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A Systemic Approach to Logistics System Design

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Abstract

A general model for a transportation/distribution system (composed of a distribution center, a number of destinations and a fleet of vehicles) and a solution technique is presented. The model incorporates operating policies into design of physical structure together with groupings of destinations. Design variables are storage capacities at destinations and vehicle capacities. Operating policies refer to selection of vehicle routing patterns. The objective function to be minimized is the cost of owning and operating the system per unit time. A solution to the model includes a nonlinear set partitioning problem as the most difficult part. Dynamic programming, although theoretically appealing, can solve this problem but proves to be very time consuming. Therefore, a very efficient heuristic algorithm is developed. The model and its solutions are illustrated on an example transportation/distribution system.

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1 Introduction

A general model of a transportation/distribution system arising from real life operations is developed for a systemic analysis and decision–making purpose. A transportation/distribution system is composed of a distribution center (a manufacturing center or a central depot), a number of destinations (demand points) and a fleet of vehicles to transport and deliver several products or an aggregate of products from the center to the destinations. The destinations agree and obey the distribution, transportation and inventory policy established for them by the distribution and transportation management. The distribution center supplies the product and is characterized by the loading rate of the products to the vehicles, its supply rate, and its storage capacity. Each destination is represented by its demand rate, its storage capacity, and transport distances to the distribution center and other destinations. Similarly, the relevant properties of a vehicle are its hauling capacity, its cruising speed, and its unloading rate of the product to the storage facilities at a destination.

Transportation/distribution systems have been the subject of research in Management Science/Operations Research for four decades. The pioneering work of Ramser and Dantzig [24] tackles the most restricted operational problem of such systems called "Truck Dispatching". It is concerned with the delivery of a product in ordered quantities to destinations by trucks on a daily basis. Clarke and Wright [10] extended their formulation and proposed an efficient heuristic algorithm for the optimum routing problem. These initial works established guiding examples of the subsequent research, and opened an avenue of research, labeled later as the vehicle routing problem (VRP). VRP is briefly defined as a problem of minimizing distances of routes for delivery vehicles to supply a number of destinations from a center or multiple centers. Several excellent articles on VRP, its extensions and solution techniques exist in the literature.

Incorporating the transportation and inventory holding costs into the objective function extends the scope of the operational problems. Burns et al. [7] compares direct shipping to peddling (trucks delivering items to more than one customer), subsequently the total cost includes the total transportation and inventory holding costs. Muckstadt and Roundy [22] develops a mathematical model for planning and shipping for each of the items stocked at the warehouse and N retailers. Their total cost is the fixed cost of placing an order, cost of shipments and the receipt at the retailers. An integrated approach towards inventory control and transportation planning was proposed by Anily and Federgruen [2]. Their model requires partitioning (based on a property called "*monotone*") the set of destinations to L subsets or regions (number of vehicles). They produced easily computable lower and upper bounds for the minimal system–wide costs. Hall [18] notices that their objective function underestimates the benefit of assigning a retailer to multiple tours. Anily and Federgruen [3] respond by discussing their understanding of loss of optimality incurred by the partitioning method. A further development by Anily and Federgruen [4] includes a variant of their previous model involving vehicle routing costs and central inventories and a structured partitioning method is introduced to solve this model. Structured partitioning refers to a partitioning based on the ordering of the elements of the set of destinations and/or partitioning a set subject to a predetermined number of parts in the partition. Further Anily and Federgruen [5] describe how to combine deliveries into efficient routes. The total cost function includes the fixed cost of placing an order and the inventory holding cost. Replenishment strategy is to serve all the retailers in a region in an efficient route when a shipment is sent to that region. Mixed integer formulation of Federgruen and Zipkin [14] combine vehicle routing with inventory allocation problem. Ronen [25] develops a mixed 0–1 integer programming formulation for short–term scheduling of vessels where objective function includes unit shipping costs, cost of unloading time, and cost of taking a specific route. Laporte et al [21] incorporate a design feature into the problem by considering locating distribution centers within a given set of sites and constructing delivery routes at a minimum cost on an integer linear programming model. Burns et al [7] present an analytical model where the cost only includes dispatching cost, fixed cost at a customer and cost of traveling distance, but not the inventory costs. Federgruen and Simchi–Levi [13] mention integrating inventory control and vehicle routing. Vidal and Goetschalckx [28] give a review of production–distribution models concentrating on mixed integer programming models.

Herer and Roundy [19] observe that an important aspect of an inventory/distribution problem is the significant amount of delivery costs. Bramel and Simchi–Levi [8] describe an integrated logistics model that considers the long–run average transportation and inventory costs. A case study of a fuel oil distributor is described by Campbell and et al [9]. The objective is to minimize the average daily distribution cost without causing stockouts at any of the customers. The policy determines a schedule for serving each customer, the delivery quantity, and the delivery route.

Viswanathan and Mathur [29] propose a model for integrating routing and inventory decisions in a one–warehouse multiproduct distributing system. Long–run average inventory and transportation costs are minimized by a heuristic solution method. The products are delivered from the warehouse to the retailers by vehicles that combine deliveries to several retailers into efficient vehicle routes (VRP). This work may be considered one step towards a more general transportation and distribution system but it does not take the design variables into consideration.

All in all, these publications revolve around VRP. One or two missing elements of a real transportation and distribution system have been introduced from time to time, but all of these works have fallen short of comprehending all significant aspects of a TDS systemically (wholisticly). A wholistic design certainly requires other design features such as storage capacities together with operational variables as decision variables. Thus the work we present here is not another research on VRP, but a systems approach to designing a transportation/distribution system (TDS) in which VRP is a subproblem (an insignificant subproblem of TDS). Here we particularly emphasize the necessity of taking into consideration the interaction between structural design and operating policies.

The main thrust and contribution of this article is to develop a model for a transportation and distribution system that includes the design variables, operational variables and the interactions between these two sets of variables. This is in a way extending the previous research into a more general model for the transportation and distribution systems. Obviously, the design of the system affects the operational policy, and conversely operational characteristics such as frequency of deliveries, vehicle routing, etc, and the interactions among them need to be considered in designing such a system. Hence the structural design variables are optimized together with the following operational decision variables:

i) Partitioning the set of destinations such that a delivery is made to each part by one vehicle trip.

ii) Frequency of delivery to each part.

iii) Size of shipment to each destination in each part.

iv) Routing of the vehicles for each trip (traveling salesperson problem).

A general model for such a problem is constructed and presented in Section 2. The objective function is the (variable) cost of owning and operating the transportation and distribution system per unit time, and is composed of capital and inventory holding costs, vehicle loading and unloading costs and the cost of transport to the groups of destinations. Inclusion of capital and inventory holding costs, vehicle loading and unloading costs, the cost of transport to the groups of destinations relates to the design and operational features in the model. One set of constraints represents the fact that the amount of shipment to a group cannot exceed the vehicle capacity. Since the number of groups is not known in advance, the number of constraints varies in the model.

