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A Numerical Comparison of Different Methods Applied to the Solution of Problems with Non Local Boundary Conditions

Abdesslam Boutayeb and Abdelaziz Chetouani

Département de mathematiques et informatique
Faculté des sciences, Oujda, Morocco
x.boutayeb@menara.ma, abd.chetouani@menara.ma

Abstract

This paper is devoted to the numerical comparison of methods applied to solve problems with nonlocal boundary conditions. Four numerical methods are compared, namely, finite differences method, Galerkin method, Keller Box scheme and orthogonal spline collocation method.

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1 Introduction

Partial differential equations with non local boundary conditions and partial integro-differential equations occur in many fields of science and engineering. A detailed description of the occurrence of such equations is given in [7]. We present in the sequel a non exhaustive list of phenomena governed by this kind of equations.

1. Diffusion problems
   Certain chemical processes are governed by the equations

$$\begin{cases} u_t = u_{xx}, \quad (x,t) \in I \times J, \\ u(x,0) = f(x), \\ u_x(1,t) = g(t), \quad t \in J, \\ \int_0^b u(s,t)ds = F(t), \quad b \in I, t \in J, \end{cases}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $I = (0,1)$, $J = [0,T]$, b is a given constant. For instance if u denotes the concentration, $F(t)$ is the mass in the area $0 < x < b$ at time t. Similar equations occur in biochemistry, in this case $b = 1$ and F is constant, the model is called a model with conservation of protein.

2. Quasi-static theory of thermoelasticity
Day [6] has shown that the entropy is governed by the equations

$$\begin{cases} b(x)u_t = (a(x)u_x)_x, & (x,t) \in I \times J, \\ u(x,0) = u_0(x), \\ u(0,t) = \int_0^1 f(x)u(x,t)dx, & t \in J, \\ u(1,t) = \int_0^1 g(x)u(x,t)dx, & t \in J, \end{cases}$$

3. **Quantum mechanics, nuclear reactor dynamics**
It is governed by the following equations

$$\begin{aligned} \frac{du}{dt} &= -\int_0^c \alpha(s)T(s,t)ds, \quad t > 0, \\ T_t &= (b(x)T_x)_x - q(x)T + \eta(x)\sigma(u(t)), \end{aligned}$$

subject to the initial conditions

$$u(0) = u_0, \quad T(x,0) = f(x), \quad x \in [0,c]$$

and the boundary conditions

$$\begin{aligned} d_1T(0,t) + d_2T_x(0,t) &= 0, \quad t > 0, \\ d_3T(c,t) + d_4T_x(c,t) &= 0, \quad t > 0, \end{aligned}$$

where $|d_1| + |d_2| > 0$ et $|d_3| + |d_4| > 0$, x denotes the position along the reactor, t denotes the time, $u(t)$ is the logarithm of the total reactor power, $T(x,t)$ is the deviation of the temperature, $\alpha(x)$ the ratio of the coefficient of reactivity, $\eta(x)$ is the fraction of the power generated at x, $q(x)$ is a constant multiple of the coefficient of external thermal conductivity and $\sigma(u) = -1 + e^u$.

4. **Population dynamics**
If $u(x,t)$ denotes the population density at time t with respect to age x, $s(x,t)$ the survival rate of the healthy population, $d(x,t)$ the death rate of the healthy population, $q(x,t)$ the survival rate of the disabled population, $d'(x,t) = d(x,t) + \delta(x,t)$ the death rate of the disabled

population, $\beta(x,t)$ the fertility rate of healthy women, $\gamma(x,t)$ the death rate of disabled women.

Let $r(x,t) = \frac{\gamma(x,t)}{u(x,t)} \times 100\%$ be the percentage of the disabled population aged x at time t, $c(x,t)$ the handicapping rate of the healthy population, $e(x,t)$ the rehabilitating rate of the disabled population and A the final age. The following equations were proposed to model the dynamics of a disabled population [3].

$$\begin{cases} u_t + u_x = -mu, \quad (x,t) \in [0,A], t \in J, \\ r_t + r_x = \delta r^2 - \epsilon r + \omega, \quad (x,t) \in [0,A], t \in J, \\ u(0,t) = \int_0^A budx, t \in J \\ u(x,t) = u_0(x), x \in [0,A]. \end{cases}$$

where $m(x,t) = d'x,t) + \delta(x,t)r(x,t)$, $b(a,t) = (\beta(x,t)(1-r(x,t)) + \gamma(x,t))$ and $\epsilon(x,t) = (c(x,t)s(x,t) + e(x,t)q(x,t) + \delta(x,t))$, $\omega(x,t) = c(x,t)s(x,t)$.

