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Nonlinear and Oblique Boundary Value Problems for the Lamé Equations

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Abstract

In this paper, we generalized B. Merouani's [10], M. Dilmi's [5] and P. Grisvard's [7] works in the case when the Lame's system is perturbed with Diriclet boundary condition on one part and non linear condition one on the other part. To achieve this goal, we consider the approached problem and we solved this last problem by using Brézi's contraction method [4].

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1 Introduction and position of the problem

We consider a homogeneous, elastic and isotropic body, occupying bounded open Ω of $\mathbb{R}^n$ $(n = 2, 3)$, with sufficiently smooth boundary $\Gamma = \partial\Omega$, which is assumed to consist of two disjoint parts $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, with meas $(\Gamma_1) > 0$.

It is about solving the following problem in $H^2(\Omega)^n$:

$$\begin{cases} -Lu + \alpha u = f & \text{in} \quad (\Omega) \\ u = 0 & \text{on} \quad (\Gamma_1) \\ -\sigma(u)\nu + P(u) \in \beta(u) & \text{on} \quad (\Gamma_2) \end{cases} \tag{1.1}$$

where $f \in L^2(\Omega)^n$, P a differential operator, of the first order to coefficients lipschitziens, β a monotoneous maximal graph as $0 \in \beta(0)$ and α is a real positive on which we will bring precisions.
L designates the elasticity system :

$$\mu\Delta + (\lambda + \mu)\nabla div,$$

where λ and μ are elasticity coefficients with $\lambda > 0$ and $\lambda + \mu \geq 0$, σ designate the stress tensor, with $\sigma = (\sigma_{ij}), i, j = 1...n$. The σ_{ij} elements are given by the Hooke's law :

$$\sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda tr(\varepsilon(u))\delta_{ij},$$

where $\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ the symmetric deformation velocity tensor.

When the operator $P \equiv 0$, $\alpha = 0$ and Ω is a plane polygon, the problem (1.1) in this case is studied precisely by B. Merauani [10].
In order to solve this problem, we considered the approached problem where we replaced β by the approaching of Yosida β_ξ and we solved for this problem from a method of Brézis's contraction [4]. At first time, we considere the following problem approached :

$$\begin{cases} -Lu_\xi + \alpha u_\xi = f & \text{in} \quad (\Omega) \\ u_\xi = 0 & \text{on} \quad (\Gamma_1) \\ -\sigma(u_\xi)\nu + P(u_\xi) = \beta_\xi(u_\xi) & \text{on} \quad (\Gamma_2) \end{cases} \tag{1.2}$$

where β_ξ is the approaching of Yosida of β, assumptions on the other factors are the same for (1.1).
For (3.3.4) verified, by a priori inequality based on the integration by parts of Grisvard-Looss [8] and by going , there exist a only solution u belonging has $H^2(\Omega)^n$ of (1.1).

2 Preliminary

Lemma 2.1. *Let Ω be opened of class C^2, P a tangent operator of the first order, for all $v \in L^1(\Omega)^n$ such that $Pv \in L^1(\Omega)^n$ we get :*

$$\int_{\Gamma} Pvds \leq c_1 \int_{\Gamma} |v|\, ds \tag{2.1}$$

where $c_1$depends of the opened and of coefficients P.

Lemma 2.2. *Let Ω be open bounded subset of $\mathbb{R}^n$ with Lipschtzienne boundary Γ, if $u \in H^1(\Omega)^n$ and if β is function uniformly Lipschtzienne then $\beta(u)$ belongs to $H^1(\Gamma)^n$.*

Theorem 2.3. *Let Ω be open bounded subset of $\mathbb{R}^n$ of class $C^{0,1}$, $\forall \;\; \epsilon > 0$ there exist $c_2(\varepsilon)$ such that :*

$$\|v\|^2_{L^2(\Gamma)^n} \leq \varepsilon \|\nabla v\|^2_{L^2(\Omega)^{n\times n}} + c_2(\varepsilon) \|v\|^2_{L^2(\Omega)^n}, \forall v \in H^1(\Omega)^n. \tag{2.2}$$

