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Free Surface Flow with Surface Tension and Gravity

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Abstract

Two-dimensional free surface flow of poured liquid from a container is computed numerically. The flow is assumed to be steady. The liquid is inviscid and incompressible. Both effects of gravity g and surface tension T are considered. The flow is a jetlike with two free surfaces, far downstream. We compute the solutions numerically via a series truncation method for different values of the Weber number and the Froude number.

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1. Introduction

Free surface flows past a bluff obstacle in the presence of gravity and surface tension are difficult to compute. Even for the simplest geometries in steady flow, analytical solutions are rare and solutions are computed numerically. Though numerical solutions are also difficult to obtain, and accurate solutions to problems with relatively simple geometrical configuration have been obtained only recently or in many cases have not been achieved yet. The difficulty of these problems arises from the nonlinear boundary condition at the unknown free surface of the flow. Here, a two dimensional free surface flow on a semi infinite plate is considered. The flow is supposed uniform, with a constant velocity *U* and constant depth *H*, far upstream.

Under the effect of gravity, the flow is poured at the edge of the plate and falls down like a jet far downstream.

As we shall see, the flow is characterized by two parameters: the Froude number F defined by

$$F^2 = \frac{U^2}{gH}, \tag{1}$$

and the Weber number α defined by

$$\alpha = \frac{\rho U^2 L}{T} \tag{2}$$

Here, *g* is the acceleration due to gravity, *T* is the surface tension and *ρ* is the density of the fluid. When the effects of surface tension and gravity *g* are neglected, the classical exact solution can be found via the hodograph transformation, due Birkhoff [1].

Different combinations and some varieties of this problem have been considered. Authors [2,8] considered the flow over weirs, or over lips or spouts of various shapes. Goh and Tuk [5] have calculated the jetlike with two free streamlines from a spout consisting of two horizontal plates. Weir flows and water falls have been computed accurately by Dias and Tuck [4], and pouring flows by Vanden Broeck and Keller [2] considering the effect of gravity and neglecting the surface tension. They computed solutions for different values of Froude number. In the present article, the pouring flow geometry is simplified to be able to consider the combined effects of gravity and surface tension. The flow is considered to be uniform far upstream on a flat horizontal semi infinite plate there is poured at the edge and falls down like a jet. Our results show that there is a solution for all Weber number $\alpha \geq 50$, and for $F \geq 2$.

The problem is formulated in section 2, the numerical procedure is described in section 3 and we conclude this work by a discussion of the results in section 4.

2. Formulation of the problem

The formulation of the problem follows closely that of Vanden Broeck and Keller [2]. Consider two-dimensional flow of inviscid, irrotational and incompressible fluid over a semi infinite horizontal wall $O'O$. The flow is supposed to start uniformly, with a constant velocity *U* and a constant depth *H*, from infinity at the point O'. At the edge *O*, the fluid is assumed to flow underneath the plate around

the point *O* and along the segment *OS*. At the point *S*, under the influence of gravity, the fluid separates from the plate and falls down to infinity like a jet.
We introduce cartesian coordinates with the *x* axis along the horizontal plate $O'O$ and the *y* axis perpendicularly upward through the edge *O* of the plate. Beyond the edge *O* of the plate, the flow becomes a jet falling down to infinity (points *C* and *J*). Gravity is acting in the negative *y* direction (see Fig. 1).
We assume that at the separation point *S* the free surface makes an angle "γ" with the plate (the angle γ is in the flow). We denote by *u* and *v* the components of the velocity in the *x* and *y* direction, respectively.
We define dimensionless variables by taking *U* as the unit velocity and *H* as the unit length. We denote the velocity potential by $\phi(x, y)$ and the stream function $\psi(x,y)$. By introducing the complex variables $z= x+ iy$, the complex potential function $f = \phi + i\psi$, is an analytical function in the flow domain. Without loss of generality, we choose $\phi=0$ at *S*. The free surface *ABC* defines a stream line on which we require ψ=1. The horizontal plate $O'O$, the segment *OS* and the free surface *SJ* define another stream line on which $\psi=0$. Hence, in the [*f*]–plane, the flow is defined by the infinite strip $0<\psi<1$. The complexe velocity is then defined by:

