Steady-States of Nonlinear Oscillators Based on Homotopic Process with Generalized Inverses

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Abstract
For autonomous oscillators, we propose an algorithm to compute their steady-state responses by homotopic process with generalized inverses. In the Newton iterations of the process the computing procedure is almost the same as the well known Aprille-Trick method except now replacing standard inverses by generalized inverses. The approach reported here, which needs not to have a priori knowledge of equations, is a unified Newton-like algorithm to determine periodic responses for autonomous or non-autonomous cases. Numerical examples are provided to further illustrate the validity of the homotopic algorithm.

Keywords: steady-state response; nonlinear oscillator; Newton iteration; generalized inverse; homotopic method; convergence region; circuit simulation

1 Introduction
Let us consider the autonomous system

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(x), \quad t \in (0, T), \\
x(0) &= x(T),
\end{align*}
\]

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where \( x \) and \( f \) are \( n \)-vectors, \( f \) is continuously differential derivative, and the period \( T \) is unknown. Usually, the function \( f \) is nonlinear. The steady-state analysis of (1) is to determine the unknown period and the corresponding periodic solution or response.

The system or oscillator (1) becomes a periodic boundary problem (i.e., the non-autonomous system) if \( f(x) \) is set to be \( f(x, t) \) and the period \( T \) is known. That is,

\[
\begin{aligned}
\frac{dx(t)}{dt} &= f(x, t), \quad t \in (0, T), \\
x(0) &= x(T),
\end{aligned}
\]

(2)

where \( f(x, t) \) is a continuous \( T \)-periodic function and has first continuously partial derivative with respect to \( x \). For such problem, the known methods of computing periodic transient responses are the Newton iterative process and its variants \([1-5]\). The quasi-linearization of periodic problems is an iterative process in function space \([6]\). The Aprille-Trick method in \([2]\) is an important and classic approach. It also suits for computations of oscillators if some condition on initial value is known in advance, see \([3]\).

Generally speaking, the computation of (1) is a difficult task. It is in fact an undetermined problem, because there are \( n + 1 \) unknown arguments but only \( n \) equations. Often, one sets a new known condition on some element of initial vector in advance, the standard algorithms like the Aprille-Trick method with some modification are then applied to solve the left \( n \) unknown arguments. The simple technique of generalized inverses was used to computer the steady-state responses of dynamic systems in \([7-8]\). The gradient method is based on the optimized strategy in \([9]\). The Newton-Picard procedure in \([10]\) needs also a priori condition on initial value.

In the paper we adopt the homotopic iterative process with generalized inverses to numerically solve (1). In the process, not like the Aprille-Trick method, the Newton iterations are now modified by replacing the standard inverses with generalized inverses. The new modified Newton iterations could converge to the wanted periodic solutions or responses without priori knowledge on oscillators. Numerical examples are shown to illustrate the convergence behaviors of the iterative process.
Steady-states of nonlinear oscillators

2 Homotopic Process with Newton Iterations by Generalized Inverses

We first transform (1) into a form of (2) with singular Jacobian matrix. Let \( s = t/T \) and \( y(s) = x(Ts) \), we have

\[
\begin{aligned}
\frac{dz(s)}{ds} &= g(z), \quad s \in (0, 1), \\
z(0) &= z(1),
\end{aligned}
\]

where

\[
g(z) = \begin{bmatrix} T f(y) \\ 0 \end{bmatrix} \quad \text{for} \quad z(s) = \begin{bmatrix} y(s) \\ T(s) \end{bmatrix}.
\]

The Jacobian matrix of (3) or \( g(z) \) is

\[
\nabla_g(z) = \begin{bmatrix} T \nabla_f(y) & f(y) \\ 0 & 0 \end{bmatrix},
\]

where \( \nabla_f(y) \) is the Jacobian matrix of \( f(y) \) or (1). The matrix in (5) is always singular for any value \( z \). In the paper we also need to make a basic assumption on (1) or (3) like in [3].

**Assumption.** For (1) or (3) there exists at least an asymptotically stable \( T \)-periodic solution for (1) or 1-periodic solution for (3).

2.1 Newton iteration with generalized inverse

The system (3) is now the same as (2) in form except \( g(z) \) is independent on \( s \). For such system, the Aprille-Trick method could be used to solve (3) if we simply replace standard inverses by generalized inverses. The main object of solving (3) is to find its initial values. By advantage of such a true initial value \( z^* \) we can then obtain the corresponding periodic response by solving initial value problem.

