On the Analysis of a Biological System: 
Compartment Model Approach

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Abstract

In this paper, we are interested in control of a substance which circulate among organs in leaving being. We determine, under certain conditions, the optimal control which steers such systems from an initial state to a desired one. The linear quadratic optimal control problem of such systems is analyzed using the Hilbert Uniqueness Method (HUM). To illustrate our approach, some examples and numerical simulations are given.

Keywords: Compartment models, controllability, optimal control, impulsive commands

1 Introduction

There is no exact definition in the literature of a compartment model. For Jacquez [6]: "a compartment system is a system which is made of a finite number of macroscopic sub-systems called compartments exchanging material". According to Legay [12]: "a compartment system is a set of two or more compartments communicating and among which one or more determined elements circulate. The number of compartments and circulation rules make up the system rules".

Different biological systems are modelled as compartment systems: in "cancer chemotherapy" (see, for instance [10], [14], [18]), in "pharmacokinetics" and "computer-assisted clinical pharmacokinetics" (see, for instance [4], [5]), in "blood glucose control for diabetic patients" [15].

In this work, we adopt a compartmental model to determine the optimal manoeuvre which allows a biological system to reach a predefined profile. More precisely, we consider a system made of n compartments exchanging a given
substance. It is known (see [3], [7], [17]) that if we allocate a number from 1 to \(n\) and if \(x_{ji}(t)\) is the amount of substance transferred from compartment \(i\) to compartment \(j\) and \(x_i(t)\) the amount of substance contained in compartment \(i\) at time \(t\), then the evolution in time of the vector 
\[
x(t) = (x_1(t), x_2(t), ..., x_n(t))^T,
\]
is described by the differential equation
\[
\begin{cases}
\dot{x}(t) = Ax(t) + BU(t) \\
x(0) \text{ is given}
\end{cases}
\]
where \(A = (k_{ij})_{1 \leq i,j \leq n}\) is an \(n \times n\) square matrix; \(k_{ij}\) is the proportionality ratio between \(\frac{d}{dt}x_{ji}(t)\) and \(x_i(t)\) (the model we adopt is based on the fact that \(k_{ij}\) is constant).

\(U(t) = (U^1(t), U^2(t), ..., U^n(t))^T\) where \(U^i(t)\) is the rate of external input of the substance to compartment \(i\) at time \(t\); \(U(t)\) represent then the control variable.

\(B\) is the \(n \times n\) diagonal matrix given by
\[
B = \begin{pmatrix}
b_1 & 0 & 0 & \cdots & 0 \\
0 & b_2 & 0 & \cdots & \vdots \\
0 & 0 & \ddots & \ddots & 0 \\
\vdots & \vdots & 0 & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & b_n
\end{pmatrix}
\]
where \(b_i = 1\) when compartment \(i\) receives the substance from outside and \(b_i = 0\) when it doesn’t. (for example: for cancer chemotherapy, in the 2-compartment model made of different phases of the cell-cycle; blocking agents like Cyclophosphamide act during synthesis (compartment 1) and killing agents like Taxol act during mitosis (compartment 2) (see [9]).

In order for the mathematical model to be representative of various situations, it is incorrect to assume that the control \(U(t)\) is a continuous function in time. Therefore we take into consideration both impulsive and continuous controls: For example, in the case of treating a patient, an injection can be interpreted as impulsive control and a perfusion is the continuous one. We assume that the control \(U(t) = (u(t), v(t))\) where \(v(t)\) is continuous in time, and \(u(t) = (u^1(t), u^2(t), ..., u^n(t))^T\) with \(u^i(t)\) is a sequence of impulsive controls \((u^i_k)_{k}\), where every action \(u^i_k\) has a time support \([t^i_k, t^i_k + \varepsilon^i_k]\) (for example, in case of treating a patient, \(t^i_k \in [0, T]\) means time of taking medicine meant to compartment \(i\) and \(\varepsilon^i_k\) means the necessary time for compartment \(i\) to absorb the medicine).

In other words, we consider the system described by
\[
\begin{cases}
\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 v(t) & 0 \leq t \leq T \\
x(0) \text{ is given}
\end{cases}
\]

(2)
where $A$, $B_1$, $B_2 \in \mathcal{L}(\mathbb{R}^n)$; $B_1$ and $B_2$ are diagonal matrices; $u \in \mathcal{E}$ and $v \in L^2(0, T; \mathbb{R}^n)$; $\mathcal{E}$ is the set of impulsive controls (the set $\mathcal{E}$ will be highlighted in section 2).

we investigate the optimal control $(u^*, v^*) \in \mathcal{E} \times L^2(0, T; \mathbb{R}^n)$ such that

\[
\begin{align*}
1) & \quad x_{(u^*, v^*)}^x(T) = x_d \\
2) & \quad \| (u^*, v^*) \| = \min \{ \| (u, v) \| / x_{(u, v)}^x(T) = x_d \}
\end{align*}
\]

where $x_{(u^*, v^*)}^x(T)$ is the solution of system (2) corresponding to the control $(u^*, v^*)$ at time $T$ and $\| . \|$ is the usual norm of $L^2(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^n)$. To illustrate this work, some examples are given.

The section 3 of this paper is devoted to the study of the problem of linear quadratic optimal control for such systems, i.e, we investigate the control $u = \sum_{i=0}^{N-1} u_i \chi_{[t_i, t_{i+1}]} \in \mathcal{E}_N$ which minimizes the cost functional

\[
J(u) = < x(T), G x(T) > + \sum_{i=0}^{N-1} < x(t_i), M x(t_i) > + \int_0^T < u(t), R u(t) > dt
\]

where $x(t_i)$ is the solution of system (1) corresponding to the control $u$ at time $t_i$, $G$, $M$ and $R$ are self-adjoint and non-negative with $< R u, u > \geq \alpha \| u \|^2$ for some $\alpha > 0$ and all $u \in \mathcal{E}_N$.

The technic used for this is similar to HUM method (see [1], [2], [8], [13]), we adapt the technic of [16] to our system, the optimal control is given by inversion of some isomorphism in an adequate Hilbert space.

## 2 Mathematical Modelling of the problem

Let’s consider the system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 v(t) \quad 0 \leq t \leq T \\
x(0) & \text{ is given}
\end{align*}
\]

(3)

where $A = (k_{ij})_{1 \leq i,j \leq n}$ is an $n \times n$ square matrix; $B_1$ and $B_2$ are $n \times n$ diagonal matrix.