Many authors have investigated the existence of solutions of this kind of problems [5, 6, 11]. For the numerical aspect, different numerical methods were used [1, 2, 4, 8, 9].

In what follows, we consider the general form of initial boundary value problem

$$\begin{cases} u_t - (a(x)u_x)_x - b(x)u_x = F(u,x,t) \\ 0 \le x \le 1 \end{cases} \tag{1}$$

subject to the non local boundary conditions

$$\begin{cases} u(0,t) = \int_0^1 K_0(x)u(x,t)dx + g_0(t) \\ u(1,t) = \int_0^1 K_1(x)u(x,t)dx + g_1(t) \\ 0 \le t \le T \end{cases} \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{3}$$

where $u_t = \frac{\partial u(x,t)}{\partial t}$, $u_x = \frac{\partial u(x,t)}{\partial x}$.

The functions F, K_0, K_1 are assumed sufficiently smooth. several authors have studied numerical methods of the problem (1-3) and its variants, see for instance [1, 2, 7].

In previous contributions, the authors considered a family of numerical methods for the numerical solution of a linear parabolic problem with boundary conditions containing integrals over the interior of the interval. Global extrapolation procedures in space only and in both space and time were also discussed [1].

A Galerkin approximation (θ-method) was devoted to a semi linear parabolic problem, the existence and convergence were proved for $\theta \leq 1/2$.

Finally, Keller box scheme and orthogonal spline collocation methods were also studied [4].

In this paper, recalling different methods considered previously for various problems with nonlocal boundary conditions, the authors propose a numerical comparison of four methods applied to solve the same problem.

2 Numerical methods

2.1 Finite Differences & Global Extrapolations [1]

The initial boundary value problem (1-3) was considered with $a(x) = 1, b(x) = 0$ and $F(u, x, t) = f(x, t)$. Let $\mathbf{U}_m^n$ denotes $u(x_m, t_n)$, the integrals are approximated by the trapezoidal rule.
Assume that the function $u(x, t)$ is sufficiently smooth, $\frac{\partial^2 u(x,t)}{\partial x^2}$ is replaced by

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{h^2}\left[u(x-h,t) - 2u(x,t) + u(x+h,t)\right] + O(h^2) \quad \text{as } h \longrightarrow 0. \quad (4)$$

This approximation yields a system of ordinary differential equations, given by

$$\begin{aligned} \frac{d\mathbf{U}(t)}{dt} &= A\mathbf{U}(t) + \mathbf{b}(t), \\ \mathbf{U}(0) &= \mathbf{f}_0, \end{aligned} \qquad (5)$$

where the matrix A of size $(M-1) \times (M-1)$ is given by

$$A = \frac{1}{h^2}\begin{bmatrix} a_1 & a_2 & & \cdots & \cdots & a_{M-1} \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ c_1 & c_2 & & \cdots & \cdots & c_{M-1} \end{bmatrix}, \qquad (6)$$

the coefficients a_i and c_i, $i = 1, \dots, M-1$ are seen to be equal to (see [1]):

$$a_1 = \frac{h(1-\frac{h}{2}K_{MM})K_{O1} + \frac{h^2}{2}K_{0M}K_{M1}}{1-\frac{h}{2}(K_{00}+K_{MM})+\frac{h^2}{4}(K_{00}K_{MM}-K_{0M}K_{M0})} - 2,$$

$$a_2 = \frac{h(1-\frac{h}{2}K_{MM})K_{02} + \frac{h^2}{2}K_{0M}K_{M2}}{1-\frac{h}{2}(K_{00}+K_{MM})+\frac{h^2}{4}(K_{00}K_{MM}-K_{0M}K_{M0})} + 1,$$

$$a_j = \frac{h(1-\frac{h}{2}K_{MM})K_{0j} + \frac{h^2}{2}K_{0M}K_{Mj}}{1-\frac{h}{2}(K_{00}+K_{MM})+\frac{h^2}{4}(K_{00}K_{MM}-K_{0M}K_{M0})}, \quad j = 3, \dots, M-1,$$

