The Laplace Order and Ordering
of Reversed Residual Life

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Abstract

The purpose of this paper is to study new stochastic comparisons based on the Laplace transform order of reversed residual life. Some basic properties of the new order are given. We also provide several preservation properties of it under the reliability operations of convolution, mixture and parallel system.

Keywords: Laplace transform order, Laplace transform order of reversed residual life, mean reversed residual life order, convolution, mixture, parallel system

1 Introduction

Suppose $X$ is a non-negative random variable having absolutely continuous distribution $F_X(x)$ with density $f_X(x)$. Then the ordinary Laplace transform of the density function $f_X$ is given by

$$L_X(s) = \int_0^\infty e^{-su} f_X(u)du, \quad s > 0.$$ 

The Laplace transform of $F_X$ is defined by

$$L_X^*(s) = \int_0^\infty e^{-su} F_X(u)du.$$
It is easy to check that if $F_X$ is continuous then
\[
L^*_X(s) = \frac{1}{s} L_X(s), \quad s \geq 0.
\]

Throughout this paper $X$ and $Y$ are two random variables with distribution functions $F_X$ and $F_Y$, respectively. Denote by $L_X$ the Laplace transform of $F_X$, and by $\overline{F}_X = 1 - F_X$ the corresponding survival function. We use a similar notation for all other distribution functions. Moreover, we will use the term increasing in place of non-decreasing, and decreasing in place of non-increasing.

Given two random variables $X$ and $Y$, $X$ is said to be smaller than $Y$ in the Laplace transform order (denoted by $X \leq_{Lt} Y$) if $L_X(s) \geq L_Y(s)$, for all $s > 0$. Clearly, $X \leq_{Lt} Y \iff L^*_X(s) \geq L^*_Y(s)$, for all $s > 0$ if both $F_X$ and $F_Y$ are continuous.

Also, it can be checked that $M \leq_{Lt} N$ is equivalent to $L^*_M(s) \geq L^*_N(s)$ for all $s > 0$, or
\[
M \leq_{Lt} N \iff \sum_{n=0}^{\infty} \epsilon^n Q_M(n) \geq \sum_{n=0}^{\infty} \epsilon^n Q_N(n), \quad \epsilon \in (0, 1),
\] (1.1)
if $M$ and $N$ are non-negative integer-valued random variables with distribution probabilities $Q_M(n) = P(M < n)$ and $Q_N(n) = P(N < n)$, respectively.

The Laplace transform can be interpreted in several ways when the random variable represents the lifetime of a system or a unit, which yields several applications of Laplace transform order. Applications, properties and interpretations of the Laplace transform order can be found in Alzaid et al. (1991), Klefsjo (1983), Shaked and Wong (1997) Denuit (2001) and Belzunce et al. (1999).

For any random variable $X$, let
\[
X(t) = [t - X \mid X < t], \quad t \in \{x : F_X(x) < 1\},
\]
denote a random variable whose distribution is the same as the conditional distribution of $t - X$ given that $X < t$. When the random variable $X$ denote the life time ($X \geq 0$, with probability one) of a unit, $X(t)$ is known as reversed residual life (or time since failure or inactivity time, or idle time), (see, for
instance, Chandra and Roy (2001), Block et al. (1998), Nanda et al. (2003), Kayid and Ahmad (2004), Ahmad et al. (2005)).

In this case given a random variable $X_t$, then

$$L^*_X(t)(s) = \int_0^t \frac{e^{su}F_X(u)du}{e^{st}F_X(t)} = s > 0.$$

In this paper, based on this measure we will deal with a new ordering called Laplace transform ordering of reversed residual life. In Section 2, we present definitions, notations and basic properties used throughout the paper. Several preservation properties of stochastic comparisons and some aging notions based on the Laplace transform order of reversed residual lives under the reliability operations of mixture, convolution and parallel system are studied in Section 3. Finally, in Section 4 we provide a preservation property under monotone transformation of the new order.

