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Variational Inequalities and Optimization Problems

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Abstract

In this paper we survey the relationships between scalar and vector variational inequalities (of differential type) and the underlying optimization problem. We show that the variational inequalities of Stampacchia type can be viewed as necessary optimality conditions, while the variational inequalities of Minty type can be considered as sufficient optimality conditions. Concerning this last statement, a gap is observed between the scalar and the vector case and possible fulfilments of this gap are investigated.

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Introduction

Variational inequalities, formulated, between the end of 60' and the beginning of 70' of previous century by the italian mathematician G. Stampacchia provide a very general framework for a wide range of mathematical problems among which, rather under general hypoteses, optimization ones. Moreover, they have shown to be important models in the study of equilibrium problems [17], in the engineering sciences (equilibrium problems in a traffic network) and in the economic sciences (oligopolistic market equilibrium problems) [1],[13][16]. Such problems, in fact, play a crucial role in the theory of complex systems and for this reason, recently, have been presented many variational formulations of these problems.

The objective of this paper is deepen the analysis of variational inequalities, provide a brief survey of the known results, either in the scalar and in the vector case, also with regard to their links with optimization problems and investigate possible fulfilments of the gap between the scalar and the vector case. The paper is structured as follows. In section 1 some known results about Stampacchia and Minty scalar variational inequalities are recalled, in section 2 their links with scalar optimization problems are analysed while in section 3 the vector variational inequalities and their links with vector optimization are presented. The fourth section, finally, is devoted to two different approaches to vector variational inequalities. The typical setting of the results due to Stampacchia is given by infinite dimensional spaces; recently, instead, thanks to the studies of F. Giannessi has been deepened also the analysis of the problem formulated in finite-dimensional space.

1. Stampacchia and Minty scalar variational inequalities

We introduce Stampacchia and Minty variational inequalities in finite dimensional spaces.

Definition 1.1: *Let K be a nonempty subset of R^n and let F be a function from R^n to R^n. A Stampacchia variational inequality* (for short $SVI(F,K)$) *is the problem to find an $x^* \in K$ such that:*

$$SVI\,(F,K) \qquad < F(x^*),\ y - x^* > \ \geq 0 \qquad \forall\, y \in K$$

where $< \cdot\,,\, \cdot >$ denotes the inner product in R^n.

The problem was first introduced and studied by Stampacchia in 1964; subsequently, however, many other papers have appeared dealing with theoretical aspects and with applications of this problem [1],[16].The vector x^*, solution of $SVI\,(F,K)$, is called *Stampacchia equilibrium point of the map F on K.*

We now introduce some equivalent formulations of $SVI\,(F,K)$ in the case in which the domain is an open set or a convex and closed set. If the domain K is an open set, then the solution of$SVI\,(F,K)$ is equivalent to that of a system of equations, as shows the following result:

Proposition 1.1: *Let $K \subseteq R^n$be an open set and let be $F : K \to R^n$. The vector $x^* \in K$ is a solution of $SVI\,(F,K)$ if and only if x^* solves the system of equation $F\,(x^*) = 0$.*

Proof: If $F(x^*) = 0$, then $SVI\,(F,K)$ holds with equality

$$< F(x^*), x - x^* > = 0 \qquad \forall\, x \in K$$

Conversely if x^* is a solution of $SVI(F,K)$ and K is an open set, exists $\delta > 0$ such that $\beta(x^*,\delta) \subset K$ and so, by supposition,

$$< F(x^*), x - x^* > \geq 0 \qquad \forall\, x \in \beta\ (x^*,\delta)$$

But $\forall\, x \in \beta(x^*,\delta)$also $(2x^* - x) \in \beta(x^*,\delta)$and then

$$< F(x^*),\ 2x^* - x - x^* > \ = \ < F(x^*),\ x^* - x > \geq 0 \qquad \forall\, x \in \beta\ (x^*,\delta)$$

Therefore $\qquad < F(x^*), x - x^* > = 0 \qquad \forall\, x \in \beta\ (x^*,\delta)$
and that is equivalent to condition x^* solves $F(x^*) = 0$.

Many classical economic equilibrium problems have been formulated as systems of equations, since market clearing conditions necessarily equate the total supply with the total demand. Note that systems of equations, however, preclude the introduction of inequalities, which may be needed, for example, in the case of non negativity assumptions on certain variables such as price.
If, instead, K is a convex and closed set, an equivalent geometric formulation of $SVI\,(F, K)$ can be given introducing the concepts of normal cone and generalized equation.