Theorem 2.4. *Let Ω be open bounded subset of $\mathbb{R}^n$ of class $C^{0,1}$, $\forall \;\; \epsilon > 0$ there exist $c_3(\varepsilon)$ such that :*

$$\|\nabla \omega\|^2_{L^2(\Gamma)^{n\times n}} \leq \varepsilon \|\omega\|^2_{H^2(\Omega)^n} + c_3(\varepsilon) \|\omega\|^2_{H^1(\Omega)^n}, \forall \omega \in H^2(\Omega)^n. \tag{2.3}$$

3 Main Results

3.1 Resolution of the approach problem

Let $\beta_\xi = \frac{I-(I+\xi\beta)^{-1}}{\xi}$ be regularized Yosida of β, one will pose
$\gamma_\xi = -(I+\xi\beta)^{-1}$ the solving of β, which is a contracting application.
Let's consider the following problem :

$$\begin{cases} -Lv + \alpha v = f & (\Omega) \\ v = 0 & (\Gamma_1) \\ -\sigma(v)\nu + P(v) - \dfrac{v}{\xi} = \dfrac{\gamma_\xi(u)}{\xi} & (\Gamma_2) \end{cases} \tag{3.1.1}$$

where $u \in L^2(\Gamma_2)^n$, assumptions on the another factors are the same that the one concerning (1.2). One will remark that a fixed point of (3.1.1) (a.e. if one finds a solution v *such that* : $v/\Gamma = u$) will give a solution of (1.2).

Theorem 3.1.1. *The problem (3.1.1) admits an unique solution $v \in K$, where*

$$K = \{w \in H^1(\Omega)^n : w = 0 \text{ on } \Gamma_1\}.$$

Proof. We solve (3.1.1) by the same variational method, for all $w \in K$, one has :

$$\int_\Omega -\frac{\partial \sigma_{ij}(v)}{\partial x_j}.w_i dx + \alpha \int_\Omega v_i.w_i dx = \int_\Omega f_i.w_i dx, \forall w \in K$$

Using the Green formula, we deduces that :

$$\int_\Omega \sigma_{ij}(v).\frac{\partial w_i}{\partial x_j} dx - \int_{\Gamma_2} (\sigma_{ij}(v).\nu_j).w_i ds + \alpha \int_\Omega v_i.w_i dx = \int_\Omega f_i.w_i dx, \forall w \in K$$

Therefore

$$\int_\Omega \sigma(v).\varepsilon(w) dx - \int_{\Gamma_2} (\sigma(v).\nu).w ds + \alpha \int_\Omega v.w dx = \int_\Omega f.w dx, \forall w \in K$$

We pose in $K \times K$:

$$a(v,w) = \int_\Omega \sigma(v).\varepsilon(w) dx + \alpha \int_\Omega v.w dx - \langle Pv, w \rangle_{H^{-\frac{1}{2}}(\Gamma_2)^n \times H^{\frac{1}{2}}(\Gamma_2)^n} + \frac{1}{\xi} \int_{\Gamma_2} v.w ds$$

this form is continuous bilineair, and if it is coercive.

We take $v \in H^2(\Omega)^n \cap K$ and we does

$$a(v,v) = \int_\Omega \sigma(v).\varepsilon(v) dx + \alpha \int_\Omega v^2 dx - \int_{\Gamma_2} Pv.v ds + \frac{1}{\xi} \int_{\Gamma_2} v^2 ds.$$

Conidering the integral of border $\int_{\Gamma_2} Pv.v ds$, by lemma 2.1 we have :

$$\int_{\Gamma_2} Pv.v ds = \int_{\Gamma_2} P(\frac{v^2}{2}) ds \quad \text{and} \quad \int_{\Gamma_2} Pv.v ds \leq c_1 \int_{\Gamma_2} v^2 ds$$