### 2 Definitions, notations and characterization

In the context of reliability theory several orders have been introduced to compare two lifetime distributions. These orders have been found useful for modelling, or the design of better systems. We see below some of these orders (see, Shaked and Shanthikumar (1994), Nanda et al. (2003) and Kayid and Ahmad (2004)).

Given a random variable $X$, with distribution function $F_X$, $X_t = (X - t | X \geq t)$, denotes the additional residual life where $t \in (0, l_X)$ and $l_X = \sup\{t : F_X(t) < 1\}$. Based on the stochastic comparison of $X_t$, the hazard rate order ($X \leq_{HR} Y$) can be defined as

$$X_t \leq_{ST} Y_t,$$

for all $t$, where the stochastic ordering ($\leq_{ST}$) is defined as

$$X \leq_{ST} Y \iff \overline{F}_X(t) \leq \overline{F}_Y(t), \text{ for all } t.$$

**Definition 2.1.** Let $X$ and $Y$ be two non-negative random variables. The random variable $X$ is said to be smaller than $Y$ in the
(i) mean residual life order (denoted by $X \leq_{MRL} Y$) if, and only if,
\[
\frac{\int_t^\infty F_Y(x)dx}{\int_t^\infty F_X(x)dx} \text{ increasing in } t \text{ for all } t \in (0, l_X) \cap (0, l_Y);
\]

(ii) reversed mean residual life order (denoted by $X \leq_{RMRL} Y$) if, and only if,
\[
\frac{\int_0^t F_Y(x)dx}{\int_0^t F_X(x)dx} \text{ increasing in } t \text{ for all } t \in R^+.
\]

(iii) increasing concave order (denoted by $X \leq_{icv} Y$) if, and only if,
\[
\int_0^x F_X(u)du \leq \int_0^x F_Y(u)du, \text{ for all } x.
\]

Let us observe that the $\leq_{MRL}$ order are more informative than the $\leq_{ST}$ order, since it compare the underlying systems at any time $t$ in contrast to the global comparison offered by the orders $\leq_{ST}$. However, it is reasonable to presume that in many realistic situations the random variable is not necessarily related to the future but can also refer to the past. For instance, consider a system whose state is observed only at certain preassigned inspection times. If at time $t$ the system is inspected for the first time and it is found to be 'down', then the failure relies on the past, i.e. on which instant in $(0, t)$ it has failed. It thus seems natural to introduce a notion that is dual to the Laplace transform order of residual lives $\leq_{Lt-rl}$, introduced by Belzunce et al. (1999) and furthered by several authors, including Gao et al.(2003), Kayid and Ahmad (2004), in the sense that it refers to past time and not to future time.

Following this idea we introduc a new partial order based on the Laplace transform order of reversed residual life.

**Definition 2.2.** Let $X$ and $Y$ be two non-negative random variables. The random variable $X$ is said to be smaller than $Y$ in the Laplace transform order of reversed residual lives (denoted by $X \leq_{LT-RRL} Y$) if
\[
X(t) \geq_{LT} Y(t) \text{ for all } t \in R^+,
\]

Observe that, by definition of $\leq_{LT-RRL}$ order, it holds $X \leq_{LT-RRL} Y$, if and only if $L_{X(t)}^*(s) \geq L_{Y(t)}^*(s)$ for all $t, s \geq 0$. Actually, an equivalent condition for $LT - RRL$ order is given in Ahmad and Kayid (2005), and is the following.
Proposition 2.1. Let $X$ and $Y$ be two continuous non-negative random variables with distribution functions $F$ and $G$, respectively, then for all $t \geq 0$ and $s > 0$

$$X \leq_{LT-RRL} Y \iff \frac{\int_0^t e^{su} F(u) du}{\int_0^t e^{su} G(u) du} \text{ is decreasing in } t \geq 0. \quad (2.1)$$

On the other hand, in the literature, many non-parametric aging classes of distributions have been defined (cf. Barlow and Proschan (1975)). In particular, decreasing reversed hazard rate ($DRHR$) and increasing reversed mean residual life ($IRMR$) classes of distributions has been studied by many researchers in the recent past.