$$c_j = \frac{\frac{h^2}{2}K_{M0}K_{Oj} + h(1-\frac{h}{2}K_{00})K_{Mj}}{1-\frac{h}{2}(K_{00}+K_{MM})+\frac{h^2}{4}(K_{00}K_{MM}-K_{0M}K_{M0})}, \quad j = 1, \dots, M-3,$$

$$c_{M-2} = \frac{\frac{h^2}{2}K_{M0}K_{0M-2} + h(1-\frac{h}{2}K_{00})K_{MM-2}}{1-\frac{h}{2}(K_{00}+K_{MM})+\frac{h^2}{4}(K_{00}K_{MM}-K_{0M}K_{M0})} + 1,$$

$$c_{M-1} = \frac{\frac{h^2}{2}K_{M0}K_{0M-1} + h(1-\frac{h}{2}K_{00})K_{MM-1}}{1-\frac{h}{2}(K_{00}+K_{MM})+\frac{h^2}{4}(K_{00}K_{MM}-K_{0M}K_{M0})} - 2.$$

where $K_{0j} = K_0(x_j), \quad K_{Mj} = K_1(x_j), \quad j = 0, 1, \dots, M.$
The vector $\mathbf{b}(t)$ has the form

$$\mathbf{b}(t) = \begin{pmatrix} \gamma_0(t) + F_1(t) \\ F_2(t) \\ \vdots \\ F_j(t) \\ \vdots \\ F_{M-2}(t) \\ \gamma_1(t) + F_{M-1}(t) \end{pmatrix}, \tag{7}$$

with

$$\gamma_0(t) = \frac{\frac{h}{2}K_{0M} \quad g_1(t) + (1-\frac{h}{2}K_{MM})g_0(t)}{h^2(1-\frac{h}{2}(K_{00}+K_{MM}) + \frac{h^2}{4}(K_{00}K_{MM}-K_{0M}K_{M0}))},$$

$$\gamma_1(t) = \frac{\frac{h}{2}K_{M0} \quad g_0(t) + (1-\frac{h}{2}K_{00})g_1(t)}{h^2(1-\frac{h}{2}(K_{00}+K_{MM}) + \frac{h^2}{4}(K_{00}K_{MM}-K_{0M}K_{M0}))},$$

$F_j(t) = F(x_j, t), \quad \mathbf{f}_0 = (f_1, \dots, f_{M-1})^T, \quad f_j = u_0(x_j), \quad j = 1, \dots, M-1,$

and $\mathbf{U(t)}$ is of size $(M-1)$ and given by

$$\mathbf{U}(t) = [\mathbf{U_1(t)}, \dots, \mathbf{U_{M-1}(t)}]^{\mathbf{T}},$$

2.2 Reformulation and Keller Method

Consider again the initial boundary value problem (1-3) with $b(x) = 0$. The rectangular $\overline{R} = [0, 1] \times [0, T]$ is subdivided on $R_{hk} = \overline{I}_{hk} \times \overline{J}_{hk}$ where $\overline{I}_{hk}$ denotes the set $\{x_i\}_{i=0}^{N+1}$ where

$$0 = x_0 < x_1 < \ldots < x_{N+1} = 1,$$

and $\overline{J}_{hk}$ denotes

$$0 = t_0 < t_1 < \ldots < t_M = T.$$

Set

$$h_i = x_i - x_{i-1}, \quad i = 1, \ldots, N+1, \quad h = \max_i(h_i),$$

$$k_n = t_i - t_{n-1}, \quad n = 1, \ldots, M.$$

In order to apply the keller box scheme, the problem (1-3) is transformed as follows

$$v(x,t) = a(x)\frac{\partial u}{\partial x},$$

introducing the auxillary functions $W(x,t)$ and $G(x,t)$, defined by

$$\begin{aligned} W(x,t) &= \int_x^1 K_0(s)u(s,t)ds, \\ G(x,t) &= \int_0^x K_1(s)u(s,t)ds, \end{aligned}$$

taking into account these functions, the problem (1-3) becomes

$$\begin{cases} u_t - v_x = f(x,t), & v(x,t) = a(x)\frac{\partial u}{\partial x}, \\ W_x(x,t) = -K_0(x)u(x,t), & G_x(x,t) = K_1(x)u(x,t), \\ u(0,t) = W(0,t) + g_0(t), & u(1,t) = G(1,t) + g_1(t), \\ W(1,t) = G(0,t) = 0, & u(x,0) = u_0(x), \\ v(x,0) = a(x)u_0'(x). \end{cases} \tag{8}$$