Definition 1.2: *If $C \subseteq R^n$ is a convex and closed set, the normal cone to at a point $x^* \in C$* is:

$$N_C\,(x^*) = \{x \in R^n : < x, y - x^* > \leq 0 \quad \forall\, y \in C\}$$

It is easily seen that the normal cone is closed and convex. Then, if K is convex, $x^* \in K$ is a solution of $SVI\,(F, K)$ if and only if:

$$-F(x^*) \in N_K(x^*),$$

that is, if and only if $0 \in F\,(x^*) + N_K\,(x^*)$ and so $SVI\,(F, K)$ is equivalent to a generalized equation.
An alternative formulation of the Stampacchia variational inequality(equivalent only under monotonicity and continuity hypotheses) has been proposed by G.J. Minty. The variational inequality which he formulated is known as *Minty variational inequality.*

Definition 1.3 *Let be K a nonempty subset of R^n and let F be a function from K to R^n. A Minty variational inequality (for short $MVI(F, K)$) is the following problem: to find an $x^* \in K$ such that:*

$$< F(y)\,,\ x^* - y > \leq 0 \qquad \forall\ y \in K$$

Any solution of $MVI(F, K)$is called a *Minty equilibrium point* of the map F over K. It is important underline that, while in $MVI(F, K)$ is considered the value assumed by F in every $y \in K$, in $SVI(F, K)$ the function F is estimated only in the given point $x^* \in K$.

A well-known Lemma, formulated by Minty in 1967, states the equivalence of the two alternative formulations (the one presented by Stampacchia and the one introduced by Minty) under continuity and monotonicity assumptions of involved function. In other words Minty's lemma gives a complete characterization of the solutions of $MVI(F, K)$ in terms of the solution of $SVI(F, K)$, when the set K is convex and the operator F is continuous and monotone.

Minty's Lemma: *Let be$F : K \to R^n$ with $K \subseteq R^n$.*

i) *If F is continuous on K and K is convex, then every $x^* \in K$ which solves $MVI\,(F, K)$ is also a solution of $SVI\,(F, K)$.*

ii) If, instead, F is monotone on the convex set K, that is if

$$< F(x) - F(y), \ x - y > \geq 0 \qquad \forall x, y \in K$$

then every $x^* \in K$ which solves $SVI\,(F, K)$ is also a solution of $MVI(F, K)$.

Remark 1: It can be observed that for the implication$MVI \Rightarrow SVI$ only the convexity of K and the continuity of F are used, while for the reverse implication only the monotonicity of F is exploited. Such hypothesis, contained in the point ii), can be weakened with the concept of pseudomonotonicity; in other words the implication $SVI \Rightarrow MVI$ is still true if F it is pseudomonotone, i.e.:

$$< F(x), x - y > \leq 0 \ \Rightarrow \ < F(y), x - y > \leq 0 \qquad \forall x, y \in K$$

One of the crucial problems in variational inequalities theory, on which is focused an important part of research, is the existence of a solution. Many classical results ensure that S, the solution set of $SVI(F, K)$, is a nonempty set. The following theorem by *Hartman and Stampacchia* requires a convex and compact set K and a continuous function F.

Theorem 1.2: *If K is a nonempty convex and compact subset of R^n* and *$F : K \to R^n$ is a continuous function, there exists an $x_0 \in K$ solution of $SVI\,(F, K)\,, i.e. S \neq \emptyset$.*

In general, the variational inequality problem $SVI\,(F, K)$can have more than one solution. If instead F is strictly monotone, then the problem $SVI\,(F, K)$ can have at most one solution.

Theorem 1.3: *If K is a nonempty convex and compact subset of $R^n$and $F : K \to R^n$ is strictly monotone on K, then the problem $SVI\,(F, K)$ has at most one solution.*

The hypotheses of continuity of F and of compactness of K do not ensure, instead, the existence of a solution for $MVI\,(F, K)$; they don't ensure, that is, that M, the solution set of $MVI\,(F, K)$, is a nonempty set.

In the case in which some solution of two variational inequalities exists, that is, in the case in which $S \neq \emptyset$ or $M \neq \emptyset$, to calculate such solutions, we can use the so called gap functions. Given:

$$H\,(x, y) = < F\,(x)\,, \ x - y > \qquad \forall\, x, y \in K$$

we consider the followings functions

$$s\,(x) = \max\{\, H\,(x, y) : \ y \in \ K\}$$

$$m\,(y) = \min\{\, H\,(x, y) : x \in K\}$$

The functions $s\,(x)$ and $m\,(y)$ are called gap functions, respectively, for SVI and MVI.

It is easy to verify that:

$$m(y) \leq 0 \leq s(x) \qquad \forall\, x, y \in K$$

The following proposition [14] characterizes the solution sets, S and M, in terms of gap functions $s(x)$ and $m(y)$.

Proposition 1.2: $S = \{a \in K : s(a) = 0\}$ *and* $M = \{a \in K : m(a) = 0\}$.