The following two propositions gives most characterizations of $DRHR$ and $IRMR$ classes.

Proposition 2.2. For $0 < t_1 < t_2$, the following statements are equivalent.

(a) $X$ has decreasing reversed hazard rate;

(b) $X_{(t_1)}$ is smaller than $X_{(t_2)}$ in stochastic ordering;

(c) $X_{(t_1)}$ is smaller than $X_{(t_2)}$ in hazard rate ordering;

(d) the distribution function of $X$ is logconcave.

Proposition 2.3. The random variable $X$ is $IRMR$ if, and only if, any one of the following conditions holds:

(i) $[t - X \mid X < t] \leq_{RMRL} [s - X \mid X < s]$ whenever $t \leq s$.

(ii) $X \leq_{RMR} [t - X \mid < t]$ for all $t > 0$ (when $X$ is a non-positive random variable).

(iii) $X + t \leq_{RMR} X + s$ whenever $t \leq s$.

Next we introduce a new aging class based on the Laplace transform order.

Definition 2.3. A non-negative random variable $X$ is said to have increasing reversed residual lives in the Laplace order, denoted by $X \in IRRL_{Lt}$ if

$$X_{(t)} \leq_{Lt} X_{(s)}, \text{ for all } s > t \geq 0.$$
The following implications are well known or easy to prove as indicated below:

\[ X \leq_{RHR} Y \Rightarrow X \leq_{LT-RRL} Y \Rightarrow X \leq_{RMRL} Y, \]  
\hspace{1cm} (2.2)

and

\[ DRFR \Rightarrow IRRL_{LT} \Rightarrow IRMR. \]

The implications \( X \leq_{RHR} Y \Rightarrow X \leq_{LT-RRL} Y \) follows from Theorem 3.B.6 of Shaked and Shanthikumar (1994) and \( X \leq_{LT-RRL} Y \Rightarrow X \leq_{RMRL} Y \) follows from (3.B.2) of Shaked and Shanthikumar (1994) and from Kayid and Ahmad (2004).

From the definition it could be thought that the new order is equivalent to one of the orders (\( \leq_{LT-R} \) and \( \leq_{R-LT-R} \)) introduced by Shaked and Wong (1997). However this is not true in general. Recall that \( \leq_{RH} \not \Rightarrow \leq_{LT-R} \) and \( \leq_{LT-R} \not \Rightarrow \leq_{ICV} \) (see Shaked and Wong, 1997), therefore from (2.2) and from \( \leq_{RMRL} \not \Rightarrow \leq_{ICV} \) (see Kayid and Ahmad, 2004) neither \( \leq_{LT-R} \) nor \( \leq_{R-LT-R} \) is equivalent to \( \leq_{LT-RRL} \).

One may refer to Shaked and Shanthikumar (1994), and Muller and Stoyan (2002) for the hazard rate order (\( \leq_{HR} \)), the reversed hazard rate order (\( \leq_{RHR} \)), the mean residual life order (\( \leq_{MRL} \)), and other common used stochastic orders. Also, one can refer to, Nanda et al. (2003) for the reversed mean residual life order (\( \leq_{RMRL} \)) and increasing reversed mean residual life aging class (\( IRMR \)) and Shaked and Shanthikumar (1994) for the decreasing reversed failure rate aging class (\( DRFR \)) and other aging notions.

3 Preservation properties of the LT-RRL order

In this section, we describe some basic properties of the \( LT - RRL \) order, which will be used frequently in the sequel, and then give preservation properties of this order under the operations of mixture, convolution, monotone transformation and series system.
3.1 Basic properties

From (1.1) and Definition 2.2, we state the next proposition without proof. Its proof follows a similar argument to that of Proposition 2.1 of Ahmad and Kayid (2005).