Keller Box Scheme (KBS) [10] is applied to the problem (8). Hence, the following system of equations arises :

$$\begin{cases} v_{i-1/2,n} = a_{i-1/2}\nabla u_{i,n}, \nabla v_{i,n-1/2} = \partial_t u_{i-1/2,n} - f_{i-1/2,n-1/2}, \\ \nabla W_{i,n} = -K_{i-1/2,0}u_{i-1/2,n}, \nabla G_{i,n} = K_{i-1/2,1}u_{i-1/2,n}, \\ u_{i,0} = u_0(x_i), v_{i0} = a(x)u_0'(x_i), \\ u_{0n} = W_{0n} + g_0(t_n), u_{Mn} = G_{Mn} + g_1(t_n), \\ W_{Mn} = G_{0n} = 0, \end{cases} \tag{9}$$

where the operators ∇ and ∂_t are given by

$$\begin{cases} x_{i-1/2} = \frac{1}{2}(x_i + x_{i-1}), \quad t_{n-1/2} = \frac{1}{2}(t_n + t_{n-1}), \\ \Phi_{i-1/2,n} = \frac{1}{2}(\Phi_{i,n} + \Phi_{i-1,n}), \quad \Phi_{i,n-1/2} = \frac{1}{2}(\Phi_{i,n} + \Phi_{i,n-1}), \\ \nabla\Phi_{i,n} = \frac{(\Phi_{i,n}-\Phi_{i-1,n})}{h_i}, \quad \partial_t\Phi_{i,n} = \frac{(\Phi_{i,n}-\Phi_{i,n-1})}{k_n}, \end{cases} \tag{10}$$

The resulting linear system is of the form

$$L_i^n \overrightarrow{\omega}_{i-1,n} + R_i^n \overrightarrow{\omega}_{i,n} = \overrightarrow{b}_{i,n}.$$

where $\overrightarrow{\omega}_{i,n} = \begin{pmatrix} u_{i,n} \\ v_{i,n} \\ W_{i,n} \\ G_{i,n} \end{pmatrix}$, $\overrightarrow{b}_{i,n} = \begin{pmatrix} 0 \\ b_{i,n} \\ 0 \\ 0 \end{pmatrix}$ and

$$L_i^n = \begin{pmatrix} -1 & -\dfrac{h_i}{2a_{i-1/2}} & 0 & 0 \\ -\dfrac{h_i}{k_n} & -1 & 0 & 0 \\ 1 & 0 & -\dfrac{2}{h_i K_{i-1/2,0}} & 0 \\ 1 & 0 & 0 & \dfrac{2}{h_i K_{i-1/2,1}} \end{pmatrix},$$

$$R_i^n = \begin{pmatrix} 1 & -\dfrac{h_i}{2a_{i-1/2}} & 0 & 0 \\ -\dfrac{h_i}{k_n} & 1 & 0 & 0 \\ 1 & 0 & \dfrac{2}{h_i K_{i-1/2,0}} & 0 \\ 1 & 0 & 0 & -\dfrac{2}{h_i K_{i-1/2,1}} \end{pmatrix},$$

taking into account the boundary conditions $u_{0,n} - W_{0,n} = g_0(t_n)$, $u_{N+1,n} - G_{N+1,n} = g_1(t_n)$, $W_{N+1n} = G_{0,N+1} = 0$ and set $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\overrightarrow{g}_{0n} = \begin{pmatrix} g_{0n} \\ 0 \end{pmatrix}$, $\overrightarrow{g}_{1n} = \begin{pmatrix} g_{1n} \\ 0 \end{pmatrix}$.