For arbitrary value \( z_0 \), from it to start we have a solution trajectory \( z(s) \) by the differential equation in (3). The extreme point of the trajectory at \( s = 1 \) is \( z(1) \). Obviously, \( z(1) \) is a function of \( z_0 \). Namely, we may set \( z(1) = F(z_0) \). In other words, to search for the initial values of (3) is just equal to solve a nonlinear equation in vector space \( \mathbb{R}^{n+1} \). The nonlinear equation is

\[
G(z_{in}) = 0,
\]

where \( G(z_{in}) = F(z_{in}) - z_{in} \). Obviously, we have \( \nabla G = \nabla F - I_{n+1} \) where \( \nabla G \) and \( \nabla F \) are the Jacobian matrices of \( G \) and \( F \), and \( I_{n+1} \) is the identity matrix in \( \mathbb{R}^{(n+1)\times(n+1)} \).
For $\nabla_g(z)$ in (5) where $z(s)$ is a solution trajectory of (3) from some given initial guess, we define its state transition matrix $\Phi_{n+1}$ by

$$\begin{cases}
\frac{d\Phi_{n+1}(z(s), s)}{ds} = \nabla_g(z(s))\Phi_{n+1}(z(s), s), & s \in (0, 1), \\
\Phi_{n+1}(z(0), 0) = I_{n+1}.
\end{cases}$$

(7)

By the elementary knowledge on ordinary differential equations in [11 - 12], we have $\nabla F = \Phi_{n+1}(z(1), 1)$. Thus, it follows

$$\nabla G = \Phi_{n+1}(z(1), 1) - I_{n+1}.$$  

(8)

Based on the special structure of $\nabla_g(z)$, by (7) we may write $\nabla G$ in partitioned form as

$$\nabla G = \begin{bmatrix} H(1) \\ 0 \end{bmatrix},$$

(9)

where $H(1) \in \mathbb{R}^{n \times (n+1)}$ is

$$H(1) = [E(1), h(1)].$$

(10)

In (10), we have $E(1) = \Phi_n(y(1), 1)$ where the matrix $\Phi_n$ is defined as

$$\begin{cases}
\frac{d\Phi_n(y(s), s)}{ds} = T(s)\nabla_f(y(s))\Phi_n(y(s), s), & s \in (0, 1), \\
\Phi_n(y(0), 0) = I_n
\end{cases}$$

(11)

in which as before $[y^t(s), T(s)]^t (= z(s))$ is a solution trajectory of (3) from some given initial guess and $I_n$ is the identity matrix in $\mathbb{R}^{n \times n}$. Moreover, in (10) we can know $h(1) = f(y(1))$ where $h(1)$ is the value of some $n$-vector function $h(s)$ ($s \in [0, 1]$) at $s = 1$.

In fact, based on (7) and (8) the function $h(s)$ satisfies

$$\begin{cases}
\frac{dh(s)}{ds} = T(s)\nabla_f(y(s))h(s) + f(y(s)), & s \in (0, 1), \\
h(0) = 0.
\end{cases}$$

(12)

Its solution is

$$h(s) = \Phi_n(y(s), s) \int_0^s \Phi_n^{-1}(y(\tau), \tau)f(y(\tau))d\tau, & s \in [0, 1],$$

(13)

see [13]. Since $\frac{dT(s)}{ds} = 0$ for $s$, by

$$\frac{dy(s)}{ds} = T(s)f(y(s)), & s \in (0, 1)$$

(14)
in (3) we have
\[ \frac{d}{ds} \left( \frac{dy(s)}{ds} \right) = T(s) \nabla_f(y(s)) \frac{dy(s)}{ds}, \quad s \in (0, 1). \]

It follows \( T(s)f(y(s)) = \Phi_n(y(s), s)(T(0)f(y(0))) \) according to (11) and (14). Since \( T(s) \equiv T \neq 0 \) for \( s \), we know
\[ f(y(s)) = \Phi_n(y(s), s)f(y(0)), \quad s \in [0, 1]. \tag{15} \]

By (13) and (15), now we yield
\[ h(s) = s\Phi_n(y(s), s)f(y(0)) = sf(y(s)), \quad s \in [0, 1]. \tag{16} \]

That is, it confirms the validity of \( h(1) = f(y(1)) \).

The basic iterative process of the paper is described by
\[ z^{(k+1)}_\text{in} = z^{(k)}_\text{in} - \nabla^+_G(z^{(k)}_\text{in})G(z^{(k)}_\text{in}), \quad k = 0, 1, \ldots, \tag{17} \]
where \( z^{(0)}_\text{in} \) is initial guess and \( \nabla^+_G \) is the generalized inverse of \( \nabla_G \). For any matrix, it is known that its generalized inverse (i.e., the Moore-Penrose inverse) always uniquely exists, see [14]. The above process is similar to the Aprille-Trick method except we now use generalized inverses not like therein the standard inverses for Jacobian matrices are necessary.