then

$$\begin{aligned} a(v,v) &\geq 2\mu \int_\Omega \varepsilon_{ij}(v).\varepsilon_{ij}(v) dx + \lambda \int_\Omega \varepsilon_{kk}(v).\varepsilon_{kk}(v) dx + \alpha \int_\Omega v^2 dx - c_1 \int_{\Gamma_2} v^2 ds \\ &\geq 2\mu \sum_{i,j} \int_\Omega |\varepsilon_{ij}(v)|^2 dx + \alpha \int_\Omega v^2 dx - c_1 \int_{\Gamma_2} v^2 ds \\ &\geq 2\mu \|\varepsilon(v)\|^2_{L^2(\Omega)^{n\times n}} + \alpha \int_\Omega v^2 dx - c_1 \int_{\Gamma_2} v^2 ds \end{aligned}$$

by Korn's inequality we get :

$$a(v,v) \geq k \|v\|^2_{H^1(\Omega)^n} + \alpha \int_\Omega v^2 dx - c_1 \int_{\Gamma_2} v^2 ds$$

the lemma 2.2 gives that :

$$a(v,v) \geq k \left\| v \right\|^2_{H^1(\Omega)^n} - c_1 \varepsilon \int_\Omega \left| \nabla v \right|^2 dx + (\alpha - c_1.c_2(\varepsilon)) \int_\Omega v^2 dx$$
$$a(v,v) \geq (k - c_1 \varepsilon) \int_\Omega \left| \nabla v \right|^2 dx + (\alpha - c_1.c_2(\varepsilon)) \int_\Omega v^2 dx.$$

If we choose,

$$\varepsilon < \frac{k}{c_1} \quad \text{and} \quad \alpha \geq c_1.c_2(\varepsilon) = \alpha_1 \tag{3.1.2}$$

then

$$a(v,v) \geq cte \left\| v \right\|^2_{H^1(\Omega)^n}, \ \forall v \in H^2(\Omega)^n \cap K$$

as $H^2(\Omega)^n \cap K$ is dense in K one has :

$$a(v,v) \geq cte \left\| v \right\|^2_{H^1(\Omega)^n}, \forall v \in K,$$

this show sthat the form $a(v,w)$ is coercive.

Let's consider the following expression :

$$S(w) = \int_\Omega f.w dx - \frac{1}{\xi} \int_{\Gamma_2} \gamma_\xi(u).w ds$$

this form is linear and continuous.
By the theorem of Lax Milgram there exist a unique solution $v \in K$ as :

$$a(v,\omega) = S(w), \ \ \forall w \in K \tag{3.1.3}$$

Let's remark if (3.1.1) and (3.1.3) are equivalent, the implication (3.1.1) $\Longrightarrow$ (3.1.3) is easily verifiable, has (3.1.3) $\Longrightarrow$ (3.1.1).

If $w \in \mathcal{D}(\Omega)^n$, we have

$$\int_\Omega \sigma(v).\varepsilon(w) dx + \alpha \int_\Omega v.w dx = \int_\Omega f.w dx$$
$$\Longrightarrow -Lv + \alpha v = f \ \text{ in the sense of } \mathcal{D}'(\Omega)^n$$

if we take $\omega \in H^2(\Omega)^n \cap K$ then

$$-\int_\Omega Lv.w dx = \int_\Omega \sigma(v).\varepsilon(w) dx - \int_\Gamma (\sigma(v).v).w ds$$

by Lions- Magenes [9] and v is solution of the (3.1.3), we get

$$\int_\Omega \sigma(v).\varepsilon(w)dx + \alpha\int_\Omega v.wdx - \langle Pv, w\rangle_{H^{-\frac{1}{2}}(\Gamma_2)^n\times H^{\frac{1}{2}}(\Gamma_2)^n} + \frac{1}{\xi}\int_{\Gamma_2} v.wds$$

$$= -\int_\Omega Lv.wdx + \langle\sigma(v).\nu, w\rangle_{H^{-\frac{1}{2}}(\Gamma_2)^n\times H^{\frac{1}{2}}(\Gamma_2)^n} - \langle Pv, w\rangle_{H^{-\frac{1}{2}}(\Gamma_2)^n\times H^{\frac{1}{2}}(\Gamma_2)^n} +$$

$$\frac{1}{\xi}\int_{\Gamma_2} v.wds + \alpha\int_\Omega v.wdx = \int_\Omega f.wdx \;\; -\int_{\Gamma_2}\frac{\gamma_\xi(u)}{\xi}.wds$$

$$\Longrightarrow \sigma(v).\nu - Pv + \frac{v}{\xi} = -\frac{\gamma_\xi(u)}{\xi}$$

Then there exist $v \in H^1(\Omega)^n$ solution of (3.1.1).