Possible solution techniques are discussed in Section 3. An interesting aspect of the solution method is a nonlinear set partitioning problem. A survey of past research clearly indicates that set partitioning is an intrinsic and inherent aspect of the transportation and distribution system (TDS) representing an important set of operational variables. The partitioning problem in this article is quite different from the structured set partitioning problem of Anily and Federgruen [3] or [4] and from that of Fisher and Jaikumar [15] who propose a heuristic for determining cluster–first route–second for solving the set partitioning problem. Beasley and Christofides [6] basically build parts of a partition along a vehicle route. Introduction of a nonlinear set partitioning problem is another contribution of this article to the existing literature.

An exhaustive search for set partitioning is prohibitive (if not impossible) for even small size problems. Dynamic programming provides another possible approach to set partitioning problem. Rosenblatt and Kaspi [26] propose a dynamic programming algorithm to partition a set of items into groups. The error in their method was detected and improved dynamic programming algorithms were proposed by Queyranne [23], Goyal [17], and Schwartz [27]. In addition, Queyranne provides an estimate for the number of arithmetic operations and space requirements for his dynamic programming algorithm ($O(3^n)$ operations and $O(2^n)$ space with n elements). Unfortunately, even moderate size problems cannot be handled by this approach. Taking this difficulty into account, we here develop an efficient heuristic algorithm to find near optimal solutions quickly and easily to the nonlinear set partitioning problem. A partial proof to this heuristic algorithm is provided by means of Optimality Principle of dynamic programming. There is no other known method (other than exhaustive search or dynamic programming) for solving such a nonlinear set partitioning problem. Hence, introduction of an efficient heuristic method for solving nonlinear set partitioning problem becomes a further contribution of this research to the existing literature. Also, investment decisions for the future of the transportation and distribution system require acquiring vehicles of operationally meaningful optimal capacity. Optimization of the vehicle capacity is outlined in Section 4.

Section 5 presents an implementation of the model and the solution method. A real life application of an earlier version of the model and its solution method was attempted by Doğrusöz [11], and [12]. Later a refined version of the model and solution method was implemented to a real life problem by Yüceer [30]. An adaptation of this problem forms the basis for the example problem of a transportation distribution system (TDS) in this article. The numerical experimentation confirms the computational efficiency and illustrates clearly the ease of obtaining good solutions with the proposed heuristic algorithm. The concluding section summarizes the findings of this research together with the results of numerical experimentation.

2 Model Construction

In this section a general model and its variants will be developed for a TDS containing a single source that supplies a single or an aggregate of the products, a set of destinations, and a fleet of identical vehicles for transporting of this product from the source to the destinations. The facilities at the destinations are assumed to have identical loading/unloading characteristics.

The model that will be constructed and presented below can be classified, in general terms, as a deterministic mathematical programming model. The objective is the minimization of the total cost (capital cost, vehicle loading, unloading and transport, and customers storage costs and inventory holding costs) per unit time, and the following design and operational decision variables are under consideration.

1. Fixed groups of customers to each of which deliveries are made in one vehicle trip.
2. Vehicle routing or more specifically TSP.
3. Replenishment cycle length to each group and subsequently replenishment quantity to each destination in each group.
4. Storage capacities of customers.
5. Vehicle size.

Variables involved are listed and defined in the following subsections.

2.1 Uncontrollable variables

These are the variables whose values and/or geographic locations are assumed uncontrollable but whose values are known (given or estimated) and inherent to the TDS. There are m destinations (customers or demand points) that are supplied from a single source.

a_0 = the source (a manufacturing center or a depot).

$A = \{a_1, a_2, \ldots, a_m\}$ = the set of destinations.

$I = \{1, 2, \ldots, m\}$ = index set of elements of A.

$I_0 = I + \{0\}$.

$s(a)$ = demand rate of the product at a destination $a \in A$.

d_{ij} = distance between a_i and a_j, $i, j \in I_0$ ($d_{ij} = d_{ji}$ is assumed).

h = combined unit cost of owning storage facilities (capital cost) and holding inventories at customer sites.

2.2 Design and operational variables

These are the variables which pertain the design and operational features of a TDS.

C = hauling capacity of a vehicle.

A_j = a subset of the set A representing a collection of destinations to be supplied in a single trip of the vehicle, $A_j \subseteq A$. (Also a collection, a group, a part of the partition.)

t_j = replenishment cycle length for the subset A_j.

δ_j = vehicle route for group A_j.

$R(a)$ = buffer stock (safety stock) level at every $a \in A$.

$q(a) = t_j s(a) + R(a)$ = storage capacity at $a \in A_j \subseteq A$.

$R(a)$ is predetermined outside the model, $q(a)$ is expressed as a function of t_j. For notational convenience, A_j's and t_j's are represented compactly by the following notation.

$J = \{1, 2, \ldots, n\}$ where $1 \leq n \leq m$.

$P = (A_1, A_2, \ldots, A_n)$ = a partition of the set A, where A_j is a part (a subset of A) and $A_i \cap A_j = \emptyset$ for all $i \neq j \in J$, and $\cup_{j \in J} A_j = A$. In this notation, the number n represents the number of parts in the partition P and $1 \leq n \leq m$.

$T = (t_1, t_2, \ldots, t_n)$ = replenishment cycle vector corresponding to the partition P.

2.3 Consequence Variables

These are the variables whose values are determined or calculated as a consequence of the choices on the decision variables.

$s(A_j) = \sum_{a \in A_j} s(a)$ = demand rate of the subset A_j for $j \in J$.

$s(A) = \sum_{j \in J} s(A_j) = \sum_{a \in A} s(a)$ = total demand rate of the system.

$\beta_1(C)$ = fixed setup time for loading of a vehicle of capacity C at the source.

β_2 = loading time per unit quantity of the product at the source.

τ_1 = fixed setup time (including waiting or other times if necessary) for unloading of a vehicle at a destination.

τ_2 = unloading time per unit quantity of the product from the vehicle to the customer's storage facility.

$k_1(C)$ = cost of owning (capital cost) and operating a vehicle of capacity C per unit time.

$k_2(C)$ = unit mileage cost of a vehicle of capacity C (e.g. $ per mile).

$d(A_j)$ = traveling salesperson distance traveled by the vehicle through route δ_j for $j \in J$.

In principle, β_1, τ_1 and τ_2 could be viewed dependent on hauling capacity C and the destination $a \in A$. For all practical reasons, however, it is not unreasonable to assume them not affected by C and $a \in A$. In fact, β_2 is a feature of the loading facility at the source (distribution center) rather than the vehicle, and unloading facilities at the destinations and/or vehicles can be assumed identical. Consequently, β_2, τ_1 and τ_2 are assumed constant.