$$\xi(z) = u - iv = \frac{df}{dz}$$

Since the flow is potential, the conditions of the flow are such the normal derivative on the boundary vanish, (that is $v = 0$ at $y = 0$) on the free surfaces *ABC* and *SJ*, the Bernoulli's equation has to be satisfied, that is

$$\frac{1}{2}q^2 + gy + \frac{T}{\rho}K = \frac{1}{2}U^2 + g. \tag{3}$$

Here *K* is the curvature of the free surface and *T* is the surface tension and *q* is the flow speed.
In dimensionless variables, Bernoulli's equation in the entire fluid takes the form

$$\frac{1}{2}q^2 + \frac{1}{F^2}y + \frac{1}{\alpha R} = \frac{1}{2} + \frac{1}{F^2}. \tag{4}$$

where *F* is the Froude number defined by (1), α is the Weber number defined by (2). In order that the curvature be well defined we introduce the function $\tau - i\theta$ as

$$\xi = u - iv = e^{\tau - i\theta}. \tag{5}$$

In these new variables (4) becomes

$$e^{2\tau} + \frac{2}{\alpha}\exp(\tau)\left|\frac{\partial\theta}{\partial\phi}\right| + \frac{2}{F^2}(y-1) = 1 \qquad \text{on} \quad \psi = 1 \ \text{ and } \ (\psi = 0, \phi > 0). \tag{6}$$

The kinematic condition on the solid boundaries can be written as

$$\begin{cases} \operatorname{Im}\xi = 0, \theta = 0 \quad on\ \psi = 0,\ \phi < \phi(O), \\ \operatorname{Im}\xi = 0, \theta = \pi \quad on\ \psi = 0,\ \phi(O) < \phi < 0 \end{cases} \tag{7}$$

We shall seek $(\tau - i\theta)$ as an analytic function of $f = \phi + i\psi$ in the strip $0 < \psi < 1$ (see Fig.2a), satisfying the condition (7).

2. Numerical procedure

We define a new variable t by the relation

$$f = \frac{1}{\pi}\log\left(\frac{(1+t)^2}{2(1+t^2)}\right) \tag{8}$$

This transformation maps the flow domain into the upper half of the unit disc in the complex t plane (Fig. 2b) so that the points A, S and J, O are mapped into the points -1, 1 and i, t_O.

3.1. Local behavior of ξ at $t = t_O$, $t = 1$ and $t = i$.

The flow is uniform except at the points O, S, J where it is a flow around or into an angle. Hence $\xi = u - iv$ is regular except at those points and a local analysis is required.

3.1.1. Asymptotic behavior at $t = t_O$.

We have a flow around an angle of 2π, hence the flow is described by the complex potential function in the z plane as

$$f \approx a\, z^{\frac{1}{2}} \quad \text{if} \quad z \to O.$$

Using the transformation (8), this condition becomes

$$\xi = O\left(\frac{1}{t - t_O}\right) \qquad \text{as} \quad t \to t_O$$

3.1.2. Asymptotic behavior at the separation point $t = 1$.

The flow makes an angle of γ at S, thus,

$$f \approx a\,(z - z_S)^{\frac{\pi}{\gamma}} \qquad \text{if} \quad z \to z_S.$$

in terms of the variable t, we write this condition as

$$\xi = O\left((t-1)^{1-\frac{\gamma}{\pi}}\right) \qquad \text{as} \quad t \to 1.$$

3.1.3. Asymptotic behavior at $t = i$.

Downstream, as $\phi \to +\infty$, the flow becomes a jet and ξ increases like $f^{1/3}$.