The solutions of two variational inequalities $SVI(F,K)$ and $MVI(F,K)$ give a saddle point of $H(x,y)$. It is known, indeed, the following result:

Proposition 1.3:

1.(x_0, y_0) is a saddle point for $H(x,y) \Leftrightarrow \begin{cases} x_0 \text{ solves the } MVI(F,K) \\ y_0 \text{ solves the } SVI(F,K) \end{cases}$

2.(x_0, y_0) is a saddle point for $H(x,y) \Rightarrow H(x_0, y_0) = 0$

From the previous proposition it follows that a method to find the solutions of $SVI(F,K)$ and $MVI(F,K)$ can be based on the search of the saddle points of the function $H(x,y)$. Furthermore, the knowledge of one solution of $SVI(F,K)$ can be useful to search the solutions of $MVI(F,K)$ and reverse. In fact, supposing to know a solution y_0 of $SVI(F,K)$ with $F(y_0) \neq 0$ for the point 2 the set $\{x \in K : H(x, y_0) = 0\}$ contains the solutions of $MVI(F,K)$ A similar condition can be obtained for the solutions of $SVI(F,K)$, starting from the solution x_0 of $MVI(F,K)$.

2. Relations between SVI,MVI and extremal problems

It is interesting the study of the relations between variational inequalities and optimization problems. Variational inequalities are, in fact, considered as related to a scalar optimization problem in which the objective function is a primitive of the operator involved in the inequality itself. In other words, definitions 1.1 and 1.3 can be put in relationship with problems of the type:

$$P(f,K) \qquad \min_{x \in K} f(x)$$

where $K \subseteq R^n$ and $f : R^n \to R$.

We recall that a point $x^* \in K$ is a solution of $P(f,K)$ if:

$$f(x) - f(x^*) \geq 0 \qquad \forall\, x \in K$$

while a point $x^* \in K$ is a strong or strict solution of $P(f,K)$ if:

$$f(x) - f(x^*) > 0 \qquad \forall\, x \in K \backslash \{x^*\}$$

The connections between minimum problems and the variational inequalities $SVI(F,K)$ and $MVI(F,K)$ have been widely studied in the case in which K is a convex set and the objective function $f : R^n \to R$, defined and differentiable on a open set containing K,is a primitive of F, that is $f'(x) = F(x)$.In other words, the easiest way to relate the variational inequalities of Stampacchia and Minty to minimization problems is to consider variational inequalities of differential type. It is possible, indeed, to consider the following variational inequalities:

- To find a point $x^* \in K$ such that:

$$< f'(x^*)\ , y - x^* > \geq 0 \qquad\qquad \forall\, y \in K$$

- To find a point $x^* \in K$ such that:

$$< f'(y)\ ,\ x^* - y > \leq 0 \qquad\qquad \forall\, y \in K$$

Such problems are denoted, respectively with:

$$SVI\,(f', K) \text{ and } MVI\,(f', K).$$

In the scalar case several results which state relations between solutions of a Stampacchia or Minty variational inequality of differential type and the underlying minimization problem are known. We recall, briefly, that if $x^* \in K \subseteq R^n$, with K convex and nonempty, is a solution of the primitive minimization problem:

$$P\,(f, K) \qquad\qquad \min_{x \in K} f(x)$$

for some function $f : R^n \to R$, differentiable on an open set containing the convex set K, then x^* solves $SVI\,(f', K)$, as stated by the following result:

Proposition 2.1[4],[14]:*Let K be a convex subset of $R^n$and let$f : R^n \to R$ be differentiable on an open set containing K.*

i) If $x^* \in K$ is a solution of $P\,(f, K)$, then x^* solves $SVI\,(f', K)$.

ii) If f is convex and $x^* \in K$ solves $SVI\,(f', K)$, then x^* is a solution of $P\,(f, K)$.

In other words if $F(x)$ is the gradient of the differentiable function $f : R^n \to R$ and if K is convex , then $SVI\,(f', K)$ is a necessary optimality condition for the minimization of the function f over the set K, condition which becomes also sufficient if f is convex.

If, instead, $x^* \in K$ is a solution of $MVI\,(f', K)$ then x^* is also solution of $P\,(f, K)$. More precisely, $MVI\,(f', K)$ is a sufficient optimality condition which becomes necessary if f is convex.

Proposition 2.2: *Let K be a convex subset of R^n and let $f : R^n \to R$ be differentiable on a open set containing K.*

i) If $x^* \in K$ is a solution of$MVI\,(f', K)$, then x^* is a solution of $P\,(f, K)$.

ii) If f is convex and x^* is a solution of $P(f,K)$, then x^* solves $MVI(f',K)$.

