Rectilinear Glass-Cut Dissections
of Rectangles to Squares

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Abstract
We study a problem of dissecting a rectangle into a minimum number of pieces which may be reassembled into a square. The dissection is made using only rectilinear glass-cuts, i.e., vertical or horizontal straight-line cuts separating pieces into two.

1 Introduction
A glass-cut of a rectangle is a cut by a straight-line segment that separates the rectangle into two pieces. A rectilinear glass-cut is a glass-cut that is either vertical or horizontal. A rectilinear glass-cut dissection of a rectangle $R$ to a rectangle $R’$ is a sequence of rectilinear glass-cuts on $R$ such that the resulting pieces can be reassembled to form the rectangle $R’$. Clearly, a sequence of $n$ rectilinear glass-cuts produces $n + 1$ pieces (see Figure 1).
We address the problem of finding rectilinear glass-cut dissections with minimum number of pieces of a rectangle $R$ into a square $S$ of the same area. Because of scaling, without loss of generality we will suppose that $S$ is a unit square and $R$ is a rectangle of width $r$ and height $\frac{1}{r}$ (i.e., a $r \times \frac{1}{r}$ rectangle). It is known that the problem has no solution for an irrational value of $r$ (see Stillwell [5]). Therefore we suppose that $r = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers with $m > n$. For the purpose of ease of analysis we will scale the problem to the equivalent version of dissection of an $m^2 \times n^2$ rectangle into an $mn \times mn$ square, and the glass-cuts are at integer positions. Our dissection algorithm in Section 2 cuts such a rectangle into a number of pieces never exceeding $m + 1$ which may be reassembled to form the square.

Dissections of unit area rectangles into a unit square are always possible if the cuts are not necessarily rectilinear. Namely, a unit area rectangle of dimensions $a \times b$ can always be dissected to a unit square using at most $\lceil a/b \rceil + 2$ pieces. This beautiful result, due to Montucla, is described in [2]. We illustrate with an example.

**Example 1.1** The rectangle of dimensions $25 \times 9$ can be dissected to a square of dimensions $15 \times 15$ using four pieces as depicted in Figure 2.

A simple rectilinear glass-cut dissection of an $\frac{m}{n} \times \frac{n}{m}$ rectangle to a unit square can be obtained as follows. Dissect the rectangle into $n$ rectangles of dimension $\frac{m}{n} \times \frac{1}{m}$. Lay these $n$ rectangles into a single rectangle of width $1/m$ and height $m$. Dissect this new rectangle into $m$ rectangles of width $1/m$ and height 1. These $m + n - 1$ rectangular pieces can now be assembled to form a unit square. We illustrate with an example.

**Example 1.2** There is a rectilinear glass-cut dissection of a $25 \times 9$ rectangle into a $15 \times 15$ square with seven rectangular pieces. The pieces are illustrated in Figure 3.

### 2 A New Dissection Algorithm

**Definition 2.1** Let $p(m, n)$ be the minimum number of pieces in dissecting the $m^2 \times n^2$ rectangle into the $mn \times mn$ square. For convenience we also define
Figure 2: Montucla’s dissection of a $25 \times 9$ rectangle into a $15 \times 15$ square using the four pieces $A, B, C, D$.

Figure 3: Dissection of a $25 \times 9$ rectangle into a $15 \times 15$ square. The dissection uses seven rectangular pieces $A, B, C, D, E, F, G$ that can be assembled to form the unit square.

$p(m, 0) = 0$.

It is easy to prove that $p(m, 1) = m$. In the sequel we assume that $n > 1$. First we prove the following lemma.

**Lemma 2.1** If $m > n$ then

$$p(m, n) \leq 2 \cdot \left\lfloor \frac{m}{n} \right\rfloor + p(n, m \mod n).$$
Proof. We start with a rectangle $R$ of dimensions $m^2 \times n^2$. The dissection is in two steps.

Step 1: In the first step we dissect the original rectangle $R$ with vertical glass-cuts (see Figure 4). Each piece is a rectangle with dimensions $(mn) \times n^2$, which gives rise to $\lfloor m^2/(mn) \rfloor = \lfloor m/n \rfloor$ such rectangles. It also leaves two “surplus” rectangles to be dissected: one, denoted by $A$, with dimensions $(mn) \times (mn - \lfloor m/n \rfloor n^2)$ (this is part of the $m^2 \times n^2$ rectangle) and one, denoted by $B$, with dimensions $(m^2 - \lfloor m/n \rfloor mn) \times n^2$ (this is part of the $mn \times mn$ square).

Step 2: In the second step we rotate the rectangle $B$ 90 degrees counterclockwise. The resulting rectangles have dimensions $mn \times rn$ and $n^2 \times rm$, where $r = m - \lfloor m/n \rfloor n$. We now perform the following dissection (see Figure 5). We dissect $A$ into $\lfloor mn/n^2 \rfloor = \lfloor m/n \rfloor$ rectangles each of dimension $n^2 \times r$. The remaining rectangle in $A$ is in fact an $rn \times rn$ square. These pieces are placed in $B$ one on top of the other. It is easy to see that the remaining rectangle has dimensions $n^2 \times r^2$. If $R'$ is the rectangle with dimensions $n^2 \times r^2$ we see that the original dissection problem of converting the rectangle $R$ into a square has been transformed into the problem of converting the rectangle $R'$ into a square at an extra cost of $2\lfloor m/n \rfloor$ rectangles. This completes the proof of Lemma 2.1.