Using the transformation (8), this condition becomes

$$\xi = O\left(\left(\left(-\ln c\left(t^2+1\right)\right)^{\frac{1}{3}}\right)\right) \text{ as } \quad t \to 1.$$

Where *c* is a constant chosen between 0 and 0.5. The only role of the constant *c* is to prevent the argument of the logarithm from being equal to 1 when $t=0$.
We now have determined the local behaviour of the flow near the singularities, we seek $\xi(t)$ in the form

$$\xi(t) = (t-1)^{1-\frac{\gamma}{\pi}}\left(-\ln c\left(t^2+1\right)\right)^{\frac{1}{3}}\left(\frac{1}{t-t_O}\right)\exp(\Omega(t)) \tag{9}$$

The function $\Omega(t)$ is bounded and continuous on the unit circle and analytic in the interior of the unit disk. The conditions (7) show that $\Omega(t)$ can be expanded in the form of a Taylor expansion in even powers of *t*. Hence, we write

$$e^{\tau - i\theta} = \xi(t) = (t-1)^{1-\frac{\gamma}{\pi}}\left(-\ln c\left(t^2+1\right)\right)^{\frac{1}{3}}\left(\frac{1}{t-t_O}\right)\exp\left(\sum_0^{\infty} a_n t^{2n}\right). \tag{10}$$

By choosing all the coefficients a_n to be real, the function (10) satisfies (7). The coefficients a_n have to be determined to satisfy (6).
The problem to be solved is to find ξ as an analytic function of *t* satisfying the kinematic boundary condition (7) on the real diameter $t \in [-1, 1]$ and the dynamic boundary condition (6) on the free surfaces. Points on the free surface are represented by $t = e^{i\sigma}$, where $\sigma \in \left[0, \frac{\pi}{2}\right[$ on the lower portion and $\sigma \in \left]\frac{\pi}{2}, \pi\right]$ on the upper portion.
Using (10) we rewrite (6) in the form

$$e^{2\tau}\frac{\pi\cos(\sigma)}{tg(\sigma/2)}\tau_\sigma + \frac{1}{F^2}e^{-\tau}\sin\theta + \frac{\pi}{\alpha}e^{\tau}\frac{\pi\cos(\sigma)}{tg(\sigma/2)}\left[e^{\tau}\left|\frac{\pi\cos(\sigma)}{tg(\sigma/2)}\right|\theta_\sigma\left|\frac{\pi\cos(\sigma)}{tg(\sigma/2)}\right|\right] = 0 \tag{11}$$

Here $\tau(\sigma)$ and $\theta(\sigma)$ denote the values of τ and θ on the free surfaces *ABC* and *SJ*.
We solve the problem numerically by truncating the infinite series in (11) after *N* terms. We find the *N-1* coefficients a_n and the angle γ by collocation. Thus we introduce the *N* mesh points

$$\sigma_I = \frac{\pi}{N}\left(I - \frac{1}{2}\right),\ I = 1,\ldots,N. \tag{12}$$

Using (10) we obtain $[\tau(\sigma)]_{\sigma=\sigma_I}$, $[\theta(\sigma)]_{\sigma=\sigma_I\ I}$ and $\left[\frac{\partial\tau}{\partial\sigma}\right]_{\sigma=\sigma_I}$, $\left[\frac{\partial\theta}{\partial\sigma}\right]_{\sigma=\sigma_I}$ in terms of coefficients a_n. Thus, we obtain N nonlinear algebraic equations of N unknowns $(a_n,\ n=1, ..., N-1)$ and the angle γ. The Weber number and the Froude number are parameters.

The resulting system is solved using Newton's method. The shape of the free surface is obtained by integrating numerically the relations:

$$dz = \frac{1}{\xi} df = \frac{1}{\xi}\frac{df}{dt} dt \tag{13}$$

along the free surface, yielding

$$\begin{cases} \dfrac{\partial x}{\partial\sigma} = exp(-\tau(\sigma))\dfrac{tg(\sigma/2)}{\pi\, cos(\sigma)} cos(\theta(\sigma)) \\ \dfrac{\partial y}{\partial\sigma} = exp(-\tau(\sigma))\dfrac{tg(\sigma/2)}{\pi\, cos(\sigma)} sin(\theta(\sigma)) \end{cases} \tag{14}$$

Most of the calculations were done and presented with $N=58$.