Remark 2: If, in point i) of Proposition 2.2, we suppose that x^* is a "strict solution" of $MVI(f',K)$, i.e.:

$$< f'(y), y - x^* > \ > 0 \qquad \forall y \in K \quad y \neq x^*$$

then it is possible to prove that x^* is the unique solution of $P(f,K)$.

Remark 3: In both propositions the convexity of f is necessary to prove only one of the implications. Such hypothesis can be weakened with the pseudo-convexity.

The result of the proposition 2.2 leads to some deeper relationships between the solutions of $MVI(f',K)$ and the corresponding primitive minimization problem. It seems that an equilibrium modelled through a $MVI(f',K)$ is more regular that one modelled through a $SVI(f',K)$. This conclusion leads to argue that if $MVI(f',K)$ admits a solution and the operator F admits a primitive $f(f' = F)$,then f has some regularity property. Analogously, the primitive minimization problem enjoys some regularity property (star-shapedness of the level sets of the objective function and Tykhonov well-posedness, if the solution is strict).

Definition 2.1: *A function f defined on R^n is said increasing along the rays starting from the point x^*(for short $f \in IAR(x^*)$) if the restriction of this function on the ray $R_{x^*,x} = \{x^* + \alpha x / \alpha \geq 0\}$ is increasing for each $x \in R^n$.*

Definition 2.2: *If $K$is a nonempty subset of R^n, we define the kernel of K the set:*

$$Ker\, K = \{x \in R^n : x + t(y - x) \in K, \forall y \in K, \forall t \in [0,1]\}$$

A nonempty set K is star-shaped if $Ker\, K \neq \emptyset$.

Definition 2.3: *Let be $K \subseteq R^n$a star-shaped set and $x^* \in Ker\, K$. A function $f$defined on K is said increasing on K along the rays that start from x^*(for short,$f \in IAR(K,x^*)$) if, for each $x \in K$, the restriction of f on $R_{x^*x} \cap K$ is increasing.*

The next result gives some properties of the functions which are increasing along rays. Such properties can be considered as extensions of analogous properties holding for convex functions.

Proposition 2.3 [6]: *Let be$K \subseteq R^n$a star-shaped set, $x^* \in Ker\, K$ and $f \in IAR(K,x^*)$. Then: i)x^* is a solution of $P(f,K)$.*

ii) no point $x \in K, x \neq x^*$, can be a local strict solution of $P(f,K)$.

Definition 2.4:*Let be $K \subseteq R^n, x^* \in Ker\, K$.A function f defined on an open set containing K is said to be radially lower semicontinuous in K along rays starting at x^* (for short $f \in RLSC(K,x^*)$), if for each $x \in K$, the restriction of f on the interval $R_{x^*,x} \cap K$ is lower semicontinuous.*

In [6] it is stated the following result:

Proposition 2.4: *Let be* K *a star-shaped set and* $x^* \in Ker\ K$.

i) If $x^* \in K$ *is a solution of* $MVI\,(f', K)$ *and* f *is differentiable on an open set containing* K *and* $f \in RLSC\,(K, x^*)$*, then* $f \in IAR\,(K, x^*)$.

ii) If $f \in IAR\,(K, x^*)$ *and* f *is differentiable on an open set containing* K *and* $f \in RLSC\,(K, x^*)$*, then* x^* *is a solution of* $MVI\,(f', K)$.

From propositions 2.3 and 2.4 we deduce the following corollary which extends a classical result according to which, if K is a convex set, every solution of $MVI\,(f', K)$ is also solution of $P\,(f, K)$:

Corollary 2.4: *Let be* $x^* \in Ker\ K$ *and let* f *be differentiable on an open set containing* K *and* $f \in RLSC\,(K, x^*)$. *If* x^* *solves* $MVI\,(f', K)$*, then* x^* *solves* $P\,(f, K)$.

From proposition 2.4 it follows also that the levels sets of f are star-shaped:

Proposition 2.5: *If* $f : K \subseteq R^n \to R, n > 1$*, is such that there exists a solution* x^* *of* $MVI\,(f', K)$ *and* K *is star-shaped at* x^**, then all the nonempty level sets of* f :

$$lev_c\, f = \{x \in K : f\,(x) \leq c\}$$

are star-shaped at x^*.

The existence of a solution of $MVI\,(f', K)$ can be also put it in relation to well-posedness of the respective minimum problem $P\,(f, K)$. Before, we recall that any sequence $\{x_k\} \subseteq K$ is called *a minimizing sequence for* $P\,(f, K)$ if satisfies the property:

$$f\,(x_k) \to \inf_K f\,(x) \text{ implies } x_k \to x^*$$

Definition 2.5: The *problem* $P\,(f, K)$ *is Tykhonov well-posed when:*

i) admits a unique solution x^*.

ii) every minimizing sequence for $P\,(f, K)$ *converges to* x^*.