**Proposition 3.1.** Let $M$ and $N$ be two nonnegative integer-valued random variables with distribution probabilities $Q_M(n)$ and $Q_N(n)$, respectively. Then

$$
\sum_{k=0}^{n} \frac{\epsilon^k Q_N(k-1)}{\sum_{k=0}^{n} \epsilon^k Q_M(k-1)}
$$

is increasing in $n \in \{0, 1, \ldots\}$ for each $\epsilon \in (0, 1)$.

Observe that

$$
X \in \text{IRRL}_{LT} \Leftrightarrow X(t) \leq_{LT-RRL} X(t') \text{ for all } t' > t \geq 0.
$$

From (2.1), we easily obtain the following useful proposition.

**Proposition 3.2.** Let $X$ be a non-negative continuous random variable with distribution function $F$, and denote

$$
\phi(t) = \int_{-\infty}^{t} e^{su} F(u) \, du \text{ for all } t.
$$

Then

$$
X \in \text{IRRL}_{LT} \Leftrightarrow \phi(t) \text{ is logconcave in } t \in R^+ \text{ for all } s \geq 0.
$$

Special attention should be paid to the logconcavity and logconvexity of a non-negative function $\zeta$. The logconcavity of $\zeta(t)$ in $t \in R$ is equivalent to that $\zeta(x - y)$ is $TP_2$ in $(x, y) \in R^2$, whereas the logconvexity of $\zeta(t)$ in $t \in R$ does not in general imply that $\zeta(x + y)$ is $TP_2$ in $(x, y) \in R^2$. The definition of $TP_2$ is given in the paragraph before Theorem 3.1. However, if $\zeta(t)$ is logconvex [logconcave] in $t \in R^+$, then $\zeta(x + y)$ [\zeta(x + 1/y)] is $TP_2$ in $(x, y) \in R^2$. These facts will be used in the sequel.

**Proposition 3.3.** Let $X$ be a non-negative random variable. Then for any non-negative random variable $Y$ independent of $X$

$$
X \in \text{IRRL}_{LT} \Rightarrow X \leq_{LT-RRL} X + Y.
$$
Proof. Suppose that $X \in IRRL_{LT}$ and $Y$ is any non-negative random variable, and fix $s \geq 0$. Let $F_W$ denote the distribution function of a random variable $W$, and denote

$$
\phi_W(t) = \int_{-\infty}^{t} e^{su} F_W(u) du, \quad \text{for all } t \in R^+.
$$

Then it follows from Proposition 3.2 that $\phi_X(t)$ is log-concave in $t \in R$; that is $\phi(t-v)/\phi_X(t)$ is increasing in $t \in R$ for each $v \geq 0$. Also, by Fubini’s Theorem, we have

$$
\phi_{X+Y}(t) = \int_{-\infty}^{t} e^{su} \left( \int_{0}^{\infty} F_X(u-v)dF_Y(v) \right) du
$$

$$
= \int_{0}^{\infty} \left( \int_{-\infty}^{t} e^{su} F_X(u-v) du \right) dF_Y(v)
$$

$$
= \int_{0}^{\infty} \phi_X(t-v) e^{sv} dF_Y(v).
$$

Therefore,

$$
\frac{\phi_{X+Y}(t)}{\phi_X(t)} = \int_{0}^{\infty} e^{sv} \frac{\phi_X(t-v)}{\phi_X(t)} dv \quad \text{is increasing in } t \in R.
$$

Thus, by Proposition 2.1, we get $X \leq_{LT-RRL} X + Y$.

### 3.2 Mixture and convolution

The following result shows that the $(LT - RRL)$ order is preserved under convolutions, when appropriate assumptions are satisfied.

**Theorem 3.1.** Let $X_1, X_2$ and $Y$ be three non-negative random variables, where $Y$ is independent of both $X_1$ and $X_2$, and let $Y$ have density $g$. If $X_1 \leq_{LT-RRL} X_2$ and $g$ is log-concave then $X_1 + Y \leq_{LT-RRL} X_2 + Y$.