2.4 Computation of Total Cost

The objective function is the cost of owning (capital costs) and operating the distribution system per unit time (excluding the fixed costs). This is the sum of four cost components; (*i*) the total cost of loading the vehicles at the source per unit time, (*ii*) the total cost of transporting the product from source to the destinations in each group (part) over all groups per unit time, (*iii*) the total cost of unloading the appropriate replenishment quantity at each destination in each group over all groups per unit time, and (*iv*) the total cost of owning storage facilities and holding inventories at the destinations per unit time.

The total cost of loading the vehicle for all parts is the sum of the costs of loading the vehicle for each part. The loading time of the vehicle for a part is the sum of the fixed setup time at the source and the time to load $t_j s(A_j)$ amount of the product for the subset A_j and is equal to $\beta_1(C) + \beta_2 t_j s(A_j)$.

Then the total cost of loading the vehicle for all parts per unit time is given as follows.

$$f_1 = k_1(C) \sum_{j \in J} \left(\beta_2 s(A_j) + \frac{\beta_1(C)}{t_j} \right) \tag{1}$$

The total transport (mileage) cost per unit time is a function of the total distance traveled and is given below.

$$f_2 = k_2(C) \sum_{j \in J} \frac{d(A_j)}{t_j} \tag{2}$$

The total unloading cost of the vehicle of capacity C is the sum of the unloading costs of the product at the destinations of each group. The unloading time of the product at a destination is the sum of a setup time τ_1 and unloading time $\tau_2 s(a) t_j$ of quantity $s(a)t_j$ which yields $(\tau_1 + \tau_2 s(a) t_j)$ where $a \in A_j$, $j \in J$. Then the total unloading cost per unit time is obtained as the sum of the terms $(\tau_1/t_j + \tau_2 s(a))$ over all $a \in A_j$ and given next.

$$f_3 = k_1(C) \sum_{j \in J} \sum_{a \in A_j} \left(\tau_2 s(a) + \frac{\tau_1}{t_j} \right) \tag{3}$$

If h' is the unit holding cost and h'' is the unit cost of owning the storage facility at A_j, then $h = h'' + 0.5h'$. The delivery quantity to the part A_j is $t_j s(A_j)$, then the total cost of owning storage facilities and holding inventories at destinations per unit time is given by the following expression.

$$f_4 = \sum_{j \in J} h t_j s(A_j) \tag{4}$$

After some algebraic manipulations the objective function is expressed as a function of decision variables P, T, and C by the following function. An important characteristic of the objective function is the consideration of only the variable operating costs per unit time.

$$f(P,T,C) = k_1(C)\,(\beta_2 + \tau_2)\, s(A) + \\ + \sum_{j \in J} \left\{ \frac{1}{t_j} \left[k_1(C) \left(\beta_1(C) + \sum_{a \in A_j} \tau_1 \right) + k_2(C) d(A_j) \right] + h t_j s(A_j) \right\} \tag{5}$$

The total amount of shipment cannot exceed the vehicle capacity C, hence the following constraint must be satisfied.

$$t_j s(A_j) - C \leq 0 \text{ for all } j \in J \tag{6}$$

A general mathematical model is expressed as follows:

$$\min f(P,T,C)$$

subject to

$$t_j s(A_j) - C \leq 0 \text{ for all } j \in J$$

$$t_j \geq 0 \text{ for all } j \in J \tag{7}$$

This general model attempts to optimize P, T, and C simultaneously. The index set J is not known a priori. The number of constraints (6) and (7) varies with the partition P, the replenishment vector T, and the vehicle capacity C. It is not practical to solve the model in this form. Several manageable versions of the model will be presented by selecting some operational and design features as decision variables and the remaining as the parameters in the model.

2.5 Variant 1. Fixed hauling capacity

An efficient operation and improved design of storage capacities of the physical distribution system is sought for a fixed hauling capacity. In this case, $k_1(C) = k_1$, $k_2(C) = k_2$, $\beta_1(C) = \beta_1$ and the first term of (5) are constants for a fixed C. Accordingly, the model is restated, after dropping off the constant term, as follows.

$$\min f(P,T) = \sum_{j \in J} \left\{ \frac{1}{t_j} \left[k_1 \left(\beta_1 + \sum_{a \in A_j} \tau_j \right) + k_2 d(A_j) \right] + h t_j s(A_j) \right\} \tag{8}$$

subject to

$$t_j s(A_j) \leq C \text{ for all } j \in J \tag{9}$$

$$t_j \geq 0 \text{ for all } j \in J$$

The storage requirement at $a \in A_j$ is equal to $q(a) = t_j s(a) + R(a)$ for $j \in J$ where $R(a)$ is the predetermined buffer stock level (any method in the literature may be used for this purpose).

2.6 Variant 2. Fixed storage and hauling capacities

The capital charges will be ignored in this case and $h = 0.5h'$ is simply the inventory holding cost. If $q(a)$ is the current storage capacity at $a \in A$, then the objective function in (8) will be minimized subject to (7), (9) and the following additional constraint.

$$t_j s(a) \leq q(a) \text{ for } a \in A_j,\ j \in J \tag{10}$$

3 Solution Methods

There is a high degree of interaction between the variables P, T, and C. If partition P changes, the replenishment cycles for each destination will change. Conversely, changing replenishment cycles for destinations will cause the partition P to change. Partition P can also affect the hauling capacity of the vehicle C (if treated as a variable), and conversely C is the main determinant of the partition P. This phenomenon can be observed simply by noticing the changing number of the constraints. The objective function attains its minimum only if $d(A_j)$ is minimum for each $j \in J$ of a given partition. This requires solving the traveling salesperson problem for each part in the partition. Any method in the literature may be employed for this purpose.

It is intuitively obvious that the hauling capacity of the vehicle should be fully utilized. This means that the optimal solution must satisfy $t_j s(A_j) = C$ for all $j \in J$. This can be proved very easily.

This condition implies that T is a function of C, i.e. $T = T(C)$. Furthermore, the optimal vehicle capacity will be obtained recursively in two steps. The following lemma is borrowed from calculus.