Definition 2.6 [4]**:** *A set* $A \subseteq R^n$ *is said locally compact at* $x^* \in A$*, when there exists a closed ball centered at* x^* *with radius* δ*, said* $B\,(x^*, \delta)$*, such that* $A \cap B\ (x^*, \delta)$ *is a compact set.*

Proposition 2.6 [4]**:** *Let* $x^* \in K$ *be a solution of* $MVI\,(f', K)$. *Then, one and only one of the following alternatives holds:*

i) problem $P\,(f, K)$ *admits infinitely many solutions.*

ii) problem $P\,(f, K)$ *admits the unique solution* x^**.Moreover if* K *is locally compact at* x^**, then problem* $P\,(f, K)$ *is Tykhonov well-posed.*

From proposition 2.6 and remark 1 follows:

Corollary 2.6 [4]**:** *If* x^* *is a "strict solution" of* $MVI\,(f', K)$*, then the problem* $P\,(f, K)$ *is Tykhonov well-posed .*

A consequence of corollary 2.6 is the following result which extends to functions that belong the class $IAR\,(K, x^*)$ some classical well-posedness property of convex functions.

Proposition 2.7 [5]: *Let be K a closed subset of R^n, $x^* \in K$ and $f \in IAR\,(K, x^*)$. If $P\,(f, K)$ admits a unique solution, then $P\,(f, K)$ is Tykhonov well-posed.*

3. Vector variational inequalities and optimization problems

Many problems, for which we make use of variational inequalities, have received a scalar formulation but in reality they have a vector nature; for this reason is necessary to investigate the extension, to the vector case, of variational inequalities.

The study of vector variational inequalities was introduced, at the end of the 80', by Giannessi who has also shown that the vector extension of variational inequalities can be useful in vector optimization. After Giannessi, vector variational inequalities have been studied mainly in relation with vector optimization problems ([2], [3],[20]); so his work has been followed by numerous other works that have explored the existence conditions of the solutions and the different equivalent formulations, extending the greatest part of the scalar results to the vectorial formulation.

Giannessi has introduced, first, a vector formulation of the Stampacchia variational inequalities and later,in 1998, has proposed also a vector formulation of Minty variational inequalities. Both the variational inequalities involve a feasible region $K \subseteq R^n$, supposed convex and nonempty, a function $F : R^n \to R^{l\,x\,n}$ and a closed convex pointed cone C in R^l, with nonempty interior, which induces a partial order. We recall that given $y \in R^l$,

$$y \nleq_c 0 \Leftrightarrow y \notin -C \setminus \{0\}$$

$$y \nless_c 0 \Leftrightarrow y \notin -\,int\;C$$

It is obvious, hence , the meaning of

$$y \ngeq_c 0 \text{ and of } y \ngtr_c 0$$

The extension to the vector case of Stampacchia variational inequality leads to consider the following problem [9]:

a) To find $x^* \in K$ such that:

$$< F\,(x^*)\,,\, y - x^* >_l \nleq_c 0 \qquad\qquad \forall\, y \in K$$

where $< \cdot, \cdot >_l$ denotes a vector of l inner products of R^n.

Problem a) is called *Stampacchia vector variational inequality* (for short $SVVI(F;K)$).

Analogously the extension to the vector case of Minty variational inequality involves the following problem:

b) To find $x^* \in K$ such that:

$$< F\,(y)\,,\, x^* - y >_l \ngeq_c 0 \qquad\qquad \forall\, y \in K$$

where$< \cdot, \cdot >_l$ denotes a vector of l inner products in R^n.

Problem b) is called *Minty vector variational inequality*(for short $MVVI\,(F,K)$). For $l = 1$ a) and b) reduce to the classical Stampacchia and Minty variational inequalities.

The order introduced by C allows the distinction between strong and weak solutions of a vector variational inequality. Giannessi, in [8], has proposed the following concepts of solution.

Definition 3.1: *A vector $x^* \in K$ is said a solution of a strong Stampacchia vector variational inequality when*

$$< F\left(x^*\right) , y - x^* >_l \nleq_c 0 \qquad \forall\, y \in K$$

Definition 3.2: *A vector $x^* \in K$ is said a solution of a weak Stampacchia vector variational inequality when*

$$< F\left(x^*\right) , y - x^* >_l \nless_c 0 \qquad \forall\, y \in K$$

The set of the solutions of a strong Stampacchia vector variational inequality is denoted with $SVVI$, while that of the solutions of the weak Stampacchia vector variational inequality with $SVVI^{\omega}$. It is easy to note that

$$SVVI \subseteq SVVI^{\omega}$$

while, as it is well known, the converse is not always valid.