**Proof.** First we note that, for fixed $s \geq 0$ and $i = 1, 2$,

$$
\Phi(i, t) = \int_{0}^{\infty} e^{-sv} F_{X_i+Y}(t-v) dv
$$

$$
= \int_{0}^{\infty} e^{-sv} \int_{0}^{\infty} F_i(t-v-u)dF_Y(u) dv
$$
Laplace order

\[ = \int_0^\infty e^{-sv} \int_0^t F_i(z-v) f_Y(t-z) \, dz \, dv \]
\[ = \int_0^\infty f_Y(t-z) \int_0^t e^{-sv} F_i(z-v) \, dv \, dz \]
\[ = \int_{-\infty}^t f_Y(t-z) \psi(i, z) \, dz. \]

As shown in Proposition 2.1, the assertion follows if we prove that \( \Phi(i, t) \) is \( TP_2 \) in \((i, t)\) (see Joag-Dev et al., 1995). By the assumption \( X_1 \leq_{LT-RRL} Y_1 \) we can say that \( \psi(i, z) \) is \( TP_2 \) in \((i, z)\). Moreover, since \( Y \) has logconcave density, \( f_Y(t-z) \) is \( TP_2 \) in \((t, z)\). Therefore by the basic composition formula (Karlin, 1968) it follows that \( \Phi(i, t) \) is \( TP_2 \) in \((i, t)\). This completes the proof.

**Corollary 3.1.** If \( X_1 \leq_{LT-RRL} Y_2 \) and \( X_2 \leq_{LT-RRL} Y_2 \) where \( X_1 \) is independent of \( X_2 \) and \( Y_1 \) is independent of \( Y_2 \), then the following statements hold:

(i) If \( X_1 \) and \( Y_2 \) have log-concave densities, then \( X_1 + X_2 \leq_{LT-RRL} Y_1 + Y_2 \).

(ii) If \( X_2 \) and \( Y_1 \) have log-concave densities, then \( X_1 + X_2 \leq_{LT-RRL} Y_1 + Y_2 \).

**Proof.** The following chain of inequalities, which establish (i), follows by Theorem 3.1:

\[ X_1 + X_2 \leq_{LT-RRL} X_1 + Y_2 \leq_{LT-RRL} Y_1 + Y_2. \]

The proof of (ii) is similar.

Repeated application of Theorem 3.1, using the closure property of log-concaves under convolution, yields the following result.

**Theorem 3.2.** If \( X_1, X_2, \ldots \) and \( Y_1, Y_2, \ldots \) are sequences of independent random variables with \( X_i \leq_{LR-RRL} Y_i \) and \( X_i, Y_i \) have log-concave densities for all \( i \), then

\[ \sum_{i=1}^n X_i \leq_{LR-RRL} \sum_{i=1}^n Y_i, \quad (n = 1, 2, \ldots). \]

**Proof.** We shall prove the theorem by induction. Clearly, the result is true for \( n = 1 \). Assume that the result is true for \( p = n - 1 \), i.e.,

\[ \sum_{i=1}^{n-1} X_i \leq_{LR-RRL} \sum_{i=1}^{n-1} Y_i. \quad (4.1) \]
Note that each of the two sides of (4.1) has a log-concave density (see, e.g., Karlin, 1968, page 128). Appealing to Corollary 4.1, the result follows.

Let now $X(\theta)$ be a random variable having distribution function $F_\theta$, and let $\Theta_i$ be a random variable having distribution $G_i$, for $i = 1, 2$, and support $R^+$. The following is a closure of $LT - RRL$ order under mixture.

