

On Linear Combination of Two Classes of Triangle Summation Operators of Bernstein Type

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Abstract

To improve the convergence properties of the Lagrange interpolation operators, Bernstein constructed two classes of operators. Subsequently, many experts generalized these operators, and acquired significant advances. In this work, we generalize the previous works and construct a new family of triangle summation operators $U_n(f; \rho, \theta)$ based on the equidistant nodes by combining two classes of operators of Bernstein type linearly, i.e., $H_n(f; 2e - 1, \theta)$ and $G_n(f; 2e, \theta)$. We prove that the new family of operators converges to arbitrary continuous functions with period 2π uniformly on $(-\infty, +\infty)$ as $n \rightarrow \infty$. In particular, the new family of operators has the best convergence order and its highest convergence order is $1/n^{2\rho+4}$, where ρ is an arbitrary odd natural number. In contrast to other triangle summation operators, the convergence properties of the new family of operators are superior to others.

Keywords: triangle summation operator of Bernstein type; uniform convergence; the best convergence order; the highest convergence order

1. Introduction

Let $C_{2\pi}$ be the continuous function space of the periodic function with period 2π , and $f(\theta) \in C_{2\pi}$, then the partial summation of the Fourier series of $f(\theta)$ is given by

$$S_n(f; \theta) = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos k\theta + b_k \sin k\theta), \quad (1)$$

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where a_k, b_k are the coefficients of the Fourier series. Let $S_n(f; \theta_k^{(n)}) = f(\theta_k^{(n)})$, where

$$\theta_k^{(n)} = \frac{(2k+1)\pi}{2n+1}, \quad k = 0, 1, 2, \dots, 2n, \quad (2)$$

are the equidistant nodes, we then have

$$a_0 = \frac{1}{2n+1} \sum_{u=0}^{2n} f(\theta_u^{(n)}), \quad a_k = \frac{2}{2n+1} \sum_{u=0}^{2n} f(\theta_u^{(n)}) \cos k\theta_u^{(n)},$$

$$b_k = \frac{2}{2n+1} \sum_{u=0}^{2n} f(\theta_u^{(n)}) \sin k\theta_u^{(n)}, \quad (3)$$

and

$$S_n(f; \theta) = \frac{1}{2n+1} \sum_{k=0}^{2n} f(\theta_k^{(n)}) \left[1 + 2 \sum_{m=1}^n \cos m(\theta - \theta_k^{(n)}) \right]. \quad (4)$$

On the other hand, by simple induction, $S_n(f; \theta)$ can also be written as

$$S_n(f; \theta) = \sum_{k=0}^{2n} f(\theta_k^{(n)}) \mu_k^{(n)}(\theta), \quad (5)$$

where

$$\mu_k^{(n)}(\theta) = \frac{\sin \frac{2n+1}{2}(\theta - \theta_k^{(n)})}{(2n+1) \sin \frac{1}{2}(\theta - \theta_k^{(n)})}, \quad k = 0, 1, 2, \dots, 2n \quad (6)$$

can be deemed to the basic functions of the Lagrange interpolation operator based on (2) as the interpolation nodes, and (5) is then called the corresponding Lagrange interpolation operator.

As is well known, the Lagrange interpolation operators have extensive application in the domains of nature science, engineering technology, etc. However, from the famous Bernstein's Theorem^[1], one can know that the Lagrange interpolation operators do not converge to arbitrary continuous functions uniformly. Thus, one of the basic problems of approximation theory is how to improve the convergence properties of the Lagrange interpolation operators. The most representative works on this aspect are due to Bernstein^[1], Kis^[2], Varma^[3], and so on. Furthermore, another interesting problem is how to construct some operators to make their convergence orders the highest under the same conditions. At present, numerous significant investigations on this aspect have been made. See the review article by Shen^[4] for a comprehensive review of works up to 1983. Further references in recent years may be found in [5-12].

To improve the convergence properties of the Lagrange interpolation operators, such as (5), Bernstein^[1] introduced an operator through combining the basic functions of the Lagrange interpolation operator, i.e.,

$$A_n(f; \theta) = \frac{1}{2} \sum_{k=0}^{2n} f(\theta_k^{(n)}) \left(\mu_k^{(n)}(\theta) + \mu_{k+1}^{(n)}(\theta) \right). \quad (7)$$

But the estimation expression of the operator that approximates to $f(\theta) \in C_{2\pi}$ has not been obtained by Kis^[2] until 1973, as follows,

$$|A_n(f; \theta) - f(\theta)| \leq \frac{19}{3\pi} \omega\left(f, \frac{2\pi}{2n+1}\right). \quad (8)$$