Theorem 3.1.2. *Under the same assumptions, there exist $u_\xi \in K$ solution to the problem (1.2).*

Proof. Let's consider the following operator :

$$\begin{array}{cccc} T: L^2(\Gamma)^n & \longrightarrow & L^2(\Gamma)^n \\ u & \longmapsto & T(u) = v/_\Gamma \end{array}$$

where v solution of (3.1.1), is in it a strict contraction, in this paragraph we can take $\Gamma \in C^{0,1}$ only. Let v_1 and v_2 be solutions of the following problems :

$$\begin{cases} -Lv_1 + \alpha v_1 = f & (\Omega) \\ v_1 = 0 & (\Gamma_1) \\ -\sigma(v_1)\nu + Pv_1 - \dfrac{v_1}{\xi} = \gamma_\xi(u_1) & (\Gamma_2) \end{cases}$$
$$\begin{cases} -Lv_2 + \alpha v_2 = f & (\Omega) \\ v_2 = 0 & (\Gamma_1) \\ -\sigma(v_2)\nu + Pv_2 - \dfrac{v_2}{\xi} = \gamma_\xi(u_2) & (\Gamma_2) \end{cases}$$

with the same assumptions for (2.1), α verified (3.1.2). We pose $w = v_2 - v_1$ and we see that w is solution of

$$\begin{cases} -Lw + \alpha w = 0 & (\Omega) \\ w = 0 & (\Gamma_1) \\ -\sigma(w)\nu + Pw - \dfrac{w}{\xi} = \gamma_\xi(u_2) \; - \gamma_\xi(u_1) & (\Gamma_2) \end{cases} \tag{3.1.4}$$

which permits to write that

$$\int_\Omega (-Lw + \alpha w).wdx = \int_\Omega \sigma(w).\varepsilon(w)dx + \alpha\int_\Omega w^2dx - \int_{\Gamma_2}(\sigma(w)\nu).wds = 0$$

it implies;

$$\int_\Omega \sigma(w).\varepsilon(w)dx+\alpha\int_\Omega w^2dx-\int_{\Gamma_2} Pw.wds+\int_{\Gamma_2}\frac{w^2}{\xi}ds+\int_{\Gamma_2}\frac{\gamma_\xi(u_2)\ -\gamma_\xi(u_1)}{\xi}ds=0$$

by lemma 2.1 and theorem 2.3, we deduct that :

$$\begin{array}{cc} & -C_1\varepsilon\int_\Omega |\nabla w|^2\,dx - C_1C_2(\varepsilon)\int_\Omega |w^2|\,dx+\int_\Omega \sigma(w).\varepsilon(w)dx \\ + & \alpha\int_\Omega w^2dx+\int_{\Gamma_2}\frac{w^2}{\xi}ds \\ \leq & \frac{1}{\xi}\int_{\Gamma_2}(\gamma_\xi(u_1)-\gamma_\xi(u_2))ds \\ \Longrightarrow & (K-C_1\varepsilon)\int_\Omega |\nabla w|^2\,dx+(\alpha-C_1C_2(\varepsilon))\int_\Omega w^2dx+\int_{\Gamma_2}\frac{w^2}{\xi}ds \\ \leq & \frac{1}{2\xi}\int_{\Gamma_2}(\gamma_\xi(u_1)-\gamma_\xi(u_2))^2ds+\frac{1}{2\xi}\int_{\Gamma_2}\frac{w^2}{\xi}ds. \end{array}$$

Since γ_ξ is a contracting, and if ε and α verify (3.1.2) then

$$\begin{array}{cc} & \inf\left[\xi(K-C_1\varepsilon),\xi(\alpha-C_1C_2(\varepsilon))\right]\|w\|^2_{H^1(\Omega)^n}+\int_{\Gamma_2} w^2ds \\ \leq & \int_{\Gamma_2} |u_2-u_1|^2\,ds \end{array}$$