**Theorem 3.3.** Let $X(\theta), \theta \in R^+$ be a family of random variables independent of $\Theta_1$ and $\Theta_2$. If $\Theta_1 \leq_{lr} \Theta_2$ and if $X(\theta_1) \leq_{LT-RRL} X(\theta_2)$ whenever $\theta_1 \leq \theta_2$, then $X(\Theta_1) \leq_{LT-RRL} X(\Theta_2)$

**Proof.** Let $F_i$ be the distribution function of $X(\Theta_i)$, with $i = 1, 2$. We known that

$$F_i(x) = \int_0^\infty F_\theta(x) dG_i(\theta).$$

Again, because of Proposition 2.1, we should prove that $\Phi(i, t) = \int_0^\infty e^{-sx} F_i(t-x) dx$ is $TP_2$ in $(i, t)$. But actually

$$\Phi(i, t) = \int_0^\infty e^{-sx} F_i(t-x) dx$$

$$= \int_0^\infty e^{-sx} \int_0^\infty F_\theta(t-x) dG_i(\theta) dx$$

$$= \int_0^\infty g_i(\theta) \int_0^\infty e^{-sx} F_\theta(t-x) dx dG_i(\theta)$$

$$= \int_0^\infty g_i(\theta) \psi(\theta, t) d\theta.$$

By assumption $X(\theta_1) \leq_{LT-RRL} X(\theta_2)$ whenever $\theta_1 \leq \theta_2$, we have that $\psi(\theta, t)$ is $TP_2$ in $(\theta, t)$, while from assumption $\Theta_1 \leq_{lr} \Theta_2$ follows that $g_i(\theta)$ is $TP_2$ in $(i, \theta)$. Thus again the assertion follows from the basic composition formula.

Suppose that $X_i, i = 1, ..., n$ be a collection of independent random variables. Suppose that $F_i$ is the distributions functions of $X_i$. Let $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ be two probability vectors. Let $X$ and $Y$ be two random variables having the respective distribution functions $F$ and $G$ defined by

$$F(x) = \sum_{i=1}^n \alpha_i F_i(x) \quad \text{and} \quad G(x) = \sum_{i=1}^n \beta_i F_i(x).$$

(4.2)
The following result gives conditions under which $X$ and $Y$ are comparable with respect to the LT − RRL order. One may refer to Ahmed and Kayid (2004) and Kayid and Ahmad (2004) for a similar preservation property of the mean residual life order ($\leq_{\text{MRL}}$), the Laplace transform of residual life order ($\leq_{\text{LT − RL}}$) and mean inactivity time order ($\leq_{\text{MIT}}$), respectively. Definition, properties and applications of $\leq_{\text{MRL}}$ order and $\leq_{\text{LT − RL}}$ order can be found, for instance, in Shaked and Shanthikumar (1994), Belzunce et al. (1999) and Gao et al. (2003).

**Corollary 3.2.** Let $X_1, \ldots, X_n$ be a collection of independent random variables with corresponding distribution functions $F_1, \ldots, F_n$, such that $X_1 \leq_{\text{LT − RRL}} X_2 \leq_{\text{LT − RRL}} \ldots \leq_{\text{LT − RRL}} X_n$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ such that $\alpha \leq_{\text{DLR}} \beta$. Let $X$ and $Y$ have distribution functions $F$ and $G$ defined in (4.2). Then $X \leq_{\text{LT − RRL}} Y$.

To demonstrate the usefulness of the above results in recognizing ($LT − RRL$)-ordered random variables, we consider the following examples.

**Example 3.1.** Let $X_\lambda$ denote the convolution of $n$ exponential distributions with parameters $\lambda_1, \ldots, \lambda_n$ respectively. Assume without loss of generality that $\lambda_1 \leq \ldots \leq \lambda_n$. Since exponential densities are log-concave, Theorem 3.3 implies that $X_\lambda \leq_{\text{LT − RRL}} Y_\mu$ whenever $\lambda_i \geq \mu_i$ for $i = 1, \ldots, n$.