The controllability problem as it was defined in the previous section, may be, mathematically interpreted by the determination of a control $(u^*, v^*)$ which allows to steer the system from the initial state $x_0$ to a desired one $x_d$ at time $T$ and with minimal-costs. In others words, we investigate $u^*$ and $v^*$ such that
The solution of the system (3) is

\[
\begin{cases}
  \text{i) } u^* \in \mathcal{E} = \{ u \in L^2(0, T; \mathbb{R}^n) / u = \sum_{k=0}^{N-1} u_k \chi_{[t_k, t_k+\varepsilon_k[}, u_k \in \mathbb{R}^n \} ; \\ u^* \in L^2(0, T; \mathbb{R}^n) \\
  \\
  \text{ii) } x^{x_0}(T) = x_d \\
  \text{iii) } \| (u^*, v^*) \| = \min \{ \| (u, v) \| / (u, v) \in \mathcal{E} \times L^2(0, T; \mathbb{R}^n) \text{ and } x^{x_0}_{(u, v)}(T) = x_d \}
\end{cases}
\]

where \( x^{x_0}_{(u, v)}(T) \) is the solution of the system (2), corresponding to control \( (u, v) \) at time \( T \) and initialization \( x_0 \) and \( \| \cdot \| \) is the usual norm on \( L^2(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^n) \).

**Remark 2.1**

i) The choice of \( \mathcal{E} \) as a control space, suppose that the absorption ratios \( (\varepsilon_k)_{0\leq k \leq N-1} \) are the same for every compartment.

ii) For every \( u = \sum_{k=0}^{N-1} u_k \chi_{[t_k, t_k+\varepsilon_k[}, v = \sum_{k=0}^{N-1} v_k \chi_{[t_k, t_k+\varepsilon_k[} \in \mathcal{E} \) \)

\[
< u, v >_{L^2(0,T;\mathbb{R}^n)} = \sum_{k=0}^{N-1} \varepsilon_k < u_k, v_k >_{\mathbb{R}^n}
\]

iii) \( \| u \|_{L^2(0,T;\mathbb{R}^n)}^2 = \sum_{k=0}^{N-1} \varepsilon_k \| u_k \|^2 \)

iv) \( \mathcal{E} \) endowed with \( L^2(0, T; \mathbb{R}^n) \) topology is a Hilbert space.

The solution of the system (3) is

\[
x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}B_1u(s)ds + \int_0^t e^{(t-s)A}B_2v(s)ds, \quad t \in [0, T].
\]

We deduce that for \( u = \sum_{k=0}^{N-1} u_k \chi_{[t_k, t_k+\varepsilon_k[} \); we have

\[
x(T) = e^{TA}x_0 + \sum_{k=0}^{N-1} \int_{t_k}^{t_k+\varepsilon_k} e^{(T-s)A}B_1u_kds + \int_0^T e^{(T-s)A}B_2v(s)ds
\]

where \( \mathcal{H} \) is an operator defined from \( \mathcal{E} \times L^2(0, T; \mathbb{R}^n) \) to \( \mathbb{R}^n \) by

\[
\mathcal{H}(u, v) = \sum_{k=0}^{N-1} \int_{t_k}^{t_k+\varepsilon_k} e^{(T-s)A}B_1u_kds + \int_0^T e^{(T-s)A}B_2v(s)ds.
\]
Lemma 2.1 \( \mathcal{H} \) is a linear continuous operator, and if we consider the inner product defined on \( L^2(0,T;\mathbb{R}^n) \times L^2(0,T;\mathbb{R}^n) \); \( \mathcal{H}^* \) the adjoint operator of \( \mathcal{H} \) is defined, for all \( x \in \mathbb{R}^n \), by

\[
\mathcal{H}^* x = (\mathcal{H}_1^* x, \mathcal{H}_2^* x)
\]

where

\[
(\mathcal{H}_1^* x)(\theta) = \left\{ \begin{array}{ll}
\frac{1}{\varepsilon_k} (\int_{t_k}^{t_k + \varepsilon_k} B_1 e((T-s)A^T) u_k ds), & \theta \in [t_k, t_k + \varepsilon_k[ , \ k \in \{0,1,\ldots,N-1\} \\
0 & \text{elsewhere}
\end{array} \right.
\]

\[
(\mathcal{H}_2^* x)(\theta) = B_2 e((T-\theta)A^T) x , \ \theta \in [0,T]
\]

Proof

The linearity is obvious, moreover

\[
\|\mathcal{H}(u,v)\|_{\mathbb{R}^n} \leq \sum_{k=0}^{N-1} \int_{t_k}^{t_k + \varepsilon_k} ||e((T-s)A^T) B_1|| u_k ||ds + (\int_0^T ||e((T-s)A^T) B_2||^2 ds)^\frac{1}{2} ||f_0^T ||v(s)||^2 ds)^\frac{1}{2}
\]

\[
\leq \sum_{k=0}^{N-1} \int_0^T ||e((T-s)A^T) B_1|| u_k ||ds + (\int_0^T ||e((T-s)A^T) B_2||^2 ds)^\frac{1}{2} ||v||_{L^2(0,T;\mathbb{R}^n)}
\]

\[
\leq (\int_0^T ||e((T-s)A^T) B_1|| ds) \sum_{k=0}^{N-1} ||u_k|| + (\int_0^T ||e((T-s)A^T) B_2||^2 ds)^\frac{1}{2} ||v||_{L^2(0,T;\mathbb{R}^n)}
\]

\[
\leq (\int_0^T ||e((T-s)A^T) B_1|| ds) \sqrt{N} \left( \sum_{k=0}^{N-1} \alpha \varepsilon_k ||u_k||^2 \right)^\frac{1}{2} + (\int_0^T ||e((T-s)A^T) B_2||^2 ds)^\frac{1}{2} ||v||
\]

where \( \alpha \) is a constant verifying \( 1 \leq \alpha \varepsilon_k \) for every \( k \) (for example \( \alpha = \sup_{0 \leq k \leq N-1} \varepsilon_k \)).

Hence

\[
\|\mathcal{H}(u,v)\| \leq (\int_0^T ||e((T-s)A^T) B_1|| ds) \sqrt{N} \alpha ||u|| + (\int_0^T ||e((T-s)A^T) B_2||^2 ds)^\frac{1}{2} ||v||
\]

That establishes the continuity of \( \mathcal{H} \).