Grace to the theorems of traces, we deduct that :

$$(C_4+1)\,\|w\|^2_{L^2(\Omega)^n}\leq \|u_2-u_1\|^2_{L^2(\Omega)^n} \tag{3.1.5}$$

these inequality implies that

$$\|w\|^2_{L^2(\Omega)^n}=\|v_2-v_1\|^2_{L^2(\Omega)^n}\leq \tfrac{1}{\sqrt{C_4+1}}\,\|u_2-u_1\|^2_{L^2(\Omega)^n}$$

since $\dfrac{1}{\sqrt{C_4+1}}<1$, T is a strict contraction. Then there exist one and only one $u\in L^2(\Omega)^n$ such that $T(u)=u=v/_\Gamma$, and v is solution of

$$\left\{\begin{array}{cc} -Lv+\alpha v=f & (\Omega) \\ v=0 & (\Gamma_1) \\ -\sigma(v)\nu+P(v)-\dfrac{v}{\xi}=\dfrac{\gamma_\xi(v)}{\xi} & (\Gamma_2) \end{array}\right. \tag{3.1.6}$$

We have just shown that the existence of a v solution belonging to $H^1(\Omega)^n$ of (1.2).

Remark 3.1.3. *We have* $u\in H^{\frac12}(\Gamma)^n$ *because* $v/_\Gamma\in H^{\frac12}(\Gamma)^n$.

3.2 Regularity of the solution of (1.2)

From (3.1.1), where the u intervening in the boundary conditions is the u realizing the fixed point of T.

Theorem 3.2.1. *Let Ω be open bounded subset of $\mathbb{R}^n$ of class C^2, if $f \in L^2(\Omega)^n, g \in H^1(\Omega)^n$, P a differential operator, of the first order to coefficients lipschitziens and α* verifying *(3.1.2), then there exist an unique $v \in H^2(\Omega)^n$ solution of*

$$\begin{cases} -Lv + \alpha v = f & (\Omega) \\ v = 0 & (\Gamma_1) \\ -\sigma(v)\nu + P(v) = g & (\Gamma_2) \end{cases} \tag{3.2.1}$$

Proof. We look at that pass locally

a) For points in the inside of Ω or on Γ_1, we know that $v \in H^2(\Omega)^n$ (classic result) .

b) For points in the boundary Γ what happens?

Let σ_0 and $\Phi \in \mathcal{D}(\overline{\Omega})$ such that $\sigma_0 \in k = supp\Phi$ and $\Phi \equiv 1$ to the neighborhood of σ_0. Γ is one compact under-variety, there exists an opened of card U_i as $\sigma_0 \in U_i$ and ψ_i is an isomorphisme of class C^2 such that :

$$\psi_i : U_i \cap K \longrightarrow W_i$$

where W_i is open of $\mathbb{R}^{n-1}$.

By ψ_i we transports our problem (3.2.1) to the neighborhood of a point $X_0 = \psi_i(\sigma_0)$ belonging to$x_n = 0$. This problem becomes

$$\begin{cases} -L\omega + \alpha\omega = f' & (W_i) \\ -\sigma(\omega)\nu + P_1(\omega) = g' & (x_n = 0) \end{cases} \tag{3.2.2}$$

where $\omega = \psi_i \circ v$, $f' = \psi_i \circ f$, $g' = \psi_i \circ g$ and $P_1 = \sum_{k=1}^{n-1} a_k(X)\frac{\partial}{\partial X_k} + a_0(X)$ with $f' \in L^2(W_i)^n$, $g' \in H^{\frac{1}{2}}(x_n = 0)^n$, P_1 differential operator of the first order and α verified (3.1.2)

ω is also solution of

$$\begin{cases} -L\omega + \alpha\omega = f' & (W_i) \\ -\sigma(\omega)\nu + P_1(X_0) + (P_1(s) - P_1(X_0)) = g' & (W_i \cap x_n = 0) \end{cases} \tag{3.2.3}$$