**Example 3.2.** Let $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, \ldots, n$ be independent random variables. Let $X$ and $Y$ be $\alpha$ and $\beta$ mixtures of $X_i$’s. An application of Theorem 4.3, immediately $X \leq_{\text{LT − RRL}} Y$ for every two probability vector $\alpha$ and $\beta$ such that $\alpha \leq_{\text{DLR}} \beta$.

Another application of Theorem 3.3 is contained in following example.

**Example 3.3.** Let $X_\lambda$ and $X_\mu$ be as given in Example 3.1. For $0 \leq q \leq p \leq 1$ and $p + q = 1$, we have

$$pX_\lambda + qX_\mu \leq_{\text{LT − RRL}} qX_\lambda + pX_\mu.$$
which the $Lt - RRL$ order is closed under parallel systems. The conditions
here are the same as those given in Theorem 5.3(ii) of Alzaid et al. (1991) (see
also Theorem 3.B.7(b) of Shaked and Shanthikumar (1994)) for the Laplace
transform order.

We first recall the definition of completely monotone ($c.m$) functions. A
function $h : R_+ \to R$ is said to be completely monotone if all its derivatives
$h^{(n)}$ exist and satisfy $h^{(0)} \equiv h \geq 0$, $h^{(1)} \leq 0$, $h^{(2)} \geq 0$, .... More formally, $h$
$c.m.$ if $(-1)^n h^{(n)}(x) \geq 0$ for all $x > 0$ and $n = 0, 1, 2, ...$

**Theorem 3.4.** Let the independent non-negative random variables $X_1, X_2,...,X_n,$
$Y_1, Y_2,...,Y_n$ have the distribution functions $F_1, F_2,...,F_n$, $G_1, G_2,...,G_n,$ re-
spectively. If $X_i \leqLT-RRL Y_i$, $i = 1, 2, ..., n$, and $F_i$ and $G_i$ are completely
monotone then

$$\max\{X_1, ..., X_n\} \leqLT-RRL \max\{Y_1, ..., Y_n\}.$$ 

**Proof.** Denote $T_n = \max\{X_1, ..., X_n\}$ and $W_n = \max\{Y_1, ..., Y_n\}$. Note that,
for the maximum, it holds

$$[t - T_n \mid T_n < t] = \max\{[t - X_i \mid X_i < t], i = 1, ..., n\},$$

and

$$[t - W_n \mid W_n < t] = \max\{[t - Y_i \mid Y_i < t], i = 1, ..., n\},$$

for all $t \geq 0$. Let $F_{ijt}$ and $G_{ijt}$ are c.m. if $F_i$ and $G_i$ are c.m.

Fix $t \geq 0$. By the assumption $X_i \leqLT-RRL Y_i$, we have

$$[t - X_i \mid X_i < t] \leqLT [t - Y_i \mid Y_i < t].$$

Then, by Theorem 3.B.7(b) of Shaked and Shanthikumar (1994), we have

$$\max\{[t - X_i \mid X_i < t], i = 1, ..., n\} \leqLT \max\{[t - Y_i \mid Y_i < t], i = 1, ..., n\}.$$ 

Thus,

$$[t - T_n \mid T_n < t] \leqLT [t - W_n \mid W_n < t] \quad \text{for all } t \geq 0,$$

implying that $T_n \leqLT-RRL W_n$. This completes the proof.

Unfortunately, the proof above does not hold for more general coherent
systems.
4 Monotone transformation

We first recall the definition of completely monotone (c.m.) functions. A function \( h : \mathbb{R}_+ \to \mathbb{R} \) is said to be completely monotone if all its derivatives \( h^{(n)} \) exist and satisfy \( h^{(0)} \equiv h \geq 0, h^{(1)} \leq 0, h^{(2)} \geq 0, \ldots \). More formally, \( h \) is c.m. if \((-1)^n h^{(n)}(x) \geq 0\) for all \( x > 0 \) and \( n = 0, 1, 2, \ldots \).

The next theorem states that the \( Lt - RRL \) order is preserved under c.m. transformations. One may refer to Alzaid et al. (1991) for a similar preservation property of the \( \leq_{LT} \) order.