The final linear system is given by

$$\mathcal{A} = \begin{pmatrix} A & & & & \\ L_1^n & R_1^n & & & \\ & L_2^n & R_2^n & & \\ & & \ddots & \ddots & \\ & & & L_{N+1}^n & R_{N+1}^n \\ & & & & B \end{pmatrix} \begin{pmatrix} \overrightarrow{\omega}_{0,n} \\ \overrightarrow{\omega}_{1,n} \\ \vdots \\ \vdots \\ \overrightarrow{\omega}_{N,n} \\ \overrightarrow{\omega}_{N+1,n} \end{pmatrix} = \begin{pmatrix} \overrightarrow{g}_{0,n} \\ \overrightarrow{b}_{1,n} \\ \vdots \\ \vdots \\ \overrightarrow{b}_{N+1,n} \\ \overrightarrow{g}_{1,n} \end{pmatrix}.$$

2.3 Orthogonal spline collocation method (OSC)

Consider the quasilinear parabolic problem

$$\left\{ \begin{array}{c} u_t - u_{xx} = F(u, x, t) \\ 0 \leq x \leq 1 \end{array} \right. \tag{11}$$

subject to the initial condition (3) and the nonlocal boundary conditions (2).

Let be $(\delta_x) = (x_j)_{j=0}^J$ a partition of [0,1] such that $0 = x_0 < x_1 < ... < x_J = 1$, where $h = max_j(h_j)$, $h_j = x_j - x_{j-1}$ and $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq J$, introduce the family of partitions $(\delta_x)_{x \in H}$ where H is a set of positive numbers with $inf(H) = 0$. Let S_h denote the space of piecewise polynomials of degree r defined by

$$S_h = \{v/v \in C^1[0, 1]and v/I_j \in \mathbb{P}_r(I_j), 1 \leq j \leq J\},$$

and S_h^0, the subspace of S_h, given by

$$S_h^0 = \{v \in S_h/V(0) = V(1) = 0\}.$$

Where $\mathbb{P}_r(E)$ denotes the set of polynomials of degree at most r.

Let $\{\xi\}_{l=1}^{r-1}$ be the Gauss-Legendre points in the interval I with corresponding weights $\{\omega\}_{l=1}^{r-1}$. Let

$$\varrho(\delta_x) = \{\zeta_{jl}\}_{j,l=1}^{J,r-1}$$

be the set of Gauss points in the x-direction, where

$$\zeta_{jl} = x_{j-1} + h_j \xi_l, \quad j = 1, ..., J, l = 1, ..., r - 1.$$

define the discrete inner products and norms by

$$< f, g >_j = h_j \sum_{l=1}^{r-1} \omega_l f(\zeta_{jl}) g(\zeta_{jl}),$$

$$< f, g > = \sum_{j=1}^{J} < f, g >_j, |f|_j^2 = < f, f >_j, \|f\|^2 = \sum_{j=1}^{J} |f|_j^2.$$

The continuous-time approximation to the solution u of (11-3) is a differentiable map $U^h : [o, T] \to S_h$, such that

$$U_t^h(\xi, t) - U_{xx}^h(\xi, t) = F(\xi, t, U^h), \quad \xi \in \varrho(\delta_x), \quad 0 \leq t \leq T \tag{12}$$

$$U^h(0, t) = < K_0, U^h(t) > +g_0(t), \quad U^h(1, t) = < K_1, U^h(t) > +g_1(t), \quad 0 \leq t \leq T \tag{13}$$

with $U^h(.,0)$ a suitable approximation to u_0. We also consider a variable time step Crank-Nicholson OSC method. Let $\delta_t = \{t_n\}_{n=0}^N$ denote a partition of $[0,T]$, with $0 = t_0 < t_1 < ... < t_N = T$.

Let $\Delta t = \max_n(\Delta t_n)$ where $\Delta t_n = t_n - t_{n-1}$, then the Crank-Nicholson method is defined by a set $\{U_n^h\}_{n=0}^N \in (S_h)^{N+1}$, such that

$$\frac{U_{n+1}^h(\xi) - U_n^h(\xi)}{\Delta t_{n+1}} - U_{n+1/2}^h(\xi) = F(\xi, t_{n+1/2}, U_{n+1/2}^h(\xi)), \quad \xi \in \varrho(\delta_x), \tag{14}$$

$0 \leq n \leq N-1$, where $t_{n+1/2} = (t_{n+1} + t_n)/2$, and $U_{n+1/2}^h = (U_{n+1}^h + U_n^h)$, with nonlocal boundary conditions

$$U_{n+1}^h(0) =< K_0, U_{n+1}^h > + g_0(t_{n+1}),$$

$$U_{n+1}^h(1) =< K_1, U_{n+1}^h > + g_1(t_{n+1}), \quad 0 \leq n \leq N-1 \tag{15}$$