Under Assumption, we also learn that the matrix \( \Phi_{n+1}(z(1), 1) \) in (7) or \( \Phi_n(y(1), 1) \) in (11) due to \( T(s) \neq 0 \) for \( s \in [0, 1] \) has only one unitary eigenvalue and other eigenvalues with magnitudes less than one, see [3, 7]. It means that the unitary eigenvalue is simple for \( \Phi_{n+1}(z(1), 1) \) or \( \Phi_n(y(1), 1) \). Thus, the matrix \( H(1) \in \mathbb{R}^{n \times (n+1)} \) has \( n \) independent columns by the theorem shown in [3]. That is, the matrix \( H(1) \) is full rank where \( \text{rank}(H(1)) = n \).

By invoking the convergence results on the generalized Newton iterative methods (see [14, 15]), the iterative process in (17) will be locally convergent if the initial guess \( z^{(0)}_\text{in} \) is chosen to be near to some zero point \( z^{*}_\text{in} \) of \( G \). Then, the limit \( z_\text{in} \) of the sequence \( \{z^{(k)}_\text{in}\} \) is just \( z^{*}_\text{in} \).

To do so, now we need to show \( G(z_\text{in}) = 0 \) if \( \nabla^+_G(z_\text{in})G(z_\text{in}) = 0 \). First, we may also write \( G(z_\text{in}) \) in (6) as
\[ G(z_\text{in}) = \begin{bmatrix} G_1(z_\text{in}) \\ 0 \end{bmatrix}, \]
where \( G_1(z_\text{in}) \in \mathbb{R}^n \). By (9) and the properties on generalized inverses in [14], we have \( \nabla^+_G(z_\text{in}) = [H^+(1), 0] \). It deduces \( \nabla^+_G(z_\text{in})G(z_\text{in}) = H^+(1)G_1(z_\text{in}) \). Since \( \text{rank}(H(1)) = n \), the matrix \( H(1) \) has a singular value decomposition
$H(1) = U[D, 0]V^t$ where $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{(n+1) \times (n+1)}$ are two orthogonal matrices, and the matrix $D \in \mathbb{R}^{n \times n}$ is nonsingular. Thus, we arrive at

$$\nabla_G^t(z_{\text{in}})G(z_{\text{in}}) = V \begin{bmatrix} D^{-1} & 0 \\ 0 & \end{bmatrix} U^t G_1(z_{\text{in}}) = V \begin{bmatrix} D^{-1}U^tG_1(z_{\text{in}}) \\ 0 \end{bmatrix}.$$  

It follows $G_1(z_{\text{in}}) = 0$ and then $G(z_{\text{in}}) = 0$ if $\nabla_G^t(z_{\text{in}})G(z_{\text{in}}) = 0$.

### 2.2 Homotopic process

The homotopic process is an effective method of finding for good initial guess for Newton-like algorithm, see [16]. Often, the homotopic process with Newton iterations could enlarge the convergence region of the basic algorithm. Now, we let $\lambda$ be a step parameter. For $G(z_{\text{in}})$ in (6) and some given initial guess $z_{\text{in},0}$, we construct a new function $HG(z_{\text{in}}, \lambda)$ by

$$HG(z_{\text{in}}, \lambda) = G(z_{\text{in}}) - (1 - \lambda)G(z_{\text{in},0}), \quad \lambda \in [0, 1]. \quad (18)$$

If we denote $z_{\text{in}}(\lambda)$ as the zero point of $HG(z_{\text{in}}, \lambda)$, then $z_{\text{in}}(1)$ is a solution of (6) where $z_{\text{in}}(0) = z_{\text{in},0}$. Moreover, for any fixed $\lambda$ the Jacobian matrix $\nabla_{HG}$ of $HG$ is just $\nabla_G$.

Let $\Delta \lambda$ be a step length on the parameter $\lambda$ in $[0, 1]$ and the stop criteria value be $\epsilon$. The homotopic iterative process with generalized inverses is now simply described as below.