Lemma 1

$$\min_{(P,C)} f(P,C) \;=\; \min_{(C)} \left\{ \min_{(P)} f(P,C) \right\} \tag{11}$$

The subproblem of minimizing $f(P,C)$ for a given vehicle capacity is a partitioning problem, but not a structured partitioning problem as described by Anily and Federgruen [4] and not ordered optimal partition problem of Hwang [20]. Partitioning makes the problem a discrete optimization problem. The number of all partitions of even a small set of 10 elements is 115975. The number of all possible partitions of a set of m elements is equal to $\sum_{k=1}^{m} \mathcal{S}_m^{(k)}$ where $\mathcal{S}_m^{(k)}$ is a Stirling number of the second kind and corresponds to the number of ways of partitioning a set of m elements into k non–empty subsets, Abramowitz and Stegun [1]. The substitution of $t_j s(A_j) = C$ for all $j \in J$ transforms the problem into a partitioning problem only for a given C. Let $f'(P) = f(P,C)$ for a given C.

$$\min f'(P) = \frac{k_1 \beta_1 s(A)}{C} + \sum_{j \in J} \left\{ \frac{s(A_j)}{C} \left[k_1 \sum_{a \in A_j} \tau_1 + k_2 d(A_j) \right] + hC \right\} \tag{12}$$

Each term of the summation in (12) represents the cost of owning and operating the part A_j per unit time for a given vehicle capacity C and is reexpressed separately for each part A_j for $j \in J$ as follows.

$$F(A_j) = \frac{s(A_j)}{C}\left(k_1 \sum_{a \in A_j} \tau_1 + k_2 d(A_j)\right) + hC \tag{13}$$

Then the total cost expression (12) is rewritten as follows.

$$f'(P) = \frac{k_1 \beta_1 s(A)}{C} + \sum_{j \in J} F(A_j) \tag{14}$$

Dynamic programming is one way of obtaining the optimal partition. Let H be a subset of A, and $g_k(H)$ be the minimum cost partition of the elements in H in the stage k with the initial condition $g_0(\emptyset) = 0$. Then the following functional equation for every $a \in A \backslash H$ can be used in determining an optimal partition (where $A \backslash H$ means all the elements of A which do not belong to H).

$$g_{k+1}(H + \{a\}) = \min\{F(H + \{a\}), g_k((H + \{a\}) \backslash D) + F(D) \mid D \subset H + \{a\}\} \tag{15}$$

Stage k computes the function $g_k(H)$ for subsets of H of k elements for $k = 1, 2, \ldots, n$. The algorithm terminates when $k = n$. The algorithm generates only the dominating partitions of A. Storage and memory requirements for this approach increase exponentially and hence even the moderate size problems cannot be solved.

A heuristic procedure is here developed to obtain near optimal solutions quickly and efficiently. If a part A_j has r elements, then the average variable cost of owning and operating per destination of A_j is given by the following.

$$G(r) = \frac{F(A_j)}{r} \text{ for each } j \in J \tag{16}$$

Intuitively, if adding another destination to A_j reduces the average variable cost, then a better performing part is obtained by adding this destination. This observation suggests a way of forming the groups.

Proposition 1 *The average variable cost per destination $f'(P)/m$ is minimum for an optimal partition P^o.*

A simple corollary of this proposition is that $f'(P)/m$ can be expressed as a weighted average of the costs of owning and operating parts A_j per unit time (excluding the constant term of (16)).

$$\frac{f'(P)}{m} = \sum_{j \in J} \frac{r_j}{m} \frac{F(A_j)}{r_j} + \frac{k_1 \beta_1 s(A)}{mC}$$

where r_j is the number of destinations in part A_j, $\sum_{j \in J} r_j = m$ and r_j/m is the weight associated with the part A_j for $j \in J$. Each destination can form a group by itself. That is, one most elementary solution is to have a part $A_i = \{a_i\}$ for $i = 1, 2, \ldots, m$. The total cost of this solution is an upper bound for the minimum value of the objective function, i.e. the average cost of supplying each destination with one vehicle tour is an upper bound for the minimum value of the average variable cost per destination. In fact, Gallego and Simchi–Levi [16] point out that direct shipping policies are within 6% of optimality.

Let s represent the average demand rate for the elements of r–member group A_j, then $s(A_j) = rs$ and substituting this in (16) yields the following equation and an algebraic manipulation gives the expression (17) for the average variable cost where $d(A_j)$ is the minimum traveling distance in a tour to A_j. This means solving the traveling salesperson problem for each part.

$$G(r) = \left(\frac{rs}{C}\left(k_1 r \tau_1 + k_2 d(A_j)\right) + hC\right)/r$$

$$G(r) = \frac{s}{C}\left(k_1 r \tau_1 + k_2 d(A_j)\right) + \frac{hC}{r} \tag{17}$$

After defining $\Delta d(r) = d(A_j + \{a\}) - d(A_j)$, the first forward differences of the function $G(r)$ for $r = 1, 2, \ldots$ are given as follows.

$$\Delta G(r) = \frac{s}{C}\left(k_1 \tau_1 + k_2 \Delta d(r)\right) - \frac{hC}{r(r+1)} \tag{18}$$

The minimum of (17) occurs at the smallest integer r_0 such that $\Delta G(r_0) \geq 0$. Hence $\Delta G(r_0) = 0$ is the equation of a boundary curve between r_0 and $r_0 + 1$ in $(s, \Delta d)$ space, which can be rearranged as below.

$$\Delta d(r) = \frac{hC^2}{s k_2 r(r+1)} - \frac{k_1 \tau_1}{k_2} \tag{19}$$

The relation (19) defines a boundary curve for each value of r in $(s, \Delta d)$ space. These level curves define the regions of consecutive minimizing values of $r = 1, 2, \ldots$ in Figure 2. Each region corresponds to a minimizing group size eligibility index or number. A member has an index r if its $(s, \Delta d)$ falls in the region for r in Figure 2. An index r is an indication of a destination's eligibility of becoming a member of an optimal part containing r elements. For example, if a destination a_j is appended to a 2–destination group which yields an average demand rate of $s = 100$ for the newly formed 3–destination group, then a_j is eligible to become a member of a 3–destination group provided that the additional traveling distance (i.e. Δd) for a_j is less than 15 miles. A destination a_j is eligible to become a member of a 2–destination group when appending a_j to another destination yields an average demand rate of 100 and the additional traveling distance (Δd) is between 15 and 120 miles.

The solution procedure will start with determining an initial group size eligibility index for each destination. The index for every destination $a \in A$ is computed by using $\Delta d(r) = d(a, a') - d(a)$ and $\Delta d(r') = d(a', a) - d(a')$ for all $a \neq a'$ and is given below.

$$r = \max\left\{\min\left(\arg(\Delta d(r)), \arg(\Delta d(r'))\right)\right\} \tag{20}$$

Theorem 1 *If the initial group size eligibility index is one for a destination* $a \in A$ *and* $\arg(\Delta d(r)) = \arg(\Delta d(r')) = 1$ *for every* $a' \neq a$*, then the destination* a *is an optimal part by itself.*

The crux of the proof relies on showing that the algorithm does not permit such a destination with an initial group size eligibility index of one to form a group with any other destination or become a member of any group in Stage 2 of dynamic programming algorithm. (Details of the proof is given in the Appendix). Optimality Principle of Dynamic Programming implies that $g_{k+1}(H + \{a\}) = g_k(H) + F(a)$ in every stage $k > 2$. Heuristically any destination with an initial index of one should form a part by itself.