2.4 Galerkin Method

Let be $(\pi_h) = (x_j)_{j=0}^J$ a partition of [0,1] such that $0 = x_0 < x_1 < ... < x_J = 1$, where $h = max_j(h_j)$, $h_j = x_j - x_{j-1}$ and $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq J$, we consider a family of partitions $(\pi_h)_{h \in H}$ where H is a set of positive numbers with $inf(H) = 0$, we define the space S_h in the following way

$$S_h = \{v/v \in C^n[0,1] and v/I_j \in \mathbb{P}_r(I_j), 1 \leq j \leq J\},$$

and S_h^0, the subspace of S_h, given by

$$S_h^0 = \{v \in S_h/V(0) = V(1) = 0\}.$$

Where $\mathbb{P}_r(E)$ denotes the set of polynomials of degree at most r, $r \geq 2$ for a closed interval E and $n < r$.

We denote by k the time step, $t_m = mk, m = 0, ..., M = [T/k]$, $t_{m+\theta} = (m+\theta)k$, and $(.,.)$ the usual inner product on $L^2[0,1])$.

The θ-Galerkin approximation (θ-method), $(U^m)_{m=0}^M \subset S_h$ to the solution of (1)-(3) is defined by

$$\left(\frac{U^{m+1} - U^m}{k}, v\right) + \left(aU_x^{m+\theta}, v_x\right) - \left(bU_x^{m+\theta}, v\right) - \left(F\left(U^{m+\theta}, t_{m+\theta}\right), v\right) = 0, v \in S_h^\circ, \tag{16}$$

$0 \leq m \leq M-1$, where $U^{m+\theta} = \theta U^{m+1} + (1-\theta)U^m$. With the nonlocal boundary conditions

$$\begin{aligned} U^{m+1}(\alpha) &= \textstyle\int_\alpha^\beta K_0(x) U^{m+1}(x)dx + g_0(t_{m+1}), \\ U^{m+1}(\beta) &= \textstyle\int_\alpha^\beta K_1(x) U^{m+1}(x)dx + g_1(t_{m+1}), \end{aligned} \tag{17}$$

$0 \le m \le M-1$, $U^0 \in S_h$ will be specified later.
we define the norms $\|.\|$ and $\|.\|_\infty$, by

$$\|v\| = \sqrt{(v,v)}, \quad \|v\|_\infty = \max_{\alpha \le x \le \beta} |v(x)|.$$

Let $p_h : H_0^1(I) \longrightarrow S_h^\circ$ be the operator defined by

$$(a(z - p_h(z))_x, v_x) - (b(z - p_h(z))_x, v) = 0, \quad v \in S_h^\circ, \tag{18}$$

we note that

$$\|z - p_h(z)\|_\infty \le C\|z\|_{W^{r,\infty}(0,1)}, \quad z \in H_0^1(I), \tag{19}$$

where $H_0^1(I)$ is the usual Sobolev space on I, $\|v\|_{W^{r,\infty}(0,1)} = \sum_{i=0}^{r} \|v^{(i)}\|_\infty$ (voir [5]).
We define on S_h° the linear form

$$\begin{aligned} &\Gamma^{m+\theta}(V,W)(\chi) = \left(\tfrac{V-W}{k}, \chi\right) + (a(\theta V_x + (1-\theta)W_x), \chi_x) \\ &- (b(\theta V_x + (1-\theta)W_x), \chi) - (F(\theta V + (1-\theta)W, t_{m+\theta}), \chi)\,, \end{aligned} \tag{20}$$

and the operators L_0 and L_1

$$L_0(V) = V(\alpha) - \int_\alpha^\beta K_0(x)V(x)dx, \quad L_1(V) = V(\beta) - \int_\alpha^\beta K_1(x)V(x)dx.$$