Definition 3.3: *A vector $x^* \in K$ is said a solution of strong $MVVI(F,K)$ when:*

$$< F\left(y\right),\ x^* - y >_l \ngeq_c 0 \qquad \forall\, y \in K$$

Definition 3.4: *A vector $x^* \in K$ is said a solution of weak $MVVI(F,K)$ when:*

$$< F\left(y\right),\ x^* - y >_l \ngtr_c 0 \qquad \forall\, y \in K$$

The set of solutions of a strong Minty vector variational inequality is denoted with $MVVI$ while that of solutions of a weak Minty vector variational inequality with $MVVI^{\omega}$. It is easy to verify that:

$$MVVI \subseteq MVVI^{\omega}$$

The previous definitions can be expressed in different form if we consider the following sets:

$$\Omega\left(x\right) = \left\{ u \in R^l : u = < F\left(x\right) , y - x >_l,\ y \in K \right\}$$

$$\Theta\ \left(x\right) = \left\{ w \in R^l : w =< F\left(y\right), y - x >_l, y \in K \right\}$$

Definition 3.5: *i) A vector* $x^* \in K$ *is a solution of a strong Stampacchia vector variational inequality when:*

$$SVVI\left(F,K\right) \qquad\qquad \Omega\left(x^*\right)\cap\left(-C\right)=\left\{0\right\}$$

ii) A vector $x^* \in K$*is a solution of a weak Stampacchia vector variational inequality when:*

$$SVVI^{\omega}\left(F,K\right) \qquad\qquad \Omega\left(x^*\right)\cap\left(-\,int\;C\right)=\phi$$

where int C *is the interior of the cone* C*.*

Definition 3.6: *i) A vector* $x^* \in K$ *is a solution of a strong Minty vector variational inequality when:*

$$MVVI\left(F,K\right) \qquad\qquad \Theta\;\left(x^*\right)\cap\left(-C\right)=\left\{0\right\}$$

ii) A vector $x^* \in K$ *is a solution of a weak Minty vector variational inequality when:*

$$MVVI^{\omega}\left(F,K\right) \qquad\qquad \Theta\;\left(x^*\right)\cap\left(-\,int\;C\right)=\phi.$$

The following result, proposed by Giannessi in 1998, extends the Minty Lemma to the vector case. Before, we recall that:

Definition 3.7: *The function* $f : K \subseteq R^n \to R^l$ *is said* C *- convex when:*

$$f\left(tx+\left(1-t\right)y\right)-\left[tf\left(x\right)+\left(1-t\right)f\left(y\right)\right]\in -C \quad \forall\, x,y\in K,\, \forall\, t\in\left[0,1\right]$$

Definition 3.8:*The function* $F : R^n \to R^{l\,x\,n}$ *is* C*- monotone on* K *if:*

$$< F\left(y\right)-F\left(x\right),\; y-x >_l \in C \qquad\qquad \forall\, x,y\in K$$

Proposition 3.1: Let f *be a function defined and differentiable on an open set containing* K*and let* f' *the Jacobian of* f*. Then* f *is* C *- convex on* K *if only if* f'*is* C *- monotone on*K*.*

Then, the extension of Minty Lemma to the vector case is:

Proposition 3.2 [8]: *If* K *is convex (and nonempty) and* F *is a function continuous and* C*- monotone, then* $x^* \in K$ *is a solution of* $SVVI^{\omega}\left(F,K\right)$ *if and only if is solution of* $MVVI^{\omega}\left(F,K\right)$.

It is known that for showing $SVVI \Rightarrow MVVI$ is necessary only the hypothesis of monotony while for the reverse is required only continuity.

Various studies have shown a relationship between the theory of the vector variational inequalities and the vector optimization. Analogously to the scalar case, when the domain K is convex and the operator F is the jacobian of a vector function $f : R^n \to R^l$, differentiable on an open set containing K, the considered vector variational inequalities $SVVI\left(F, K\right)$ and $MVVI\left(F, K\right)$ are connected with the following vector optimization problem :

$$VP\left(f, K\right) \qquad\qquad C - \min_{x \in K} f\left(x\right)$$

For the considered vector optimization problem different solution concepts can be given.

Definition 3.9: *The point $x^* \in K$ is an efficient solution of the problem $VP\left(f, K\right)$ if:* $$f\left(x\right) \not\leq_c f\left(x^*\right) \qquad \forall\, x \in K$$
that is
$$f\left(x^*\right) - f\left(x\right) \notin -C\backslash\left\{0\right\} \qquad \forall\, x \in K$$

Definition 3.10: *The point $x^* \in K$ is a weakly efficient solution of problem $VP\left(f, K\right)$ if:*

$$f\left(x\right) \not<_c f\left(x^*\right) \qquad \forall\, x \in K$$

The set of the efficient points of a function$f$with respect to the region K is denoted with E, while the set of the weakly efficient points with E^ω. Is known in literature that:

$$E \subseteq E^\omega$$

while it is not necessarily true the reverse.