If the initial group size eligibility index is larger than one for a destination, then such a destination is a potential candidate to be part of a group of size two or more. The essence of the algorithm is to start with a part A_j with r elements and test all the unassigned destinations for eligibility to become additional members of A_j. If none is eligible, then A_j is set a near optimal part. Let B_R be the set of remaining (unassigned) candidates, and $B_U \subset B_R$ be the set of yet untested candidates to join A_j. This procedure as a formal algorithm to obtain a (near) optimal solution to this set partitioning problem is described below. The traveling salesperson problem (TSP) for each part A_j needs to be solved as a subproblem. In practice, however, the parts A_j are sufficiently small (containing usually less than 5 or 6 destinations) which render an exhaustive search for the solution of TSP. The cheapest insertion heuristic can be used for solving the TSP for larger size subsets. A better alternative is that the heuristic algorithm of Fisher and Jaikumar [15] may be employed to solve TSP for subsets with a large number of destinations.

Step 0. Initilization: Determine the initial group size eligibility index for each destination. A destination with an index of one is an optimal part by itself. If there are j of those, then j subsets or parts (containing a single destination) of the partition are determined. All such destinations are exempt from further search. Therefore, B_R is the set of remaining destinations ranked according to demand rates in descending order.

Step 1. If $B_R = \emptyset$, then a (near) optimal solution is obtained hence terminate, else $j := j + 1$ and go to Step 2.

Step 2. Take the first destination $a \in B_R$ (rank ordered set) and set $A_j = \{a\}$, $r = 1$, $B_R = B_R - \{a\}$, and $B_U = B_R$ where the rank ordering of B_R is preserved in B_U. Solve $\min d(A_j)$ (which is basically the traveling salesperson problem for the part A_j) and go to Step 3.

Step 3. If $B_U = \emptyset$, then A_j is set a (near) optimal part and go to Step 1, else go to Step 4.

Step 4. Form a new group $A_j + \{a'\}$ where a' is the first destination in B_U. Solve $\min d(A_j + \{a'\})$ and go to Step 5.

Step 5. Compute $s = (s(A_j) + s(a'))/(r+1)$ and $\Delta d = d(A_j + \{a'\}) - d(A_j)$. Determine group size eligibility index number r' by the region that $(s, \Delta d)$ falls in Figure 2. If $r' > r$, then $A_j := A_j + \{a'\}$, $r := r + 1$, $s(A_j) := s(A_j) + s(a')$, $B_R := B_R - \{a'\}$, $B_U := B_R$ and go to Step 3. If $r' \leq r$, then $B_U := B_U - \{a'\}$ and go to Step 3.

4 Determination of Optimal Vehicle Hauling Capacity

The vehicle hauling capacity C has been treated as a constant (or predetermined or a priori decided) so far, C can also be optimized, as a part of the design problem, by utilizing this model. Accordingly, the optimal vehicle hauling capacity will be obtained by a recursive optimization as stated in the Lemma of Section 3. The costs k_1, k_2 and the set up time β_1 are functions of the capacity C and a regression analysis on the vehicle capacity C indicates that they can be approximated by linear functions of C, as shown below.

$$k_1(C) = k_{11}C + k_{12} \tag{21}$$

$$k_2(C) = k_{21}C + k_{22} \tag{22}$$

$$\beta_1(C) = k_{31}C + k_{32} \tag{23}$$

Substituting these into the objective function and determining the optimal partition P^o for a given C generates a convex function $f(P^o(C), C)$ in tabular form. By algebraic manipulation and rearrangement of the terms, this function can be put into the following form (details are given in the appendix).

$$f(P^o(C), C) = g_0 + g_1(C) + Cg_2(C) + \frac{g_3(C)}{C} \tag{24}$$

where g_0 is a constant, and g_1, g_2, g_3 are step functions of C.

Given an optimal partition P^o, there exists an interval where P^o remains optimal and g_1, g_2, g_3 remain constant. Therefore, $f(P^o(C), C)$ is a smooth convex curve in such an interval of C. For all non–negative values of C, however, $f(P^o(C), C)$ is a union of smooth convex curve sections over C (Figure **??**). For convenience, this function may be approximated by a strictly convex function estimated through a regression analysis as demonstrated in Figure **??**. The minimum of approximating function provides a (near) optimal vehicle hauling capacity C^o for the distribution system. The exact value will be obtained by solving $\min_{(C)} \{f(P^o(C), C)\} = f(P^o(C^o), C^o)$ numerically, i.e. by a univariate search technique.

5 An Illustration

Design of an example product distribution system with 10 destinations $(a_1, a_2, \ldots, a_{10})$, a manufacturing center (a_0) and an aggregate product will be used to illustrate the model and its solution methods. This is an example of a system of delivery of aggregate petroleum products throughout a country (Turkey) over seaways by means of ship tankers. The geographic locations of the center and the destinations are given in Figure 1. The distance matrix is provided in Table 1 together with demand rates for the aggregate product at the destinations. The combined capital and inventory holding costs is \$0.12 per ton for each of the destinations.

A vehicle with a hauling capacity of 4400 tons is used in delivering the products. The parameters for this vehicle are $k_1 = \$663.91$ per hour, $k_2 = \$66.07$ per mile, $\beta_1 = 6.5$ hours per loading and $\tau_1 = 4$ hours per unloading.

The dynamic programming algorithm of Section 3 produces the optimal answer: $P^o = ((a_1, a_8, a_{10}), (a_2), (a_3), (a_4, a_7), (a_5), (a_6), (a_9))$ with a total cost of \$14397.51 per hour. This computation takes about 3.024 minutes on a PC with a Pentium 4 processor. Employment of the Theorem 1 reduces the number of elements to 6, after eliminating the destinations with initial group size

Table 1: The distance matrix (miles) and the demand rates (tons/day)

	a_{10}	a_9	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1
rate	55	288	40	120	430	53	127	202	90	40
a_0	209	431	186	52	57	203	70	169	45	316
a_1	129	188	212	368	373	519	386	304	284	
a_2	190	414	192	97	102	248	107	132		
a_3	252	474	300	221	216	325	184			
a_4	279	501	256	68	47	149				
a_5	412	634	368	168	155					
a_6	266	488	233	25						
a_7	261	458	209							
a_8	90	252								
a_9	232									

eligibility index of one (see foregoing discussion), and reduces the computation time to 0.11 seconds.