Lemma 2.1 gives an algorithm for computing a dissection of the $m^2 \times n^2$ rectangle into an $mn \times mn$ square. Consider the sequence of integers generated...
Figure 5: Step 2 of the dissection. We rotate the rectangle $B$ and dissect. The remaining rectangle $R'$ has dimensions $n^2 \times r^2$.

by the Euclidean algorithm: $r_0 = m, r_1 = n$ and

\[
\begin{align*}
    r_0 &= q_0 r_1 + r_2 & 0 \leq r_2 < r_1 \\
    r_1 &= q_1 r_2 + r_3 & 0 \leq r_3 < r_2 \\
    \vdots & & \vdots \\
    r_i &= q_i r_{i+1} + r_{i+2} & 0 \leq r_{i+2} < r_{i+1} \\
    \vdots & & \vdots \\
    r_k &= q_k r_{k+1} & r_{k+2} = 0,
\end{align*}
\]

where $r_{k+1} = \gcd(m, n) = 1$ and $k \in O(\log n)$. If we iterate Lemma 2.1 $k$ times then we obtain a dissection consisting of

\[
p(m, n) \leq 2 \sum_{i=0}^{k-1} \left\lfloor \frac{r_i}{r_{i+1}} \right\rfloor + p(r_k, r_{k+1})
\]

rectangular pieces. In computing the last term $p(r_k, r_{k+1})$ note that by the Euclidean algorithm $r_k = q_k r_{k+1}$ and hence we have to dissect a rectangle of dimensions $(q_k r_{k+1})^2 \times r^2_{k+1}$ into a square of dimensions $q_k r_{k+1} \times q_k r_{k+1}$. It is now easy to see that this last dissection can be accomplished in exactly $q_k = r_k/r_{k+1} = r_k$ rectangular pieces each of dimensions $q_k r_{k+1} \times r_{k+1}$. To sum up we have proved the following theorem.

**Theorem 2.1** An $\frac{m}{n} \times \frac{n}{m}$ rectangle can be dissected into a unit square using only rectilinear glass-cuts, and the number of pieces does not exceed

\[
2 \sum_{i=0}^{k-1} \left\lfloor \frac{r_i}{r_{i+1}} \right\rfloor + r_k,
\]  

(1)
where $r_0 = m > r_1 = n > \cdots > r_{k+1} = \gcd(m, n) = 1$ is the sequence of remainders produced by the computation of $\gcd(m, n)$ using the Euclidean algorithm.

We illustrate the previous method with an example.

**Example 2.1** There is a five piece rectilinear glass-cut dissection of the $25 \times 9$ rectangle into a $15 \times 15$ square. The dissection is depicted in Figures 6 and 7.

Figure 6: Dissection of a $25 \times 9$ rectangle into a $15 \times 15$ square using our algorithm. There are five rectangular pieces in the dissection of dimensions $15 \times 9, 9 \times 6, 6 \times 4, 3 \times 2, 3 \times 2$, respectively.

Figure 7: The square of dimensions $15 \times 15$ resulting from the dissection by assembling the five rectangles of dimensions $15 \times 9, 9 \times 6, 6 \times 4, 3 \times 2, 3 \times 2$.

Using Formula 1, we can also prove the following upper bound on the number of pieces.
Theorem 2.2 The number of pieces to dissect an \( \frac{m}{n} \times \frac{n}{m} \) rectangle to unit square does not exceed \( m + 1 \).

**Proof.** We show that for \( n > 1 \) the number of pieces obtained by the previous algorithm never exceeds \( m + 1 \). If \( n = 2 \) then it is not hard to see that \( p(m, 2) \leq 2 \lfloor m/2 \rfloor + p(2, m \mod 2) \leq m \). Hence without loss of generality we may assume \( n \geq 3 \).

We now prove by induction on \( m \) that \( p(m, n) \leq m + 1 \). If \( n \leq m/3 \) then using the induction hypothesis and since \( n \geq 3 \),

\[
 p(m, n) \leq 2 \lfloor m/n \rfloor + p(n, m \mod n) \\
\leq 2(m/n) + n + 1 \\
\leq 2(\lfloor m/3 \rfloor) + m/3 + 1 \\
= m + 1.
\]

Hence without loss of generality we may assume \( n > m/3 \), which also implies \( \lfloor m/n \rfloor = 2 \). If also \( n \leq m - 4 \) then from the induction hypothesis we have

\[
 p(m, n) \leq 2 \lfloor m/n \rfloor + p(n, m \mod n) \\
\leq 2 \cdot 2 + n + 1 \\
\leq 4 + m - 4 + 1 \\
= m + 1.
\]

Hence, without loss of generality we may assume that \( n \geq m - 3 \). If \( m > 6 \) then \( m/n < 2 \) and \( \lfloor m/n \rfloor = 1 \). Hence, if also \( n \leq m - 2 \) then

\[
 p(m, n) \leq 2 \lfloor m/n \rfloor + p(n, m \mod n) \\
\leq 2 \cdot 1 + n + 1 \\
\leq 2 + m - 2 + 1 \\
= m + 1.
\]

This reduces to the case where \( m > 6 \) and \( n \geq m - 1 \). In the case where \( m = n + 1 \) we can prove directly that \( p(m, m - 1) \leq m + 1 \). So we only need to consider the cases \( 6 \geq m > n \geq 3 \). Since \( \gcd(m, n) = 1 \) this leaves only the cases \( (6, 5), (5, 4), (5, 3), (4, 3) \). In view of Example 2.1 we have that \( p(5, 3) \leq 5 \). This and the previous observations complete the proof of the theorem. \( \square \)

### 3 Open Problem

We do not know whether or not our algorithm gives the optimal number of pieces. In fact no non-trivial lower bound is known which is valid for all possible rectilinear (and otherwise) dissections. For additional problems see also [1].
References


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