The following results extend to the vector case propositions 2.1 and 2.2 of the scalar case; they connect $SVVI\left(f', K\right)$ and $MVVI\left(f', K\right)$ to a vector optimization problem [8],[14].

Proposition 3.3 [8]: *Let $K \subseteq R^n$ be closed and with nonempty interior and let $f : R^n \to R^l$ be differentiable on a open set containing K.*

i) If $x^* \in K$ is a weakly efficient solution of $VP\left(f, K\right)$, then it is also solution of $SVVI^\omega\left(f', K\right)$, that is:

$$x^* \in E^\omega \Rightarrow x^* \in \; SVVI^\omega$$

ii) If K is a convex set, f is C - convex and $x^*$is a solution of $SVVI^\omega\left(f', K\right)$, then it is a weakly efficient solution of $VP\left(f, K\right)$, that is

$$x^* \in \; SVVI^\omega \Rightarrow x^* \in \; E^\omega$$

Then, analogously to the scalar case, also for the vector case, the Stampacchia vector variational inequality represents a necessary condition for the optimization, condition that becomes sufficient under convexity assumptions.The following proposition gives, in particular, an extension to the vector case of proposition 2.2.Particularly, Giannessi has underlined some relations between a solution of a Minty vector variational inequality and an efficient or weakly efficient solution of a problem of vector optimization, under convexity and monotonicity assumptions.

Proposition 3.4 [8]: *Let C be a polyhedral cone and let K be a convex set. If f is C-convex and differentiable on an open set containing K, an then $x^* \in K$ is a weakly efficient solution of $VP(f, K)$ if and only if it is a solution of $MVVI^{\omega}(f', K)$.*

In proposition 3.4 convexity is needed also for proving that $MVVI(f', K)$ is a sufficient condition for optimality and that is it is essential either for the necessary condition or for that sufficient, while in the scalar case, convexity is needed only in the proof of the necessary part.

Recently the research has concentrated on the possibility to establish relationships of inclusion or coincidence of the solution sets of a vector variational inequality and those of the related vector optimization problem. Proposition 3.3 gives a result of equivalence between weak solutions of a Stampacchia vector variational inequality and weakly efficient solutions of the vector optimization problem. It is not possible to establish, instead, the equivalence between the strong solutions of the vector variational inequalities and the efficient points of a vector problem optimization. The known results don't allow permit to reproduce proposition 3.3; indeed a result introduced by X.Q.Yang and C.J. Goh [20], seems to exclude it. The two authors, infact, after have proving the implication in one sense, introduce the reverse implication with incompatible hypothesis.

Proposition 3.5: *Let $f : R^n \to R^l$ be $C-$convex and differentiable. Then:*

$$SVVI \subseteq E.$$

Preposition 3.6: *Let $f : K \subseteq R^n \to R^l$ be a differentiable function and strictly C-concave. Then :*

$$E \subseteq SVVI.$$

4. Two different approaches to problems $SVVI$ and $MVVI$

In the last section we have observed that while in the scalar case Minty variational inequality of differential type represents a sufficient optimality condition without additional hypothesis, in the vector case some convexity hypotheses are need. The existing extension of MVI to the vector case doesn't allow to get, without additional hypothesis, the results that are valid in the scalar case. In other terms, the relationships between Minty vector variational inequalities and the underlying vector optimization problem extend the results known in the scalar case only under convexity hypotheses. For this reason in [4], using a technique applied for the Stampacchia vector variational inequalities, it is suggested, for the Minty vector variational inequality, a concept of solution stronger than the one in definition 3.3.

Definition 4.1: *A vector $x^* \in K$ is a (weak) solution of a convexified Minty vector variational inequality when:*

$$CMVVI^{\omega}(F,K) \qquad conv\Theta(x^*) \cap (-int\ C) = \phi$$

where conv A is the convex hull of given set A .

Definition 4.1 is linked to the weak solutions of a vector optimization problem.

Remark 4:

i) Clearly, if $l = 1$ def. 4.1 is equivalent to def.1.3;

ii) If $l \geq 2$ from definition 4.1 it follows that if $x^* \in K$ solves $CMVVI^{\omega}(F,K)$ then it is a solution also of $MVVI^{\omega}(F,K)$. The converse is not always true.