**Algorithm**

- Step 0: Given $z_{\text{in},0}$, set $z^{(0)}_{\text{in}} = z_{\text{in},0}$ and $\lambda = 0$;
- Step 1: For $l = 0, 1, \ldots, L - 1$, do
  $$z^{(l+1)}_{\text{in}} = z^{(l)}_{\text{in}} - \nabla_G^+(z^{(l)}_{\text{in}}) (G(z^{(l)}_{\text{in}}) - (1 - \lambda)G(z_{\text{in},0})),$$
  if $\|z^{(l+1)}_{\text{in}} - z^{(l)}_{\text{in}}\| < \epsilon$, go to Step 2;
- Step 2: Set $\lambda = \lambda + \Delta \lambda$, if $\lambda > 1$, stop; else set $z^{(0)}_{\text{in}} = z^{(L)}_{\text{in}}$ and go to Step 1.

Before ending the section, we mention here that without Assumption the above algorithm sometimes could also compute unstable periodic solutions of (1) or (3) as long as the iterative process can go on not break at some point.

### 3 Numerical Examples

Our numerical experiments are based on two test examples. The first one is the nerve membrane model, and the other is the van der Pol oscillator. In our computations the iterative error is defined as the sum of the squared differences of successive vectors taken over all discrete points.
Example 1: nerve membrane model

The model is written by

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= 3x_1(t) - x_1^3(t) + 3x_2(t) + \sigma, \\
\frac{dx_2(t)}{dt} &= -(x_1(t) + 0.8x_2(t) - 0.7)/3,
\end{align*}
\]  

(19)

where \(\sigma\) is a parameter, see [7, 17]. We let \(\sigma = -3\) and the time step be 0.001. By the basic algorithm with generalized inverses: the computed solution is \(x_1^* = 1.0455, \ x_2^* = 0.1750,\) and \(T^* = 9.6070\) for the first guess \(x_{1,0} = 3,\ x_{2,0} = 1.5,\) and \(T_0 = 12;\) for the second guess \(x_{1,0} = 1,\ x_{2,0} = 1,\) and \(T_0 = 5,\) the computed solution is \(x_1^* = 1.3109, \ x_2^* = 0.3226,\) and \(T^* = 9.6070.\) The numerical results are shown in Figure 1.

![Figure 1](image)

**Figure 1:** The convergence behavior (left) and computed phase (right) of Example 1 by the basic algorithm.

For \((x_{1,0}, x_{2,0}) \in [-1, 3] \times [-1, 3]\) and \(T_0 = 5,\) we respectively take the homotopic process and the basic algorithm to numerically solve (19). The spatial domain grid lengths are set to be 0.1 for \(x_1\) and \(x_2.\) The computed convergence regions are given in Figure 2 where the convergent initial guesses are marked by dots. It is obvious that the convergence region of the homotopic process is more larger than that of the basic algorithm.

Example 2: van der Pol oscillator

The well known van der Pol oscillator is modelled as

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= x_2, \\
\frac{dx_2(t)}{dt} &= \mu(1 - x_1^2)x_2 - x_1,
\end{align*}
\]  

(20)
Figure 2: The convergence regions of the basic algorithm (left) and the homotopic process (right) for Example 1.

where $\mu$ is a parameter, see also [3, 8, 18].

To compute the example, we use the basic algorithm where the time step is set to be 0.005. The initial guess is $x_{1,0} = 2$, $x_{2,0} = 1$, and $T_0 = 5$. For $\mu = 0.1$, the computed solution is $x_1^* = 0.3468$, $x_2^* = 1.6504$, and $T^* = 6.2780$; For $\mu = 0.5$, the computed solution is $x_1^* = 0.0368$, $x_2^* = 2.0116$, and $T^* = 6.3192$; The computed solution is $x_1^* = -0.3645$, $x_2^* = 1.7862$, and $T^* = 6.5153$ for $\mu = 1$. For the case of $\mu = 0.1$, the numerical results are in Figure 3.

For the van der Pol oscillator, the basic algorithm can also converge to the unstable solution $x^* = [0, 0]^T$ for $\mu = 0.1$ if the initial guess is taken as $x_{1,0} = 0.1$, $x_{2,0} = 0.1$, and $T_0 = 5$.

4 Conclusions

We have presented a simple approach to numerically solve periodic responses of nonlinear oscillators. The method of the paper needs not to have a priori knowledge on dynamic systems or oscillators. The basic algorithm is almost the same as the well known Aprille-Trick method for non-autonomous systems with known period, except we now replace standard inverses by generalized inverses. The homotopic process can remedy lacks of Newton-like algorithms on initial guesses’ restrictions. The new treatment on unknown period is easily understood by engineering fields. Moreover, by combining the Aprille-Trick method a unified approach is developed to find periodic solutions of autonomous or non-autonomous systems.
Figure 3: The convergence behavior (left) and computed phase (right) of Example 2 for $\mu = 0.1$ by the basic algorithm.

References


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