We denote

$$X_{hk} = (S_h)^{M+1} \;\text{ et }\; Y_{hk} = S_h \times ((S_h^\circ)^*)^M \times \mathbb{R}^M \times \mathbb{R}^M$$

where $(S_h^\circ)^*$ is the dual space of S_h°. clearly

$$dim(X_{hk}) = dim(Y_{hk}).$$

We also define the operator

$$\Phi_{hk} : X_{hk} \longrightarrow Y_{hk},$$

by

$$\Phi_{hk}(V^0, ..., V^M) = \Bigg(U^0 - V^0, \left(\Gamma^{m-1+\theta}(V^m, V^{m-1})\right)_{m=1}^M,$$
$$\left(L_0\left(\frac{V^m - V^{m-1}}{k}\right) - \frac{g_0(t_m) - g_0(t_{m-1})}{k}\right)_{m=1}^M,$$

$$\left(L_1\left(\frac{V^m - V^{m-1}}{k}\right) - \frac{g_1(t_m) - g_1(t_{m-1}}{k}\right)_{m=1}^{M}\Bigg), \tag{21}$$

given an initial condition $U^0 \in S_h$ that satisfies

$$L_0(U^0) = g_0(\alpha), \quad L_1(U^0) = g_1(\alpha). \tag{22}$$

It is easy to show that an element $(U^0, ..., U^M) \in (S_h)^{M+1}$ is a solution of the θ-method if and only if

$$\Phi_{hk}(U^0, ..., U^M) = 0, \tag{23}$$

The existence and convergence were treated in [2].

3 Numerical comparison of the methods

The four numerical methods were tested on the following problem

$$u_t + u_{xx} = -e^{-t}\left\{x(x-1) + \frac{\delta}{6(1+\delta)} + 2\right\}, x \in (0,1), 0 < t \leq T,$$

$$u(0,t) = u(1,t) = -\delta\int_0^1 u(x,t)dx, t > 0,$$

$$u(x,0) = x(x-1) + \frac{\delta}{6(1+\delta)}, x \in [0,1].$$

Where $|\delta| < 1$ and the exact solution, $u = u(x,t)$, is given by

$$u(x,t) = e^{-t}\left(x(x-1) + \frac{\delta}{6(1+\delta)}\right)$$

N=M	Finite differences	Galerkin($\theta = 1/2$)	Keller	OSC
4	410^{-4}	$2.2\ 10^{-4}$	2.110^{-3}	1.110^{-3}
8	2.510^{-4}	7.110^{-5}	6.10^{-4}	3.210^{-4}
16	1.310^{-4}	2.410^{-5}	1.710^{-4}	1.110^{-4}
32	7.10^{-5}	3.510^{-6}	7.810^{-5}	7.10^{-5}
64	3.510^{-5}	1.910^{-6}	6.810^{-5}	6.810^{-5}
128	2.110^{-5}	6.310^{-7}	6.710^{-5}	6.210^{-5}

Table 1: maximum errors

N=M	Finite differences	Galerkin($\theta = 1/2$)	Keller	OSC
4	.73	.86	.77	.17
8	.96	.91	.82	.48
16	1.3	9.8	1.4	1.75
32	7.5	5.2	3.2	6.4
64	800	600	11.5	38
128	5000	1000	100	150

Table 2: Cpu time

The same pattern was globally obtained when the four methods were tested on other problems with nonlocal boundary conditions [4].

4 Discussion and Conclusion

The present study is devoted to the comparison of numerical methods used to solve boundary-value problems with non local boundary conditions. Four numerical methods are considered, namely, Finite Differences Method (FDM), Galerkin Method (GM), Keller's Method (KM) and Orthogonal Spline Collocation Method (OSCM). The details concerning each method and the theoretical results yielded can be found in the corresponding references given in the previous sections. In this paper, we restrict ourselves to the comparison of the numerical results given by the four methods applied to the same problem.

The Finite Differences Method is seen to be easier to use and adaptable with respect to increasing the accuracy either by increasing the number of knots or by extrapolations. It has, however, the drawback of giving approximations at the knots point only and the fact that the discretisation step is generally constant.

The Keller's Method offers an attractive alternative to solve boundary value problems with non local conditions, avoiding the approximation of integrals and leading to the use of Almost Block Diagonal Matrices. Similarly, Orthogonal Spline Collocation Method seems to be an elegant method yielding results that are comparable to those given by the Keller's method, the two allowing for the approximation of the function and its dervative.

Finally, at a higher cost, the results given by the Galerkin method are more accurate than those given by the three other methods.

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