Being only concerned by values of functions on $\psi_i(U_i \cap K \cap \Gamma)$, we have a function $\psi_r(X) = \psi\left(\frac{x}{r}\right) \in \mathcal{D}(\overline{W_i})$ which is equal 1 on $\psi_i(K)$, it permits to avoid to reason on $\partial\psi_i(U_i \cap K \cap \Gamma)$, (3.2.3) it becomes

$$\begin{cases} -L\omega + \alpha\omega = f' & (W_i) \\ -\sigma(\omega)\nu + P_1(X_0) + \psi_r(X)(P_1(s) - P_1(X_0)) = g' & (W_i \cap x_n = 0) \end{cases}$$

We considers the following operator

$$\begin{array}{rccc} \Lambda: & H^2(W_i) & \longrightarrow & L^2(W_i) \times H^{\frac{1}{2}}(W_i \cap x_n = 0) \\ & \omega & \mapsto & (-L\omega + \alpha\omega, -\sigma(\omega)\nu + P_1(X_0)(\omega) + \\ & & & \psi_r(X)\{P_1(s)(\omega) - P_1(X_0)(\omega)\}) \end{array}$$

since

$$\begin{array}{rccc} \Lambda_1: & H^2(W_i) & \longrightarrow & L^2(W_i) \times H^{\frac{1}{2}}(W_i \cap x_n = 0) \\ & \omega & \mapsto & (-L\omega + \alpha\omega, -\sigma(\omega)\nu + P_1(X_0)(\omega)) \end{array}$$

is an isomorphisme see Lions-Magenes [9] in addition

$$\|\psi_r(P_1(X) - P_1(X_0))\|_{H^{\frac{1}{2}}(W_i \cap x_n = 0)} \to 0 \text{ when } supp\psi \to 0$$

according [11], Λ will be an isomorphisme, and into Ω we have shown that $v \in H^2_{loc}(\Gamma)^n$.

Combining a) and b) using a partition of the unit, one will have shown that $v \in H^2(\Omega)^n$.

Remark 3.2.2. *The solution of (1.2) is in* $H^2(\Omega)^n$.

According to the remark 3.1.3 and the lemma 2.2, we obtain $\gamma_\xi(u) \in H^{\frac{1}{2}}(\Gamma)^n$ by follows the solution v of (3.1.1) is in $H^2(\Omega)^n$, since v is solution of (1.2), then the solution of (1.2) is in $H^2(\Omega)^n$. We will call this solution u_ξ (α verified (3.1.2)).

3.3 Existence of a solution in $H^2(\Omega)^n$ of (1.1)

Let u_ξ be solution of (1.2) we have the

Theorem 3.3.1 (*Inégalité a priori*). *There exist a constant* C *independent of* ξ *such that*

$$\|u_\xi\|^2_{H^2(\Omega)^n} \le C\,\|f\|^2_{L^2(\Omega)^n} \tag{3.3.1}$$

for u_ξ *solution of (1.2) with* α *verified (3.3.4).*

Proof. Let u_ξ be solution of (1.2) then

$$\begin{array}{rcl} (-Lu_\xi + \alpha u_\xi, -Lu_\xi + \alpha u_\xi) & = & \|Lu_\xi\|^2_{L^2(\Omega)^n} - 2\alpha(Lu_\xi, u_\xi) + \alpha^2\|u_\xi\|^2_{L^2(\Omega)^n} \\ & = & \|Lu_\xi\|^2_{L^2(\Omega)^n} + 2\alpha\int\limits_\Omega \sigma(u_\xi).\varepsilon(u_\xi)dx \\ & - & 2\alpha\int\limits_{\Gamma_2}(\sigma(u_\xi).\nu).u_\xi ds + \alpha^2\|u_\xi\|^2_{L^2(\Omega)^n} \end{array}$$

we also have

$$-2\alpha\int\limits_{\Gamma_2}(\sigma(u_\xi).\nu).u_\xi ds = -2\alpha\int\limits_{\Gamma_2} Pu_\xi.u_\xi ds + 2\alpha\int\limits_{\Gamma_2}\beta_\xi(u_\xi).uds$$

Using the fact that β_ξ is increasing Lipchitzienne with $\beta_\xi(0) = 0$, we have

$$-2\alpha \int_{\Gamma_2} (\sigma(u_\xi).\nu) .u_\xi ds \geq -2\alpha c_1 \int_{\Gamma_2} |u_\xi|^2 ds \tag{3.3.2}$$

on the other hand, using the same method of [1], [5] and of Grisvard [6, 7], we show the existence of a constant $c_5 > 0$ such that

$$c_5 \|u_\xi\|^2_{H^2(\Omega)^n} \leq \|Lu_\xi\|^2_{L^2(\Omega)^n} \tag{3.3.3}$$

We also get (3.3.3) as in Schechter [11].