The efficiency of the heuristic algorithm of Section 3 will now be demonstrated on the same example problem. The equation (19) is computed with the given data to obtain the equation (25) and the levels curves are plotted in Figure 2.

$$\Delta d = \frac{35162.71}{sr(r+1)} - 40.19 \text{ for } r = 1, 2, \ldots \tag{25}$$

The initial group size eligibility index for each destination will be computed first. Destination a_1 has a consumption rate of 40 tons per day, and a total traveling distance from the center is 632 miles. The nearest element to a_1 is a_2. If both were visited in the same tour, the total traveling distance would be 645 miles. The additional traveling distance is $\Delta d = 13$, and the average demand rate for such a part would be $(90 + 40)/2 = 65$, since $s(a_2) = 90$. The incremental effect of combining these two destinations is given by $(s, \Delta d) = (65, 13)$ and falls in a region with an index number of 3. On the other hand, $d(a_2) = 90$ and $\Delta d(a_2, a_1) = 555$ which yields $(s, \Delta d) = (65, 465)$ and consequently an index number of 1. Therefore, $r_{12} = \min(1, 3) = 1$. Table 2 summarizes the computations and provides an index number of 2 for a_1.

Repeating similar procedure for each destination results with the initial index of one for the destinations a_3, a_5, a_6, a_9 and an index of 2 for all others.

Initialization steps states that each of a_3, a_5, a_6, a_9 is a group by itself. The

Table 2: Initial group size eligibility index computation for a_1

i	j	s	$\Delta d(i,j)$	$(s, \Delta d(i,j))$	$\Delta d(j,i)$	$(s, \Delta d(j,i))$	r
1	2	65.00	13	3	555	1	1
1	3	121.00	157	1	451	1	1
1	4	83.50	140	2	632	1	1
1	5	46.50	406	1	632	1	1
1	6	235.00	114	1	632	1	1
1	7	80.00	104	2	632	1	1
1	8	40.00	82	3	342	2	2
1	9	164.00	303	1	73	2	1
1	10	47.50	22	4	236	2	2

remaining destinations are ranked in descending order of the demand rates as given below.

destination	rank	demand rate
a_4	1	127 tons/day
a_7	2	120 tons/day
a_2	3	90 tons/day
a_{10}	4	55 tons/day
a_1	5.5	40 tons/day
a_8	5.5	40 tons/day

The first candidate to form a group with others in the list is a_4. An attempt will be made for a_4 to form a group with one or more of the remaining destinations. The next destination is the list is a_7. Hence $d(a_4, a_7) = 70 + 68 + 52 = 190$, $d(a_4) = 140$ and $d(a_7) = 104$, then $\Delta d(a_4, a_7) = 50$ and $\Delta d(a_7, a_4) = 86$ with $s = (120 + 127)/2 = 123.5$. The group size index is $r = 2$ for both cases. A tentative part $A_j = (a_4, a_7)$ is obtained. A query of adding a new destination will be done next. The next destination in the list is a_2, then $d(A_j, a_2) = 272$ and $\Delta d = 82$ with $s = 112.33$. This time $(s, \Delta d)$ yields again an index of 2, thus a_2 is not allowed to join the group (a_4, a_7). Next a_{10} will be tested. $d(A_j, a_{10}) = 70 + 68 + 261 + 209 = 608$ and $\Delta d = 608 - 190 = 418$ with $s = (127 + 120 + 55)/3 = 100.67$. An index of one is obtained from the Figure 2 for $(100.67, 418)$, and a_{10} is not allowed to join this group either. After repeating similar arguments for a_1 and a_8, a conclusion of a (near) optimal

Table 3: The computation of the total cost for the optimal partition

j	A_j	$s(A_j)$	$d(A_j)$	t_j	$F(A_j)$
1	(1,8,10)	135	721	32.59	2234.01
2	(2)	90	90	48.89	703.95
3	(3)	202	338	21.78	1675.14
4	(4,7)	247	190	17.81	1530.85
5	(5)	53	406	83.02	883.10
6	(6)	430	114	10.23	1523.61
7	(9)	288	862	15.28	4429.61
	Total	Cost			14397.51

part (a_4, a_7) is reached. These two destinations will be exempt from further search.

The first element in the remaining list is a_2 now. This element does not accept any others to form a group, since the remaining destinations are too far away, hence it is declared as a part by itself.

The highest ranked remaining element in the list is a_{10} now. The next element in the list is a_1 will be tested for eligibility of joining a_{10}. The distances are $d(a_{10}) = 418$, $d(a_{10}, a_1) = 209+129+316 = 654$, then $\Delta d = 654-418 = 236$ and $s = (55 + 40)/2 = 47.5$ and further $(s, \Delta d)$ yields an index of 2. On the other hand, $d(a_1) = 632$ and $\Delta d = 654 - 632 = 22$, then an index of 4 for $(s, \Delta d) = (47.5, 22)$ is obtained in Figure 2. This, in a sense, confirms the formation of the part $A_j = (a_{10}, a_1)$ with $r = 2$ tentatively. Finally a_8 will be tested. Traveling distance is $d(A_j, a_8) = 186 + 90 + 129 + 316 = 721$ which implies $\Delta d = 721 - 654 = 67$. Since $s = (55 + 40 + 40)/3 = 45$, an index of $r = 3$ is obtained for $(45, 67)$ from Figure 2. Consequently, (a_{10}, a_1, a_8) is declared a part. All the elements of the set A are assigned to obtain a partition with the cost computations given in Table 3 and hence the algorithm is terminated. Surprisingly (or luckily), this turns out to be exactly the same partition obtained by dynamic programming. In general, of course, the optimal partition is not guaranteed by this heuristic.

Upon determining the optimal partition, we can now make the remaining decisions. The storage capacity at destination a_6 is calculated to be 4400 tons plus the safety stock level, since it will be supplied in a single trip of the vehicle every $4400/430 = 10.23$ days. Destinations a_1 ,a_8 ,a_{10} will be supplied in a single trip of the vehicle in every $4400/(40+40+55) = 32.59$ days. Destinations

a_1 and a_8 both require $32.59 \times 40 = 1303.60$ tons of storage capacity plus a safety stock level, and a_{10} requires a capacity of $32.59 \times 55 = 1792.45$ tons and a safety stock level. The storage capacities at other destinations can be computed similarly.