On the other hand, since \( B_i^T = B_i \) for \( i = 1, 2 \); we have
\begin{equation}
< \mathcal{H}(u, v), x > = \sum_{k=0}^{N-1} \int_{t_k}^{t_k+\varepsilon_k} e^{(T-s)A} B_1 u_k ds , x > + \int_0^T e^{(T-s)A} B_2 v(s) ds , x >
\end{equation}

Thus, the adjoint of \( \mathcal{H} \) is

\[ \mathcal{H}^* x = (\mathcal{H}_1^* x, \mathcal{H}_2^* x) \]

where

\[ (\mathcal{H}_1^* x)(\theta) = \left\{ \begin{array}{ll}
\frac{1}{\varepsilon_k} \int_{t_k}^{t_k+\varepsilon_k} B_1 e^{(T-s)A} u_k ds x, & \theta \in [t_k, t_k+\varepsilon_k], k \in \{0, 1, \ldots, N-1\} \\
0 & \text{elsewhere}
\end{array} \right. \]

\[ (\mathcal{H}_2^* x)(\theta) = B_2 e^{(T-\theta)A^T} x , \quad \theta \in [0, T] \]

\[ \text{Definition 2.1} \quad \text{The system (3) is said to be controllable on } [0,T] \text{ if the operator } \mathcal{H} \text{ is surjective.} \]

\[ \text{Proposition 2.1} \quad \mathcal{H} \text{ is surjective if and only if for all } x_0, x_d \in \mathbb{R}^n \text{ it exists a control } u \in \mathcal{E} \text{ and a control } v \in L^2(0,T;\mathbb{R}^n) \text{ such that } x_{(u,v)}^{x_0}(T) = x_d , \text{ where } x_{(u,v)}^{x_0}(T) \text{ is the solution of the system (3) corresponding to control } (u, v). \]

\[ \text{Proof} \]

If \( \mathcal{H} \) is surjective then \( \exists \ u \in \mathcal{E} , \ v \in L^2(0,T;\mathbb{R}^n) \) such that \( \mathcal{H}(u,v) = x_d - e^{TA}x_0 \) hence \( x_d = x_{(u,v)}^{x_0}(T) \).

Conversely, if for \( x_0 = 0 \) and any \( x_d \in \mathbb{R}^n \), there exists \( u \in \mathcal{E} , v \in L^2(0,T;\mathbb{R}^n) \) such that \( x_{(u,v)}^{x_0}(T) = x_d \), then \( \mathcal{H}(u,v) = x_d \) that establishes that \( \mathcal{H} \) is surjective.
Remark 2.2 It follows from the previous proposition that

The system (3) is controllable on \([0, T] \iff \text{Im} \mathcal{H} = \mathbb{R}^n \iff \text{Ker} \mathcal{H}^* = \{0\}.

Proposition 2.2 The system (3) is controllable on \([0, T]\) if and only if

\[
\text{Ker} \begin{pmatrix}
\int_{t_0 + \epsilon_0}^{t_0 + \epsilon_0} B_1 e^{(T-s)A^T} ds \\
\int_{t_1 + \epsilon_1}^{t_1 + \epsilon_1} B_1 e^{(T-s)A^T} ds \\
\vdots \\
\int_{t_{N-1} + \epsilon_{N-1}}^{t_{N-1} + \epsilon_{N-1}} B_1 e^{(T-s)A^T} ds \\
B_2 \\
B_2 A^T \\
\vdots \\
B_2 (A^T)^{n-1}
\end{pmatrix}
= \{0\}
\]

Proof
It is an immediate consequence of the definition of \(\mathcal{H}^*\) and the remark 2.2.

\[\Box\]

In order to lighten the matrix condition in the previous proposition, we give the following necessary condition

Proposition 2.3

\[
\text{Ker} \begin{pmatrix}
\int_{t_0 + \epsilon_0}^{t_0 + \epsilon_0} B_1 e^{(T-s)A^T} ds \\
\int_{t_1 + \epsilon_1}^{t_1 + \epsilon_1} B_1 e^{(T-s)A^T} ds \\
\vdots \\
\int_{t_{N-1} + \epsilon_{N-1}}^{t_{N-1} + \epsilon_{N-1}} B_1 e^{(T-s)A^T} ds \\
B_2 \\
B_2 A^T \\
\vdots \\
B_2 (A^T)^{n-1}
\end{pmatrix}
= \{0\}
\]

\[\Rightarrow \text{Ker} \begin{pmatrix}
B_1 \\
B_1 A^T \\
\vdots \\
B_1 (A^T)^{n-1}
\end{pmatrix}
= \{0\}
\]

\[\Leftrightarrow \text{rank } \begin{bmatrix}
B_1 | B_1 A^T | \ldots | B_1 (A^T)^{n-1} | B_2 | B_2 A^T | \ldots | B_2 (A^T)^{n-1}
\end{bmatrix} = n\]
Proof
If we suppose that

\[ x \in \text{Ker} \begin{pmatrix} B_1 \\ B_1A^T \\ \vdots \\ B_1(A^T)^{n-1} \\ B_2 \\ B_2A^T \\ \vdots \\ B_2(A^T)^{n-1} \end{pmatrix} \]

then

\[ B_1x = B_1A^Tx = \ldots = B_1(A^T)^{n-1}x = 0 \]

by the Cayley-Hamilton theorem, there exist reals \( a_0, a_1, \ldots, a_{n-1} \) such that

\[ A^n = a_0I + a_1A + \ldots + a_{n-1}A^{n-1} \]

we deduce by immediate recurrence that, for any integer \( k \),

\[ B_1(A^T)^kx = 0 \]

thus for any \( t \in [0,T] \),

\[ B_1e^{tA^T}x = 0 \]

consequently, for any \( k = 0,1,\ldots,N-1 \)

\[ \left( \int_{t_k}^{t_{k+1}} B_1e^{(T-s)A^T} ds \right) x = 0 \]

therefore

\[ x \in \text{Ker} \begin{pmatrix} \int_{t_0}^{t_0+\epsilon_0} B_1e^{(T-s)A^T} ds \\ \int_{t_1}^{t_1+\epsilon_1} B_1e^{(T-s)A^T} ds \\ \vdots \\ \int_{t_{N-1}}^{t_{N-1}+\epsilon_{N-1}} B_1e^{(T-s)A^T} ds \\ B_2 \\ B_2A^T \\ \vdots \\ B_2(A^T)^{n-1} \end{pmatrix} \]
Remark 2.3 The reciprocal of proposition 2.3 is false. Indeed for $A^T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; we have rank$[B_1|B_1A^T|B_2|B_2A^T] = 2$ and

$$Ker \begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_{N-1} \\ B_2 \\ B_2A^T \end{pmatrix} \neq \{0\} \text{ where } M_k = \int_{t_k}^{t_{k+\epsilon_k}} B_1 e^{(T-s)A^T} \, ds$$