Using trace theorems (see [9]), inequalities (3.3.2) and (3.3.3), the theorem 2.1 and Korn's inequality we have :

$$\begin{array}{rcl}(-Lu_\xi + \alpha u_\xi, -Lu_\xi + \alpha u_\xi) & \geq & c_5 \|u_\xi\|^2_{H^2(\Omega)^n} + 2\alpha k \|u_\xi\|^2_{H^1(\Omega)^n} \\ & + & \alpha^2 \|u_\xi\|^2_{L^2(\Omega)^n} - 2\alpha c_1 \varepsilon \|\nabla u_\xi\|^2_{L^2(\Omega)^{n\times n}} \\ & - & 2\alpha c_1 c_2(\varepsilon) \|u_\xi\|^2_{L^2(\Omega)^n}\end{array}$$

this implies

$$\begin{array}{rcl}(f, f) & \geq & c_5 \|u_\xi\|^2_{H^2(\Omega)^n} + 2\alpha (k - c_1\varepsilon) \|\nabla u_\xi\|^2_{L^2(\Omega)^{n\times n}} \\ & + & (\alpha^2 - 2\alpha c_1 c_2(\varepsilon)) \|u_\xi\|^2_{L^2(\Omega)^n}\end{array}$$

if we choose

$$\varepsilon < \frac{k}{c_1} \quad \textit{and} \quad \alpha \geq \alpha_2 = \max\{\alpha_1, \alpha_2 = 2c_1.c_2(\varepsilon)\} \tag{3.3.4}$$

the inequality (3.3.1) is examined

Corollary 3.3.2 ***(Passage to the limit).*** *For (3.3.4) verified, there exists $u \in H^2(\Omega)^n$ solution of (1.1).*

Indeed, According (3.3.1) we have,

$$\|u_\xi\|^2_{H^2(\Omega)^n} \leq c \|f\|^2_{L^2(\Omega)^n},$$

then we can find a sequence $\xi_j \longrightarrow 0$ such that :

$$\begin{array}{ccc} u_{\xi_j} \longrightarrow u & \text{weakly in} & H^2(\Omega)^n \\ u_{\xi_j} \longrightarrow u & \text{strongly in} & H^1(\Omega)^n \\ u = 0 & \text{on} & \Gamma_1 \end{array}$$

and we see that :

$$\begin{array}{rcl} \sigma(u_{\xi_j})\nu \longrightarrow \sigma(u)\nu & \text{in} & L^2(\Omega)^n \\ P(u_{\xi_j}) \longrightarrow P(u) & \text{in} & L^2(\Omega)^n \end{array}$$

then,

$\beta_{\xi_j}(u_{\xi_j}) \longrightarrow -\sigma(u)\nu + P(u).$

But since

$$\beta_\xi \subset \beta \circ (1 + \xi\beta)^{-1}$$

we deduct that

$$\begin{array}{c} \beta_{\xi_j}(u_{\xi_j}) \in \beta \circ (1 + \xi_j\beta)^{-1} \\ u_{\xi_j} - (1 + \xi_j\beta)^{-1}u_{\xi_j} = \xi_j\beta_{\xi_j}(u_{\xi_j}) \underset{\xi_j \longrightarrow 0}{\longrightarrow} 0 \\ \Longrightarrow (1 + \xi_j\beta)^{-1}u_{\xi_j} \underset{\xi_j \longrightarrow 0}{\longrightarrow} u. \end{array}$$

Combining this result with the others above, we have :

$$-\sigma(u)\nu + P(u) \in \beta(u)$$

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