Determining the optimal vehicle hauling capacity requires the solution of partitioning problem for various vehicle sizes, then the values of the functions g_1, g_2, g_3 can be computed. Estimation of the parameters of the equations (21), (22) and (23) gives $k_1(C) = 0.14C + 47.91$, $k_2(C) = 0.013C + 8.87$, and $\beta_1(C) = 0.001C + 2.1$. The unloading rate of the vehicle is $\tau_1 = 0.00833$ hr/ton and the loading rate at the center is $\beta_2 = 0.00333$ hr/ton. Table 4 lists the vehicle capacity ranges, and the cost function with the values of g_1, g_2, g_3 (g_0 is omitted in the computation, since it is a constant with respect to C) together with the (near) optimal partition in every range of C. A regression analysis provides an approximating cost function $\tilde{f}(C) = 8193.8 + 3.398C + 4271683/C$. The cost curve is a union of smooth convex curve segments and approximated by the convex function $\tilde{f}(C)$ defined above. The minimum of the approximating curve occurs at $C = 1121.21$ tons. The actual minimum is at $C = 1124.38$ tons. The proximity of these values should be noted here. Thus it is easier for decision making to use the approximating function $\tilde{f}(C)$ after it is developed. The size of the vehicle is a design variable of the distribution system. An operationally meaningful investment decision is to acquire a vehicle of capacity of 1121.21 tons. In practice, however, this may not be feasible always, and a choice is to be made according to the manufacturer's design, such as 1200 tons. If the manufacturer of vehicles offers only capacities in discrete values, the problem is somewhat simplified since there is only a finite number of values of C and the value of C which minimizes the cost function $f(P^o(C^o), C^o)$ will be obtained with less effort.

6 Conclusions

A model for designing a distribution system, taking operational problems into account is presented above. Optimal solution to the model includes a nonlinear set partitioning problem along with the optimization of some continuous decision variables. In this form, the model appears to be very complex mixed combinatorial optimization problem. A sequential optimization algorithm is suggested and used in the above presentation. The most difficult stage of this algorithm is the nonlinear set partitioning. A listing of all partitions is prohibitive even for small size problems. The number of all partitions of a

Table 4: The cost function $f(P^o(C), C)$

Range of C	$f(P^o(C), C)$	Optimal partition
1–1762	$7160.74 + 3.762C + 4756007.15/C$	$(a_1)(a_2)(a_3)(a_4)(a_5)$ $(a_6)(a_7)(a_8)(a_9)(a_{10})$
1762–3798	$7320.61 + 3.642C + 4846991.29/C$	$(a_1)(a_2)(a_3)(a_4)(a_5)$ $(a_6)(a_7)(a_8, a_{10})(a_9)$
3798–4804	$8111.38 + 3.402C + 5306168.84/C$	$(a_1, a_8, a_{10})(a_2)(a_3)$ $(a_4, a_7)(a_5)(a_6)(a_9)$
4804–8000	$8634.29 + 3.282C + 5563920.58/C$	$(a_1, a_8, a_{10})(a_2, a_4)$ $(a_3)(a_5)(a_6, a_7)(a_9)$

set of m elements is a sum of Stirling numbers of the Second Kind. The dynamic programming cannot even handle moderate size problems. Table 5 shows the performance of the dynamic programming algorithm for different size problems. The computation time increases exponentially and a problem containing 10 members requires 3.024 minutes on a PC with a Pentium 4 processor and computation of $2^{10} - 1$ subsets. Even invoking the Theorem 1 may not be sufficient to reduce the computation and memory requirements for problems of moderate sizes and increase its efficiency considerably. An efficient heuristic method is needed in practice to solve the problem quickly to obtain good solutions, even if not optimal. The performance of the heuristic algorithm proposed in this paper is also given in Table 5 for comparison. The proposed heuristic takes only a fraction of a $1/100$ of second to solve a problem with 10 members and a center. A problem of size 50 is solved, on the average, within 0.06 seconds. In fact, problems of size up to 25–30 destinations can be solved by manual computations by the heuristic algorithm. Should the larger problems be considered, the efficiency and practicality of this algorithm will be more appreciated.

The model and the proposed solution algorithm is implemented to a sample distribution system with 10 destinations. The results are summarized in Tables 1– 5. In this application, heuristic set partitioning algorithm for $C = 4400$ tons vehicle capacity finds a (near) optimal solution in only three iterations. Fortunately, this solution turns out to be optimal as shown by the dynamic programming. Furthermore, the number of parts in the partition or the number of constraints in the model is determined and implicitly minimized by this algorithm. Since the number of parts in the partition is the main determi-

Table 5: Computational performance of two algorithms

	Dynamic Programming optimal solution		Heuristic Algorithm near optimal		
Number of Destinations	mean (sec)	std (sec)	mean (sec)	std (sec)	% below upper bound
5	0.14	0.03			
10	180.96	0.54	0.00	0.00	3.51
25			0.01	0.03	2.70
50			0.03	0.03	2.66
75			0.06	0.03	1.94

nant of the number of vehicles required to operate the system (that can be determined very easily by a Gannt chart), it is a useful, practical and indirect result of the model and its solution. This idea can be expanded by scheduling the distribution of the product(s) and testing such a schedule of the fleet of vehicles by means of simulation.

Although the model and solution algorithm given here is aimed at solving the design and operational problems simultaneously, the model can be used for developing operational policies and to adapt to changing conditions (most typically changes in the demand rates) for an existing system. Obviously it is also possible to guide modifications on design variables such as storage and vehicle capacities, should the changing conditions call for. A further extension is certainly the development of a multi–product transportation and distribution model.

Appendix:

The detailed proof of Theorem, stated in Section 3 without proof, and the derivation of Equation (24) is provided here. Those were delayed to the appendix to avoid disruption of continuity. The Theorem states that *if the initial group size eligibility index is one for a destination under the condition* $\arg(\Delta d(r)) = \arg(\Delta d(r')) = 1$, *then such a destination is an optimal part by itself.*

Proof: Suppose that $a \in A$ has an initial group size index of one with $\arg(\Delta d(r)) = \arg(\Delta d(r')) = 1$. Then for any $a' \neq a$, we will have to compare $F(a+a')$ with $F(a) + F(a')$ and show that $F(a+a') \geq F(a) + F(a')$, hence the statement holds. In case of equality, multiple optimum partitions exist.

$$F(a) = \frac{s(a)}{C}\left(k_1(\beta_1 + \tau_1) + k_2 d(a)\right) + hC \tag{26}$$

$$F(a') = \frac{s(a')}{C}\left(k_1(\beta_1 + \tau_1) + k_2 d(a')\right) + hC \tag{27}$$

$$F(a+a') = \frac{s(a+a')}{C}\left(k_1(\beta_1 + 2\tau_1) + k_2 d(a,a')\right) + hC \tag{28}$$

Since the total demand $s(a+a') = s(a) + s(a')$, then the sum $F(a) + F(a')$ is calculated as follows.

$$F(a) + F(a') = \frac{s(a+a')}{C} k_1(\beta_1 + \tau_1) + \frac{k_2}{C}(d(a)s(a) + d(a')s(a')) + 2hC \tag{29}$$