Indeed, we have

$$A^T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$M_k = \begin{pmatrix} [-e^{(T-s)A^T}l_{t_k}^{t_{k+\epsilon_k}}] & [-e^{(T-s)A^T}l_{t_k}^{t_{k+\epsilon_k}}] \\ 0 & 0 \end{pmatrix}$$

then

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \in Ker \begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_{N-1} \\ B_2 \\ B_2A^T \end{pmatrix}.$$

Consider $\Lambda$ the $n \times n$ matrix defined by

$$\Lambda = \mathcal{H}\mathcal{H}^*$$

we have then

$$\Lambda x = \mathcal{H}(\mathcal{H}^*x) = \mathcal{H}(\mathcal{H}^*_1 x, \mathcal{H}^*_2 x)$$

$$= \sum_{k=0}^{N-1} \left( \int_{t_k}^{t_{k+\epsilon_k}} e^{(T-s)A} B_1 (\frac{1}{\epsilon_k} \int_{t_k}^{t_{k+\epsilon_k}} B_1 e^{(T-s)A^T} \, ds ) \, ds \right) x ds$$

$$+ \int_0^T e^{(T-s)A} B_2 e^{(T-s)A^T} x \, ds$$

$$= \sum_{k=0}^{N-1} \frac{1}{\epsilon_k} \left( \int_{t_k}^{t_{k+\epsilon_k}} e^{(T-s)A} B_1 ds \right) \left( \int_{t_k}^{t_{k+\epsilon_k}} B e^{(T-s)A^T} \, ds \right) x$$

$$+ \left( \int_0^T e^{(T-s)A} B_2 e^{(T-s)A^T} \, ds \right) x$$

It’s easy to show that $\Lambda$ is a bounded self-adjoint operator.
Lemma 2.2 the system (3) is controllable on [0, T] if and only if $\text{Ker} \Lambda = \{0\}$.

Proof
It is sufficient to show the equality

$$\text{Ker} \Lambda = \text{Ker} \left( \begin{array}{c} \int_{t_0}^{t_0 + \varepsilon_0} B_1 e^{(T-s)A^T} ds \\ \int_{t_1}^{t_1 + \varepsilon_1} B_1 e^{(T-s)A^T} ds \\ \vdots \\ \int_{t_{N-1} + \varepsilon_{N-1}}^{t_{N-1}} B_1 e^{(T-s)A^T} ds \\ B_2 \\ B_2 A^T \\ \vdots \\ B_2 (A^T)^{n-1} \end{array} \right).$$

Let’s consider

$$x \in \text{Ker} \left( \begin{array}{c} \int_{t_0}^{t_0 + \varepsilon_0} B_1 e^{(T-s)A^T} ds \\ \int_{t_1}^{t_1 + \varepsilon_1} B_1 e^{(T-s)A^T} ds \\ \vdots \\ \int_{t_{N-1} + \varepsilon_{N-1}}^{t_{N-1}} B_1 e^{(T-s)A^T} ds \\ B_2 \\ B_2 A^T \\ \vdots \\ B_2 (A^T)^{n-1} \end{array} \right)$$

that implies that

$$\int_{t_k}^{t_k + \varepsilon_k} B_1 e^{(T-s)A^T} x ds = 0 , \ \forall \ k \in \{0, 1, ... N - 1\}$$

and

$$B_2 x = B_2 A^T x = ... = B_2 (A^T)^{n-1} x = 0$$

thus

$$\left( \int_{t_k}^{t_k + \varepsilon_k} e^{(T-s)A} B_1 ds \right) \left( \int_{t_k}^{t_k + \varepsilon_k} B_1 e^{(T-s)A^T} x ds \right) = 0 , \ \forall \ k \in \{0, 1, ... N - 1\}$$

and

$$\int_0^T e^{(T-s)A} B_2 (B_2 e^{(T-s)A^T} x) ds = 0$$

hence
\[ \Lambda x = 0 \]

which means
\[ x \in \text{Ker} \Lambda. \]

Conversely, if we suppose that \( \Lambda x = 0 \) then \( \langle \Lambda x, x \rangle = 0 \) and \( \langle \mathcal{H}^* x, x \rangle = 0 \), which implies that \( \| \mathcal{H}^* x \| = 0 \). Consequently,
\[
\begin{pmatrix}
\int_{t_0}^{t_1 + \varepsilon_0} B_1 e^{(T-s)A^T} ds \\
\int_{t_1}^{t_1 + \varepsilon_1} B_1 e^{(T-s)A^T} ds \\
\vdots \\
\int_{t_{N-1} + \varepsilon_{N-1}}^{t_{N-1}} B_1 e^{(T-s)A^T} ds \\
B_2 \\
B_2 A^T \\
\vdots \\
B_2 (A^T)^{n-1}
\end{pmatrix}
\]

\[ = \left\{ 0 \right\}. \]

Now, we establish the fundamental result of this section.

**Theorem 2.1** If we suppose that the system (3) is controllable on \([0,T]\), i.e.,
\[
\text{Ker} \begin{pmatrix}
\int_{t_0}^{t_1 + \varepsilon_0} B_1 e^{(T-s)A^T} ds \\
\int_{t_1}^{t_1 + \varepsilon_1} B_1 e^{(T-s)A^T} ds \\
\vdots \\
\int_{t_{N-1} + \varepsilon_{N-1}}^{t_{N-1}} B_1 e^{(T-s)A^T} ds \\
B_2 \\
B_2 A^T \\
\vdots \\
B_2 (A^T)^{n-1}
\end{pmatrix} = \{0\}
\]

then the unique control solution \((u^*, v^*)\) of the problem (P) is given by
\[
(u^*(\theta), v^*(\theta)) = \left( \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k} \int_{t_k}^{t_{k+\varepsilon_k}} B_1 e^{(T-s)A^T} ds \chi_{[t_k, t_{k+\varepsilon_k}]}(\theta) f, B_2 e^{(T-\theta)A^T} f \right)
\]