4. Results and Discussion

The numerical scheme described in section 3 is used to compute solutions for different values of the Weber number α and Froude number F, and the parameter t_O.

4.1. Flow without surface tension (waterfalls):

For $\alpha\rightarrow\infty$, $g\neq 0$ and for different values of Froude number, we compute the solutions numerically using the procedure described above. We consider $t_O=1$ when the position of the separation point S approaches the point O (Fig. 3). We write the hodograph variable ξ as

$$\xi(t) = \left(-\ln c\ \left(t^2+1\right)\right)^{\frac{1}{3}} \exp\left(\sum_0^\infty a_n t^{2n}\right) \tag{15}$$

The numerical procedure described in Sec. 3 was used to compute solutions for various values of F. Within the error $\varepsilon\approx 10^{-8}$, the coefficients of the series are decreasing and within graphical accuracy are similar to the results of Dias and Tuck [4].

Solutions were found for any value of the Froude number greater than 2. Fig. 4 shows the profile of the free surfaces for F =2. As F increases, the solution reduces to a horizontal uniform stream. The results presented in this paragraph were obtained with N =120.

4.2. Flow with surface tension effect

When the effect of surface tension is included in the free surface condition, the numerical computational shows that the algorithm converges rapidly if α increases with an error less than 10^{-6}. There exists a critical value $\alpha=\alpha^*=50$ for which we have a solution that corresponds to t_O =0.72 and N depending on t_O. For a fixed value of Weber number α and a Froude number F, the coefficients a_n of the series $\sum_{n=1}^{\infty} a_n\, t^{2(n-1)}$ were found to decrease very rapidly. For example, with α = 90, F=2 and t_O = 0.6769, we have $a_1 \sim 2.05\times10^{-1}$, $a_{10} \sim 8\times10^{-3}$, $a_{30} \sim 14.205\times10^{-5}$, $a_{50} \sim 1.036\times10^{-5}$.

Similar results obtained with different values of α and the parameters t_O, Froude number F indicate that there is a unique flow with two free surfaces, upper free surface and lower free surface for each value of $\alpha \geq \alpha^*$. Fig. 1 shows the profile of the free surface for α =90 and F=2.

Typical free surface profiles "lower free surface" are shown in Fig. 5 for different values of α and for F =2.The effect of the surface tension is more apparent on the t_O as shown in Fig. 6.

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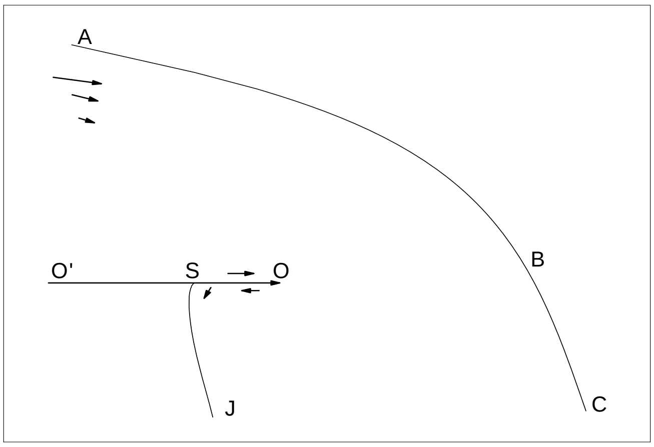

Figure 1. Liquid pouring over a wall, the flow shown here, has two free streamlines and was calculated for α =90 and F =2.

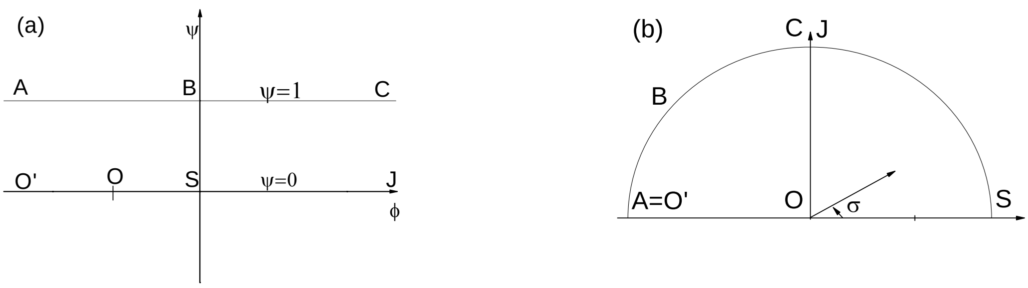

Figure 2. a-The flow configuration in the complex potential plane $f = \varphi + i\psi$.
b- The image of the strip $0 < \psi < 1$ in the t-plane is the semicircle.