**Theorem 4.1.** Let \( X \) and \( Y \) be two nonnegative random variables. Then \( X \leq_{LT-RRL} Y \) if and only if \( h(X) \leq_{LT-RRL} h(y) \) for all non-negative function \( h \) with a c.m. derivative.

**Proof.** We give the proof of the necessity only. Let \( h \) be any nonnegative function with a c.m. derivative, and suppose that \( X \leq_{LT-RRL} Y \). We have to prove that

\[
[h(X) \mid h(X) < t] \geq_{LT} [h(Y) \mid h(Y) < t] \quad \text{for all } t \in \mathbb{R}^+.
\] (4.1)

From the assumption, it follows that \( X_{h^{-1}(t)} \geq_{LT} Y_{h^{-1}(t)} \) or, equivalently,

\[
[X \mid X < h^{-1}(t)] \geq_{LT} [Y \mid Y < h^{-1}(t)] \quad \text{for all } t \in \mathbb{R}^+.
\] (4.2)

Here the inverse \( h^{-1} \) of \( h \) is taken to be the right continuous version of it defined by \( h^{-1}(u) = \sup\{x : h(x) \leq u\} \) for \( u \in \mathbb{R} \). From the definition of \( h^{-1} \) and the continuity of \( h \), it is easy to check that \( x < h^{-1}(t) \) if and only if \( h(x) < t \). Thus (4.2) can be rewritten as

\[
[X \mid h(X) < t] \geq_{LT} [Y \mid h(Y) < t] \quad \text{for each } t \in \mathbb{R}^+.
\]

By Corollary 3.2 of Alzaid et al. (1991), we get that

\[
h ([X \mid h(X) < t]) \geq_{LT} h ([Y \mid h(Y) < t]) \quad \text{for each } t \in \mathbb{R}^+.
\]

implying (4.1). This completes the proof.

The following theorem characterizes the \( LT - RRL \) order in terms of the reversed mean residual life \( \leq_{RMRRL} \) order.

**Theorem 4.2.** Let \( X \) and \( Y \) be two nonnegative random variables. Then
$X \leq_{LT-RRL} Y$ if and only if $h(X) \leq_{RMRL} h(Y)$ for each nonnegative function $h$ with a c.m. derivative.

**Proof. Necessity**: Suppose that $X \leq_{LT-RRL} Y$ and that $h$ is any non-negative function with a c.m. derivative. Then, by Theorem 3.1, we get $h(X) \leq_{LT-RRL} h(Y)$. Since the order $\leq_{LT-RRL}$ is stronger than the order $\leq_{RMRL}$ (see Kayid and Ahmad, 2004) it follows that $h(X) \leq_{RMRL} h(Y)$.

**Sufficiency**: Let $h$ be a nonnegative function with a c.m. derivative, and fix $s > 0$. Since $\eta(x) = 1 - e^{-sx}$ is nonnegative with a c.m. derivative, then it is easy to see that $\eta(h(x))$ is also nonnegative with a c.m. derivative. So, by the assumption, we have that $\eta(h(X)) \leq_{RMRL} \eta(h(Y))$; that is,

$$E[\eta(h(X)) \mid \eta(h(X)) < t] \geq E[\eta(h(Y)) \mid \eta(h(Y)) < t] \quad \text{for all } t \in R^+,$$

or, equivalently,

$$E[\eta(h(X)) \mid h(X) < t] \geq E[\eta(h(Y)) \mid h(Y) < t] \quad \text{for all } t \in R^+. \quad (3.3)$$

By the definition of the order $\leq_{LT}$, it follows from (3.3) that

$$[h(X) \mid h(X) < t] \geq_{LT} [h(Y) \mid h(Y) < t] \quad \text{for all } t \in R^+.$$

implying $h(X) \leq_{LT-RRL} h(Y)$. Therefore, the desired result follows from Theorem 4.1.

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**REFERENCES**


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