where \(f\) is the unique solution of the linear system
\[ \Lambda f = x_d - e^{TA}x_0. \]
Proof
If the system (3) is controllable, by lemma 2.2, we have
\[ \text{Ker}\Lambda = \{0\} \]
\[ \Lambda \in \mathcal{L}(\mathbb{R}^n), \] then \( \Lambda \) is bijective, hence there exists \( f \in \mathbb{R}^n \), unique solution of the equation
\[ \Lambda f = x_d - e^{TA}x_0. \]
Observe that the control \((u^*, v^*)\) may be written as follows
\[ (u^*, v^*) = \mathcal{H}^* f \in \mathcal{E} \times L^2(0, T; \mathbb{R}^n) \]
we have
\[ x_{(u^*, v^*)}^x(T) = e^{TA}x_0 + \mathcal{H}(u^*, v^*) \]
\[ = e^{TA}x_0 + \mathcal{H}\mathcal{H}^* f \]
\[ = e^{TA}x_0 + \Lambda f \]
\[ = x_d \]
then the control \((u^*, v^*)\) allows to steer the system from the initial state \(x_0\) to the desired one \(x_d\) at instant \(T\).
Let \((u, v) \in \mathcal{E} \times L^2(0, T; \mathbb{R}^n)\) another control such that \(x_{(u, v)}^x(T) = x_d\), then
\[ \mathcal{H}(u^*, v^*) = \mathcal{H}(u, v) \Rightarrow < \mathcal{H}((u^*, v^*) - (u, v)), f > = 0 \]
\[ \Rightarrow < (u^*, v^*) - (u, v), \mathcal{H} f > = 0 \]
\[ \Rightarrow < (u^*, v^*) - (u, v), (u^*, v^*) > = 0 \]
\[ \Rightarrow \|(u^*, v^*)\|^2 = < (u, v), (u^*, v^*) > \leq \|(u, v)\|\|(u^*, v^*)\|. \]
Consequently
\[ \|(u^*, v^*)\| \leq \|(u, v)\| \]
that establishes the optimality of \((u^*, v^*)\).

\[ \blacksquare \]

Example : Let us consider a 2-compartment model, then \(A\) is a \(2 \times 2\) square matrix which supposed diagonalisable, so we can write
\[ A^T = PD P^{-1} \]
where
\[ P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad P^{-1} = \frac{1}{\text{det}P} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}; \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \]
\(\lambda_1, \lambda_2\) are the eigenvalues of \(A\). Therefore
\[ e^{(T-s)A^T} = \frac{1}{\text{det}P} \begin{pmatrix} ade^{\lambda_1(T-s)} - bce^{\lambda_2(T-s)} & -abe^{\lambda_1(T-s)} + abe^{\lambda_2(T-s)} \\ cde^{\lambda_1(T-s)} - cde^{\lambda_2(T-s)} & -bce^{\lambda_1(T-s)} + ade^{\lambda_2(T-s)} \end{pmatrix} \]
If we take 
\[ B_1 = B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]
which means that the impulsive control (for example injection) and the continuous control (for example perfusion) act only on compartment 1. Let we call 
\[ M_k = \begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_{N-1} \\ B_2 \\ B_2A^T \end{pmatrix} \]
where 
\[ M_k = \int_{t_k}^{t_k+\varepsilon_k} B_1 e^{(T-s)A^T} ds \]
then 
\[ M_k = \frac{1}{\det P} \begin{pmatrix} y_k & z_k \\ 0 & 0 \end{pmatrix} \]
where 
\[ y_k = ad\alpha_k e^{\lambda_1(T-t_k)} - bc\beta_k e^{\lambda_2(T-t_k)} \]
\[ z_k = ab \left[ -\alpha_k e^{\lambda_1(T-t_k)} + \beta_k e^{\lambda_2(T-t_k)} \right] \]
with 
\[ \alpha_k = \begin{cases} \frac{1}{\lambda_1} \left( 1 - e^{-\lambda_1 \varepsilon_k} \right) & \text{if } \lambda_1 \neq 0 \\ \varepsilon_k & \text{if } \lambda_1 = 0 \end{cases}, \quad \beta_k = \begin{cases} \frac{1}{\lambda_2} \left( 1 - e^{-\lambda_2 \varepsilon_k} \right) & \text{if } \lambda_2 \neq 0 \\ \varepsilon_k & \text{if } \lambda_2 = 0 \end{cases} \]

Using proposition 2.2, the system is controllable if and only if \( \ker M = \{0\} \).

For \( N = 1 \) and \( B_2 = 0 \); \( M = M_0 \) then \( \det M = 0 \), i.e., \( \ker M \neq \{0\} \); the system is not controllable, it means that if the control acts on only one compartment, taking medicine (for example) only one time is not sufficient to lead the system to the desired state.

For \( N = 2 \), \( t_0 = 0 \), \( t_1 = \frac{T}{2} \)

\[ \det \begin{pmatrix} y_0 & z_0 \\ y_1 & z_1 \end{pmatrix} = ab(\det P)e^{\frac{\lambda_1 + \lambda_2}{2}} \left( -\alpha_1 \beta_0 e^{\frac{\lambda_2 T}{2}} + \alpha_0 \beta_1 e^{\frac{\lambda_1 T}{2}} \right) \]

if we suppose \( ab \neq 0; \varepsilon_0 = \varepsilon_1 = \varepsilon \) and \( \lambda_1 \neq \lambda_2 \) (i.e. \( \alpha_0 = \alpha_1, \beta_0 = \beta_1 \)) then \( \det \begin{pmatrix} y_0 & z_0 \\ y_1 & z_1 \end{pmatrix} \neq 0 \); in which case the system is controllable.

Using theorem 2.1, if \( \lambda_1 \neq 0, \lambda_2 \neq 0 \) and \( \lambda_1 + \lambda_2 \neq 0 \) and when the system is controllable, the optimal control \((u^*, v^*)\) which allows the system to be leads from a state \( x_0 \) to a desired final state \( x_d \) at time \( T \) is given by 
\[ u^*(\theta) = \begin{cases} \frac{1}{\varepsilon_k(\det P)} \begin{pmatrix} y_k & z_k \\ 0 & 0 \end{pmatrix} & \text{if } \theta \in [t_k, t_k + \varepsilon_k[ \\ 0 & \text{elsewhere} \end{cases} \]
\[ v^*(\theta) = \frac{1}{\det P} \begin{pmatrix} \text{ad} \ e^{\lambda_1(T-\theta)} - bc \ e^{\lambda_2(T-\theta)} & -\text{ab} \ (e^{\lambda_1(T-\theta)} - e^{\lambda_2(T-\theta)}) \\ 0 & 0 \end{pmatrix} f \]

where
\[
f = (\det P)^2 \begin{pmatrix} \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k} y_k^2 + y & \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k} y_k z_k + z \\ \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k} y_k z_k + z & \sum_{k=0}^{N-1} \frac{1}{\varepsilon_k} z_k^2 + w \end{pmatrix}^{-1} (x_d - e^{TA}x_0) \]

\[ y_k = \text{ad}\alpha_k e^{\lambda_1(T-t_k)} - bc\beta_k e^{\lambda_2(T-t_k)} \]
\[ z_k = \text{ab} \ [-\alpha_k e^{\lambda_1(T-t_k)} + \beta_k e^{\lambda_2(T-t_k)}] \]
\[ y = -a^2\beta^2 \gamma_1 + 2abcd\gamma - b^2c^2 \gamma_2 \]
\[ z = \text{ab} \ [\text{ad}\gamma_1 - (bc + ad)\gamma + bc \gamma_2] \]
\[ w = -a^2b^2(\gamma_1 - 2\gamma + \gamma_2) \]