Adding and deleting new terms, then rearranging all the terms produces the following equation.

$$\begin{aligned} F(a) + F(a') &= \frac{s(a+a')}{C}\left(k_1(\beta_1 + 2\tau_1) + k_2 d(a,a')\right) + hC \\ &\quad + \frac{k_2}{C}(d(a)s(a) + d(a')s(a')) + hC \\ &\quad - \frac{s(a+a')}{C}(k_1\tau_1 + k_2 d(a,a')) \end{aligned} \tag{30}$$

The difference $\Delta F = F(a+a') - F(a) - F(a')$ is expressed as follows.

$$\begin{aligned} \Delta F &= \frac{k_2 s(a+a')}{C}\left(d(a,a') + \frac{k_1\tau_1}{k_2} - \frac{hC}{k_2 s(a+a')}\right) \\ &\quad - \frac{k_2}{C}\left(s(a)d(a) + s(a')d(a')\right) \end{aligned} \tag{31}$$

Let $d(\bar{a}) = \max\{d(a), d(a')\}$ and the average demand rate be $\bar{s}$, then $s(a + a') = s(a) + s(a') = 2\bar{s}$, $\Delta d(\bar{a}) = d(a,a') - d(\bar{a})$ and the following inequality is obtained.

$$\Delta F \geq \frac{s(a+a')k_2}{C}\left(\Delta d(\bar{a}) - \left(\frac{hC^2}{2\bar{s}k_2} - \frac{k_1\tau_1}{k_2}\right)\right) \tag{32}$$

The index for a is one under the condition $\arg(\Delta d(r)) = \arg(\Delta d(r')) = 1$, then $(\bar{s}, \Delta d(a))$ and $(\bar{s}, \Delta d(a'))$ for every $a' \neq a$ both have to fall in the region for $r = 1$ in Figure 2. This means that the right hand side of the expression (32) is nonnegative and $\Delta F \geq 0$ or $F(a) + F(a') \leq F(a+a')$. The Optimality Principle of dynamic programming preserves $g_{k+1}(H + \{a\}) = g_k(H) + F(a)$ in every stage after the second stage.

The equation (24), $f(P^o(C), C) = g_0 + g_1(C) + g_2(C)C + g_3(C)/C$, will be derived now. Substitution of linear estimators (21, 22, and 23) of $k_1(C)$, $k_2(C)$, and $\beta_1(C)$ into the equation (5) yields the following.

$$f(P^o(C), C) = \frac{s(A)}{C}(k_{11}C + k_{12})(k_{31}C + k_{32}) + (k_{11}C + k_{12})\, s(A)(\beta_1 + \tau_1)$$
$$+ \sum_{j\in J}\left\{\frac{s(A_j)}{C}\left((k_{11}C + k_{12})\sum_{a\in A_j}\tau_1 + (k_{21}C + k_{22})\, d(A_j)\right) + hC\right\} \tag{33}$$

An algebraic manipulation of (33) yields the following equation.

$$\begin{aligned} f(P^o(C), C) &= s(A)(k_{11}k_{32} + k_{12}k_{31}) + \sum_{j\in J} s(A_j)\left(k_{11}\sum_{a\in A_j}\tau_1 + k_{21}d(A_j)\right) \\ &+ \left(\sum_{j\in J} h + s(A)k_{11}(k_{31} + \beta_1 + \tau_1)\right) C + s(A)k_{12}(\beta_1 + \tau_1) \\ &+ \frac{1}{C}\left(s(A)k_{12}k_{32} + \sum_{j\in J} s(A_j)\left(k_{12}\sum_{a\in A_j}\tau_1 + k_{22}d(A_j)\right)\right) \end{aligned} \tag{34}$$

Consequently, the equation (24) is obtained by defining the following functions.

$$g_0(C) = s(A)(k_{11}k_{32} + k_{12}(k_{31} + \beta_1 + \tau_1)) \tag{35}$$

$$g_1(C) = \sum_{j\in J} s(A_j)\left(k_{11}\sum_{a\in A_j}\tau_1 + k_{21}d(A_j)\right) \tag{36}$$

$$g_2(C) = s(A)k_{11}(k_{31} + \beta_1 + \tau_1) + \sum_{j\in J} h \tag{37}$$

$$g_3(C) = s(A)k_{12}k_{32} + \sum_{j\in J} s(A_j)\left(k_{12}\sum_{a\in A_j}\tau_1 + k_{22}d(A_j)\right) \tag{38}$$

The term $g_0(C)$ is constant with respect to C. The other terms $g_1(C)$, $g_2(C)$, and $g_3(C)$ are step functions of C.

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Figure 1: Geographic locations of the manufacturing center and the destinations

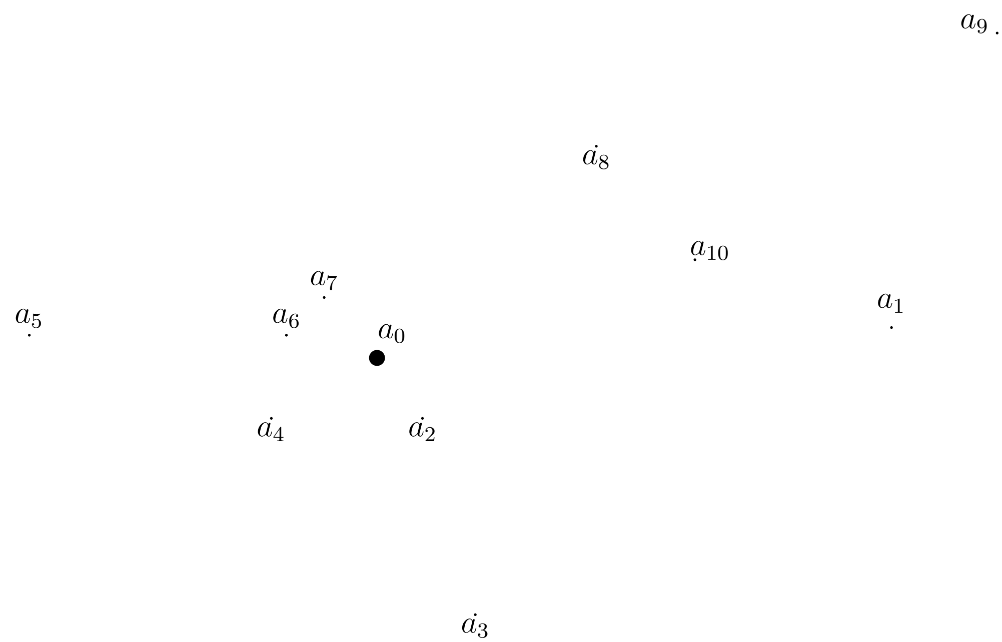

Figure 2: Group size eligibility index chart