Proposition 4.1 [**4**]: *Let $f : R^n \to R^l$ be differentiable on an open set containing K. If $x^* \in K$ is a solution of $CMVVI^{\omega}(f',K)$, then x^* is a weakly efficient solution of $VP(f,K)$.*

The converse of the proposition 4.1 can be stated under the hypothesis of $C-$convexity of f:

Proposition 4.2 [**4**]**:** *Let $f : R^n \to R^l$ be $C-$convex and differen-tiable. If $x^* \in K$ is a weakly efficient solution of $VP(f,K)$, then x^* is a solution of $CMVVI^{\omega}(f',K)$.*

Propositions 4.1 and 4.2 reproduce, for a vector minimum problem, the results known in the scalar case (see proposition 2.2).

Roughly speaking a Minty vector variational inequality is a sufficient condition for weak efficiency without any assumption on the differentiable objective function f, but it becomes also necessary under C - convexity assumptions on f.

Corollary 4.2: *Let C be a polyhedral cone and let $f : R^n \to R^l$ be C-convex and differentiable.*
If x^* solves $MVVI(f',K)$ then x^* solves $CMVVI^{\omega}(f',K)$.

A second approach useful to fill the gap between proposition 3.4 and the analogous scalar result, is obtained considering the function $\phi_{\hat{x}}$:

$$\phi_{\hat{x}}(x) = \max_{\xi \in C' \cap S} < \xi, f(x) - f(\hat{x}) >$$

where C' denotes the positive polar of $C$and S is the unit sphere in R^l and $\hat{x} \in K$. Function $\phi_{\hat{x}}$ is a nonlinear scalarizing function. Several scalarization techniques are known in vector optimization; the most common is linear scalarization.

The following proposition resumes some classical properties of function $\phi_{\hat{x}}$.

Proposition 4.3:

i)$\phi_{\hat{x}}$ *is directionally differentiable and*

$$\phi_{\hat{x},d}(x,d)' = \max_{\xi \in R_{\widehat{x}}(x)} \xi^T f'(x) \; d;$$

where: $R_{\hat{x}}(x) = \{ \xi \in C' \cap S : \phi_{\hat{x}}(x) =< \xi, f(x) - f(\hat{x}) > \}$

ii) $\phi'_x(x, .)$ *is sublinear and can be expressed as:*

$$\phi'_{\hat{x}}(x,d) = \max_{x \in \partial \phi_{\hat{x}}(x)} < v, d >$$

where $\partial \phi_{\hat{x}}(x) = conv\left(\xi^T f'(x)\;,\; \xi \in R_{\hat{x}}(x)\right)$ *denotes the convex hull of the set* $\xi^T f'(x)$*.*

It is possible consider the following problems:

$SVI(\phi'_{\hat{x}}, K)$: for a given $\hat{x} \in K$, find a point $x^* \in K$ such that:

$$\phi'_{\hat{x}}(x^*, y - x^*) \geq 0 \qquad \forall y \in K$$

$MVI(\phi'_{\hat{x}}, K)$: for a given $\hat{x} \in K$, find a point $x^* \in K$ such that:

$$\phi'_{\hat{x}}(y, x^* - y) \leq 0 \qquad \forall y \in K$$

The solutions of problem $SVI(\phi'_{\hat{x}}, K)$ coincide with the solutions of $SVI(f', K)$ as shows the following result :

Proposition 4.4 [5]: *Let* K*be a convex set. If* $x^* \in K$ *solves the problem* $SVI(\phi'_{\hat{x}}, K)$ *for some* $\hat{x} \in K$*, then* x^* *is a solution of* $SVVI(f', K)$*.Conversely, if* $x^* \in K$ *solves* $SVVI(f', K)$*, then* x^* *solves the problem* $SVI(\phi'_{\hat{x}}, K)$*.*

The next result, instead, turns attention to problem $MVI(\phi'_{\hat{x}}, K)$.

Proposition 4.5 [5]: *Let* $x^* \in K$ *be a solution of* $MVI(\phi'_{\hat{x}}, K)$*. Then* x^* *solves* $MVVI(f', K)$*.*

The converse of the previous result holds under convexity assumptions.

Proposition 4.6: *Let* K *be a convex set and let* f *be a C-convex function. If* $x^* \in K$ *solves* $MVVI(f', K)$*, then* x^**solves problem* $MVI(\phi'_{\hat{x}}, K)$*.*

The convexity assumption in the previous result cannot be dropped. Hence, when the convexity assumptions do not hold, $MVI(\phi'_{\hat{x}}, K)$ defines a stronger solution concept then $MVVI(f', K)$.

The next result states that the solutions of $MVI\,(\phi'_{\hat{x}}, K)$ are also solutions of $VP\,(f, K)$, filling so the gap left from the proposition 3.4.

Proposition 4.7

Let $x^* \in Ker\ K$ be a solution of $MVI\,(\phi'_{\hat{x}}, K)$ for some $\hat{x} \in K$. Then x^* is a weak solution of $VP\,(f, K)$.

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