**Numerical simulation**

The parameter values chosen for the model are taken from [5] and are
\[
\begin{pmatrix} -0.15 & -0.081 \\ 0.081 & -0.56 \end{pmatrix} \text{ the unit is } h^{-1}, \quad x_d = \begin{pmatrix} 1.1 \times 10^{-4} \\ 0 \end{pmatrix} \text{ the unit is g and} \]

\[ K_a = 2.3 \ h^{-1} \text{ the absorption constance.} \]

Then, in the case when there is only an impulsive control,

for \( B_2 = 0; x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \varepsilon_k = \varepsilon = \frac{1}{23} h^{-1}, \ k = 0, 1, \ldots, N-1; \) and for \( T = 120 \)

\[ \text{h and} \]

\[ N = 5, \] we have

\[ a = 1; b = 1; c = -3.31838022; d = 1.125046886; \det P = 4.443427106; \lambda_1 = -0.728757033; \lambda_2 = -0.0622429671; \alpha_k = 0.5118514633 \text{ and } \beta_k = 0.4409424737, \]

\[ k = 0, 1, \ldots, N-1. \]

The parameters \( y_k \) and \( z_k \) are given in table 1.

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_k \times 10^4 )</td>
<td>8.346315978</td>
<td>37.175556</td>
<td>165.584669</td>
<td>737.535242</td>
<td>3285.076055</td>
</tr>
<tr>
<td>( z_k \times 10^4 )</td>
<td>2.515177714</td>
<td>11.202922</td>
<td>49.899246</td>
<td>222.257605</td>
<td>989.963728</td>
</tr>
</tbody>
</table>

**Table 1.** The parameters \( y_k \) and \( z_k \).

\[ f = \begin{pmatrix} -0.1081830018 \times 10^9 \\ 0.3589923336 \times 10^9 \end{pmatrix} \]

and the optimal control \( u^* \) is given in table 2; \( u^*(\theta) = \begin{pmatrix} u_1^*(\theta) \\ 0 \end{pmatrix} \)
Table 2. The evolution of the control $u^*_1$.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\hline
u^*_1(\theta) \times 10^4 & 0.256729455 & -67.1249512 & 114.823352 & -69.0944365 & 63.5731726 \\
\hline
\end{array}
\]

3 Linear quadratic optimal control

In this section, we consider the problem of linear quadratic optimal control related to the system described by

\[
\begin{cases}
\dot{x}(t) = Ax(t) + Bu(t) & 0 \leq t \leq T \\
x(0) \text{ is given}
\end{cases}
\]

where $A \in \mathcal{L}(\mathbb{R}^n)$; $B$ is an $n \times n$ diagonal matrix;

$u \in \mathcal{E}_N = \{ u = \sum_{k=0}^{N-1} u_k \chi_{[t_k, t_k+\varepsilon_k]}, u_k \in \mathbb{R}^n, t_k = k \frac{T}{N}, t_k + \varepsilon_k < t_{k+1} \}$

Our control problem is to determine the control $u \in \mathcal{E}_N$ which minimizes the cost functional

\[
J(u) = \langle x(T), Gx(T) \rangle + \sum_{i=0}^{N-1} \langle x(t_i), Mx(t_i) \rangle + \int_0^T <u(t), Ru(t)> dt
\]

where $G$, $M$ and $R$ are self-adjoint and non-negative operators of $\mathcal{L}(\mathbb{R}^n)$ with $\langle Ru, u \rangle \geq \alpha \|u\|^2$ for some $\alpha > 0$ and all $u \in \mathcal{E}_N$.

3.1 Preliminary properties

In this subsection, we will develop an optimality system from which derives the optimal control $u^* \in \mathcal{E}_N$. For this, let we call $x_i = x(t_i)$, $0 \leq i \leq N$, so we have

\[
x_{i+1} = e^{t_{i+1}A}x_0 + \int_{t_i}^{t_{i+1}} e^{(t_{i+1}-s)A} Bu(s) ds
\]

\[
= e^{\delta A} e^{t_i A} x_0 + \int_{t_i}^{t_{i+1}} e^{\delta A} e^{(t_i-s)A} Bu(s) ds + \int_{t_i}^{t_{i+1}} e^{(t_{i+1}-s)A} Bu(s) ds
\]

\[
= e^{\delta A} x_i + \left( \int_{t_i}^{t_{i+1}+\varepsilon_i} e^{(t_{i+1}-s)A} Bu(s) ds \right) u_i
\]

Then

\[
x_{i+1} = C x_i + B_i u_i
\]

where $C = e^{\delta A}$ and $B_i = \int_{t_i}^{t_{i+1}+\varepsilon_i} e^{(t_{i+1}-s)A} Bu(s) ds$.

We can establish easily that

\[
x_i = C^i x_0 + \sum_{j=0}^{i-1} C^{i-j-1} B_j u_j \quad 1 \leq i \leq N
\]

\[
x_i = C^i x_0 + (\mathcal{H}_N(u))_i \quad 1 \leq i \leq N
\]
Where

\[ H_N : E_N \longrightarrow l^2(1, 2, ..., N, \mathbb{R}^n) \]
\[ u = \sum_{i=0}^{N-1} u_i \chi_{[t_i, t_i+\varepsilon_i]} \longrightarrow ((H_N(u))_1, (H_N(u))_2, ..., (H_N(u))_N) \]

with

\[ (H_N(u))_i = \sum_{j=0}^{i-1} C^{i-j-1} B_j u_j \quad 1 \leq i \leq N \]