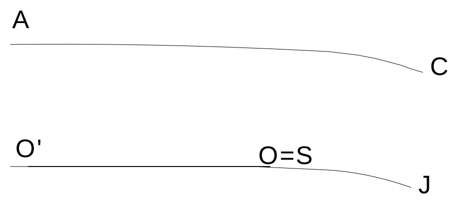

Figure 3. Computed waterfall for $F = 2$.

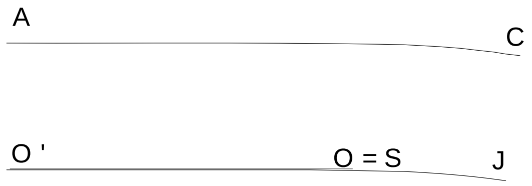

Figure 4. Computed waterfall for $F = 10$.

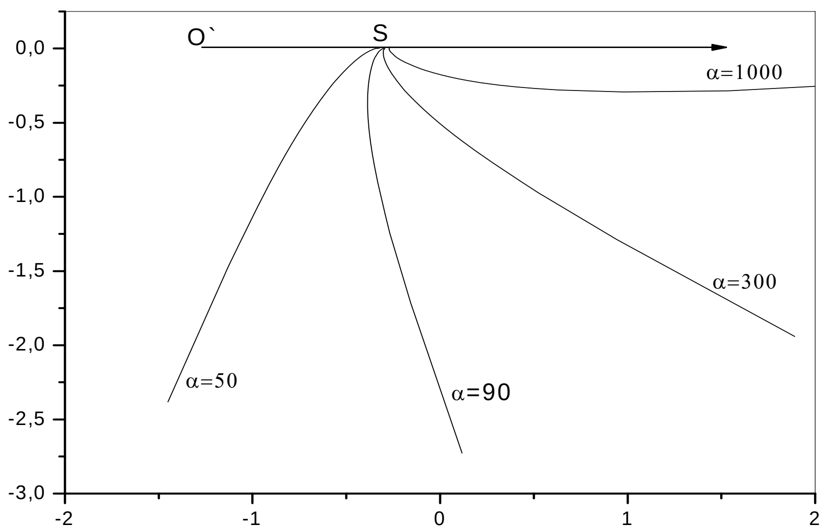

Figure 5. Free surface shapes "lower free surface" for different values of the Weber number α.

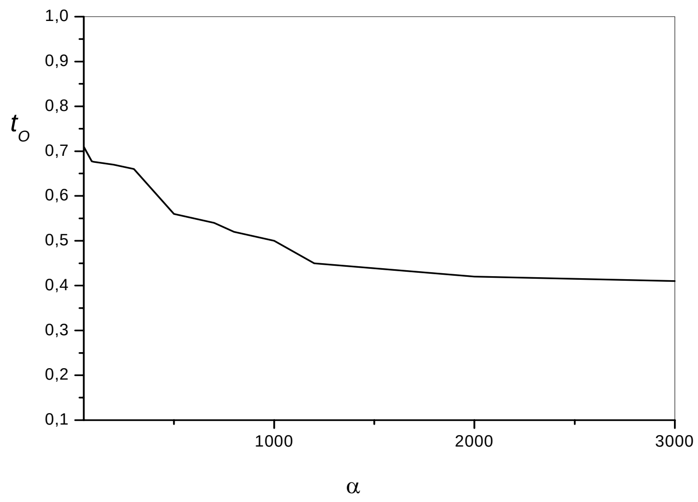

Figure 6. Plot of t_o versus Weber number α.

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