha q_2}. \tag{5.7}$$

From (5.5), we get  $\frac{d\lambda_2}{dt} - A_1\lambda_2 = -A_2e^{-\delta t}$ , whose solution is given by

$$\lambda_2(t) = \frac{A_2}{A_1 + \delta}e^{-\delta t}, \tag{5.8}$$

where  $A_1 = \frac{r_2}{K_2}x_2^* + \frac{\sigma_1x_1^*}{x_2^*}$ ,  $A_2 = (p_1 - \frac{c_1}{q_1x_1^*})\sigma_2$ .

From (5.4), we get  $\frac{d\lambda_1}{dt} - A_3\lambda_1 = -A_4e^{-\delta t}$ , whose solution is given by

$$\lambda_1(t) = \frac{A_4}{A_3 + \delta}e^{-\delta t}, \tag{5.9}$$

where  $A_3 = \frac{r_1}{K_1}x_1^* + \frac{\sigma_2x_2^*}{x_1^*}$ ,  $A_4 = p_1q_1E_1 + \frac{A_2}{A_1+\delta} + (p_2 - \frac{c_2}{q_2y^*})k\alpha y^*$ .

From (5.2) and (5.9), we get the singular path

$$(p_1 - \frac{c_1}{q_1x_1^*}) = \frac{A_4}{A_3 + \delta}. \tag{5.10}$$

Using  $x_2^* = \frac{K_2}{2r_2}\{r_2 - \sigma_2 + [(r_2 - \sigma_2)^2 + \frac{4r_2\sigma_1x_1^*}{K_2}]^{1/2}\}$  and

$y^* = \frac{\delta c_2 q_1}{q_1 p_2 q_2 (\delta + d - k\alpha x_1^*) + \alpha q_2 (p_1 q_1 x_1^* - c_1)}$ ,  $A_1, A_3$  and  $A_4$  can be written as

$$\begin{aligned} A_1 &= \frac{1}{2}(r_2 - \sigma_2) + \frac{1}{2}[(r_2 - \sigma_2)^2 + 4r_2\sigma_1x_1^*/K_2]^{1/2} \\ &\quad + \sigma_1r_2x_1^*/\{r_2 - \sigma_2 + [(r_2 - \sigma_2)^2 + 4r_2\sigma_1x_1^*/K_2]^{1/2}\}, \\ A_3 &= \frac{r_1}{K_1}x_1^* + \frac{\sigma_2K_2}{2r_2x_1^*}\{r_2 - \sigma_2 + [(r_2 - \sigma_2)^2 + 4r_2\sigma_1x_1^*/K_2]^{1/2}\}, \\ A_4 &= p_1q_1E_1 + (p_1 - \frac{c_1}{q_1x_1^*})\sigma_2/\{\frac{1}{2}(r_2 - \sigma_2) + \frac{1}{2}[(r_2 - \sigma_2)^2 + 4r_2\sigma_1x_1^*/K_2]^{1/2} \\ &\quad + \sigma_1r_2x_1^*/r_2 - \sigma_2 + [(r_2 - \sigma_2)^2 + 4r_2\sigma_1x_1^*/K_2]^{1/2} + \delta\} + \delta k\alpha q_1 c_2 \\ &\quad p_2/[q_1p_2q_2(\delta + d - k\alpha x_1^*) + (p_1q_1x_1^* - c_1)\alpha q_2] - \frac{k\alpha c}{q_2}. \end{aligned}$$

Thus (5.10) can be written as

$$F(x_1^*) = (p_1 - \frac{c_1}{q_1x_1^*}) - \frac{A_4}{A_3 + \delta} = 0.$$

There exists a unique positive root  $x_1^* = x_{1\delta}$  of  $F(x_1^*) = 0$  in the interval  $0 < x_1^* < K_1$ , if the following inequalities hold:

$$F(0) < 0, F(K_1) > 0, F'(x_1^*) > 0 \text{ for } x_1^* > 0.$$

For  $x_1^* = x_{1\delta}$ , we get  $y^* = y_\delta$  from (5.7)

We then have

$$x_{2\delta} = \frac{K_2}{2r_2} \{r_2 - \sigma_2 + [(r_2 - \sigma_2)^2 + 4r_2\sigma_1 x_{1\delta}/K_2]^{1/2}\},$$

$$E_{1\delta} = \frac{1}{q_1} \left\{ r_1 \left( 1 - \frac{x_{1\delta}}{K_1} \right) - \sigma_1 + \frac{\sigma_2 x_{2\delta}}{x_{1\delta}} - \alpha y_\delta \right\},$$

and

$$E_{2\delta} = \frac{1}{q_2} (-d + k\alpha x_{1\delta}).$$

Hence once the optimal equilibrium  $(x_{1\delta}, x_{2\delta}, y_\delta)$  is determined, the optimal harvesting effort  $E_{1\delta}$  and  $E_{2\delta}$  can be determined.

From (5.3), (5.8) and (5.9), we observe that  $\lambda_i(t)$  ( $i = 1, 2, 3$ ) do not vary with time in optimal equilibrium. Hence they remain bounded as  $t \rightarrow \infty$ .

## 6 Concluding remarks

In this paper, we have analyzed a prey-predator fishery model with prey dispersal in a two-patch environment, one is assumed to be a free fishing zone and the other is a reserved zone where fishing and other extractive activities are prohibited. we have discussed the local and global stability of the system. It has been observed that, whether in the absence or in the presence of predators, the fishing populations may be sustained at an appropriate equilibrium level. The optimal harvesting policy has been discussed using Pontryagin's maximum principle.

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