The adjoint operator \( H_N^* \) is given by

\[ H_N^* : l^2(1, 2, ..., N, \mathbb{R}^n) \longrightarrow E_N \]
\[ (x_1, x_2, ..., x_N) \longrightarrow H_N^*(x_1, x_2, ..., x_N) \]

such that

\[ H_N^*(x_1, x_2, ..., x_N)(\theta) = \frac{1}{\varepsilon_i} \sum_{k=i+1}^{N} B_i^* C^{*k-i-1} x_k \quad if \ \theta \in [t_i, t_i+\varepsilon_i] \quad 1 \leq i \leq N-1 \]

Indeed, for \( u \in E_N \) and \((x_1, x_2, ..., x_N) \in l^2(1, 2, ..., N, \mathbb{R}^n)\); we have

\[ \langle H_N u, (x_1, x_2, ..., x_N) \rangle = \sum_{k=1}^{N} \langle (H_N u)_k, x_k \rangle \]
\[ = \sum_{k=1}^{N} \sum_{i=0}^{k-1} C^{k-i-1} B_i u_i , x_k \rangle \]
\[ = \sum_{k=1}^{N-1} \sum_{i=0}^{N-1} \sum_{i=0}^{k} u_i , B_i^* C^{*k-i-1} x_k \rangle \]
\[ = \sum_{i=0}^{N-1} \sum_{k=i+1}^{N} \sum_{i=0}^{k} u_i , B_i^* C^{*k-i-1} x_k \rangle \]
\[ = \sum_{i=0}^{N-1} \varepsilon_i u_i , \frac{1}{\varepsilon_i} \sum_{k=i+1}^{N} B_i^* C^{*k-i-1} x_k \rangle \]

We deduce that

\[ J(u) = \langle x_N, G x_N \rangle + \sum_{i=1}^{N-1} \langle x_i, M x_i \rangle + \langle u, R u \rangle_{L^2(0, T, \mathbb{R}^n)} \]
\[ = \langle C x_0 + (H_N u)_N, G(C x_0 + (H_N u)_N) > + \sum_{i=1}^{N-1} \langle C^i x_0 + (H_N u)_i, M(C^i x_0 + (H_N u)_i) > + < u, R u > \]
\[ = J_0 + J(u) \]
Where

\[ J_0 = \langle C^N x_0, GC^N x_0 \rangle + \sum_{i=1}^{N-1} \langle C^i x_0, MC^i x_0 \rangle \]

\[ \bar{J}(u) = \langle (\mathcal{H}_N u)_N, G(\mathcal{H}_N u)_N \rangle + \sum_{i=1}^{N-1} \langle (\mathcal{H}_N u)_i, M(\mathcal{H}_N u)_i \rangle \]

\[ + 2(\langle (\mathcal{H}_N u)_N, GC^N x_0 \rangle + \sum_{i=1}^{N-1} \langle (\mathcal{H}_N u)_i, MC^i x_0 \rangle) + \langle u, Ru \rangle \]

Let \( D_i = M \) if \( 1 \leq i \leq N - 1 \) et \( D_N = G \) and consider the linear, auto-adjoint operator \( \bar{D} \) defined by

\[ \bar{D}: \tilde{P}^2(1, 2, ..., N, \mathbb{R}^n) \rightarrow \tilde{P}^2(1, 2, ..., N, \mathbb{R}^n) \]

\[ (x_1, x_2, ..., x_N) \rightarrow (D_1 x_1, D_2 x_2, ..., D_N x_N). \]

If we call

\[ \left\{ \begin{array}{l}
  a_i = MC^i x_0 \quad 1 \leq i \leq N - 1 \\
  a_N = GC^N x_0
\end{array} \right. \]

Then

\[ \bar{J}(u) = \langle \mathcal{H}_N u, \bar{D}\mathcal{H}_N u \rangle + 2 \langle \mathcal{H}_N u, (a_1, a_2, ..., a_N) \rangle + \langle u, Ru \rangle \]

\[ = \langle u, (\mathcal{H}_N^* D\mathcal{H}_N + R)u \rangle + 2 \langle u, \mathcal{H}_N^* (a_1, a_2, ..., a_N) \rangle \]

We deduce that the optimal control \( u^* \) of the quadratic function \( J \) is such that

\[ (\mathcal{H}_N^* D\mathcal{H}_N + R)u^* = -\mathcal{H}_N^* (a_1, a_2, ..., a_N) \]

Thus, we can establish easily that

\[ u^* = -R^{-1}\mathcal{H}_N^* (M x_1^{u^*}, ..., M x_{N-1}^{u^*}, G x_N^{u^*}) \]

\[ = -R^{-1}\mathcal{H}_N^* (D_1 x_1^{u^*}, D_2 x_2^{u^*}, ..., D_N x_N^{u^*}) \]

Therefore

\[ u^*(\theta) = -\frac{1}{\varepsilon_i} R^{-1} B_i^* \sum_{k=i+1}^{N} C^{s_k-i-1} D_k x_k^{u^*} \quad if \quad \theta \in [t_i, t_i + \varepsilon_i[ \quad 0 \leq i \leq N - 1 \]

Let’s consider the signal \((p_i)_{0 \leq i \leq N-1}\) defined by

\[ p_i = \sum_{k=i+1}^{N} C^{s_k-i-1} D_k x_k^{u^*} \quad if \quad \theta \in [t_i, t_i + \varepsilon_i[ \quad 0 \leq i \leq N - 1 \]
Then the signal $p_i$ verify
\[
\begin{cases}
p_i = C^* p_{i+1} + M x_{i+1}^0, & 0 \leq i \leq N - 2 \\
p_{N-1} = G x_N^0
\end{cases}
\] (6)

Finally, we have the following optimality system
\[
\begin{aligned}
u^*(\theta) &= -\frac{1}{\varepsilon_i} R^{-1} B_i^* p_i & \text{if } \theta \in [t_i, t_i + \varepsilon_i] \quad 0 \leq i \leq N - 1 \\
p_i &= C^* p_{i+1} + M x_{i+1}^0 & 0 \leq i \leq N - 2 \\
p_{N-1} &= G x_N^0 \\
x_{i+1}^u &= C x_i^u + B_i u_i & i = 0, \ldots, N - 1
\end{aligned}
\] (7)

Where $C = e^{\delta A}$; $B_i = \int_{t_i}^{t_i + \varepsilon_i} e^{(t_i + s)A} B ds$ and $B_i^* = \int_{t_i}^{t_i + \varepsilon_i} B e^{(t_i + s)A^T} ds$.

### 3.2 An adequate topology

The technic developed here is similar to HUM method (see [16]). For $f = (f_1, f_2, \ldots, f_N) \in \mathcal{F} = l^2(1, 2, \ldots, N, \mathbb{R}^n)$; we define the signal $z_f = (z_{f0}, z_{f1}, \ldots, z_{fN-1})$ by the difference equation
\[
z_i^f = \sum_{k=i+1}^{N} C^* i \leq j \leq N - 1
\]

The following functional defined in $\mathcal{F}$ by
\[
\|f\|_N^2 = \|f\|_F^2 + \sum_{i=0}^{N-1} \|R^{-1/2} B_i^* z_i^f\|^2
\]
is a norm in $\mathcal{F}$ equivalent to the norm $\|\cdot\|_F$ de $\mathcal{F}$. Let’s us define the operator $\Lambda_N$ by
\[
\Lambda_N : \mathcal{F} \rightarrow \mathcal{F} \\
f \mapsto f + \tilde{D}_f^\frac{1}{2} \Psi f
\]
where $\Psi f = (\Psi_i^f)_{1 \leq i \leq N}$ is given by
\[
\Psi_i^f = (H_N u_f)_i = \sum_{j=0}^{i-1} C_{i-j-1} B_j u_j^f & 1 \leq i \leq N
\]

with $u_f = \sum_{i=0}^{N-1} u_i^f \chi_{[t_i, t_i + \varepsilon_i]}$ is described by
\[
u_i^f = \frac{1}{\varepsilon_i} R^{-1} B_i^* z_i^f & 1 \leq i \leq N - 1
Lemma 3.1 \( \Lambda_N \) is a bounded, self-adjoint operator satisfying
\[
< \Lambda_N f, f > = \| f \|_N^2 ; \quad \forall f \in \mathcal{F}
\]

Proof: We have
\[
< \Lambda_N f, g > = < f, g > + \sum_{i=1}^{N} < D_i^{\frac{1}{2}} \Psi_i^f, g_i >
\]
\[
= < f, g > + \sum_{i=1}^{N} \sum_{j=0}^{i-1} \frac{1}{\varepsilon_j} < B_j^* z_j^f, R^{-1} B_j^* C^{i-j-1} D_i^{\frac{1}{2}} g_i >
\]
\[
= < f, g > + \sum_{j=0}^{N-1} \frac{1}{\varepsilon_j} < B_j^* z_j^f, R^{-1} B_j^* z_j >
\]
\[
= < f, g > + \frac{1}{\varepsilon_j} < R^{-\frac{1}{2}} B_j^* z_j^f, R^{-\frac{1}{2}} B_j^* z_j >
\]
Then
\[
< \Lambda_N f, g > = < f, \Lambda_N g >
\]
and
\[
< \Lambda_N f, f > = \| f \|_x^2 + \sum_{j=0}^{N-1} \frac{1}{\varepsilon_j} \| R^{-\frac{1}{2}} B_j^* z_j^f \|^2
\]
\[
= \| f \|_N^2 \quad \Box
\]

Finally, we prove the following main result

Theorem 3.1 The optimal control minimizing the functional \( J \) in \( \mathcal{E}_N \) is given by:
\[
u(\theta) = \frac{1}{\varepsilon_i} R^{-1} B_i^* z_i^f \quad ; \quad \theta \in [t_i, t_i + \varepsilon_i] , \quad i = 0, 2, \ldots, N - 1
\]
where \( z_i^f \) is the solution of the difference equation
\[
\begin{align*}
z_i^f &= \sum_{k=i+1}^{N-1} C^{*k-i-1} M^* f_k + C^{*N-i-1} G^\frac{1}{2} f_N \quad 0 \leq i \leq N - 2 \\
z_{N-1}^f &= G^\frac{1}{2} f_N \quad (8)
\end{align*}
\]
Moreover, the optimal cost is
\[ J(u) = \|f\|_N^2. \]

Proof: From the optimality system (9); it is enough to establish that
\[ z_i^f = -p_i \quad 0 \leq i \leq N - 1 \]
i.e.,
\[ f_i = -D_i^\frac{1}{2} x_i^u \quad 1 \leq i \leq N \]

We have
\[ \Lambda_N f = -(M^\frac{1}{2} C x_0, \ldots, M^\frac{1}{2} C^{N-1} x_0, M^\frac{1}{2} C^N x_0) \]

Then
\[ f = -(M^\frac{1}{2}(C x_0 + \Psi_1), \ldots, M^\frac{1}{2}(C^{N-1} x_0 + \Psi_{N-1}^f), G^\frac{1}{2}(C^N x_0 + \Psi_N^f)) \]
\[ = -(M^\frac{1}{2}(C x_0 + (H_N u)_1), \ldots, M^\frac{1}{2}(C^{N-1} x_0 + (H_N u)_{N-1}), G^\frac{1}{2}(C^N x_0 + (H_N u)_N)) \]
\[ = -(M^\frac{1}{2} x_1^u, \ldots, M^\frac{1}{2} x_N^u, G^\frac{1}{2} x_N^f) \]

Therefore
\[ f_i = -D_i^\frac{1}{2} x_i^u \quad 1 \leq i \leq N \]

Moreover
\[ J(u) = \langle x_N^u, G x_N^u \rangle + \sum_{i=1}^{N-1} \langle x_i^u, M x_i^u \rangle + \langle u, Ru \rangle_{L^2(0,T;\mathbb{R}^n)} \]
\[ = \langle G^\frac{1}{2} x_N^u, G^\frac{1}{2} x_N^u \rangle + \sum_{i=1}^{N-1} \langle M^\frac{1}{2} x_i^u, M^\frac{1}{2} x_i^u \rangle + \int_0^T \langle u(\theta), Ru(\theta) \rangle \ d\theta \]
\[ = \langle f_N, f_N \rangle + \sum_{i=1}^{N-1} \langle f_i, f_i \rangle + \sum_{i=1}^{N-1} \int_{t_i}^{t_{i+1}} \frac{1}{\zeta_i} \langle R^{-\frac{1}{2}} B_i^* z_i^f, R^{-\frac{1}{2}} B_i^* z_i^f \rangle \ d\theta \]
\[ = \|f\|^2 + \sum_{i=1}^{N-1} \frac{1}{\zeta_i} \langle R^{-\frac{1}{2}} B_i^* z_i^f, R^{-\frac{1}{2}} B_i^* z_i^f \rangle \]
\[ = \|f\|^2 + \sum_{i=1}^{N-1} \frac{1}{\zeta_i} \|R^{-\frac{1}{2}} B_i^* z_i^f\|^2 \]
\[ = \|f\|_N^2 \]
References


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