Further, Kis^[2] also introduced the following operator in his work, namely,

$$C_n(f; \theta) = \frac{1}{4} \sum_{k=0}^{2n} f(\theta_k^{(n)}) \left(\mu_{k-1}^{(n)}(\theta) + 2\mu_k^{(n)}(\theta) + \mu_{k+1}^{(n)}(\theta) \right), \quad (9)$$

and he carried out the estimation

$$|C_n(f; \theta) - f(\theta)| \leq \frac{13}{3\pi} \omega\left(f, \frac{2\pi}{2n+1}\right). \quad (10)$$

In 1997, He and Zhang^[5] introduced the following operator through rearranging the basic functions of the Lagrange interpolation operator, i.e.,

$$H_n(f; \rho, \theta) = \sum_{k=0}^{2n} f(\theta_k^{(n)}) m_{k,\rho}(\theta), \quad (11)$$

where

$$m_{k,\rho}(\theta) = \frac{1}{2^{\rho+1}} \sum_{i=0}^{\rho+1} \binom{\rho+1}{i} \left(\mu_k^{(n)}(\theta) + (-1)^{s+i} \mu_{k+s-1-i}^{(n)}(\theta) \right), \quad (12)$$

ρ is an arbitrary nonnegative integer, and $s = \left[\frac{\rho+1}{2} \right] + 1$, in which $[a]$ denotes the integer part of a . On $H_n(f; \rho, \theta)$, the authors carried out the following results:

- (i) $H_n(f; \rho, \theta)$ converges to arbitrary continuous functions with period 2π uniformly on $(-\infty, +\infty)$;
- (ii) $H_n(f; \rho, \theta)$ has the best convergence order if $f(\theta) \in C_{2\pi}^b$, ($0 \leq b \leq \rho$);
- (iii) The highest convergence order cannot exceed $1/n^{\rho+1}$.

In fact, as $\rho = 0$, it is easy to see that $A_n(f; \theta)$ is a special case of $H_n(f; \rho, \theta)$. Similarly, $C_n(f; \theta)$ is another special case of $H_n(f; \rho, \theta)$ as $\rho = 1$. Furthermore, Bernstein^[2] constructed another operator (it is usually called the Bernstein summation operator of the second),

$$B_n(f; \theta) = \frac{1}{2} \sum_{k=0}^{2n} f(\theta_k^{(n)}) \left(\mu_k^{(n)}\left(\theta - \frac{\pi}{2n+1}\right) + \mu_k^{(n)}\left(\theta + \frac{\pi}{2n+1}\right) \right). \quad (13)$$

On $B_n(f; \theta)$, we know that^[2]:

- (i) $B_n(f; \theta)$ converges to arbitrary continuous functions with period 2π uniformly on $(-\infty, +\infty)$;

- (ii) $B_n(f; \theta)$ has the best convergence order if $f(\theta) \in C_{2\pi}^b$, ($b = 0, 1$);
- (iii) The highest convergence order cannot exceed $1/n^2$.

2. Main results

In this paper, let ρ be a prescribed odd natural number, and $l = \frac{\rho+1}{2}$. For convenience, we rewrite $H_n(f; \rho, \theta)$ as $H_n(f; 2e - 1, \theta)$, namely,

$$H_n(f; 2e - 1, \theta) = \sum_{k=0}^{2n} f(\theta_k^{(n)}) m_{k, 2e-1}(\theta), \quad (14)$$

$$m_{k, 2e-1}(\theta) = \frac{1}{2^{2e}} \sum_{i=0}^{2e} \binom{2e}{i} \left(\mu_k^{(n)}(\theta) + (-1)^{e+1+i} \mu_{k+e-i}^{(n)}(\theta) \right). \quad (15)$$

Obviously, the expression (14) is equivalent to (11) as $e = 1, 2, \dots, l$. On the other hand, we denote the operator $G_n(f; 2e, \theta)$ by

$$G_n(f; 2e, \theta) = \sum_{k=0}^{2n} f(\theta_k^{(n)}) q_{k, 2e}(\theta), \quad (16)$$

where

$$q_{k, 2e}(\theta) = \frac{1}{2} \left(\mu_k^{(n)} \left(\theta - \frac{(2e-1)\pi}{2n+1} \right) + \mu_k^{(n)} \left(\theta + \frac{(2e-1)\pi}{2n+1} \right) \right), \quad (e = 1, 2, \dots, l+1). \quad (17)$$

It is easy to see that the operator, $B_n(f; \theta)$, given by (13), is a special case of $G_n(f; 2e, \theta)$ as $e = 1$, and the results which $B_n(f; \theta)$ possesses are also valid for $G_n(f; 2e, \theta)$.

In [6], Meng constructed an operator $D_n(f; \theta)$ by combining the operators $C_n(f; \theta)$, $B_n(f; \theta)$, and $G_n(f; 4, \theta)$, that is,

$$D_n(f; \theta) = \frac{1}{4} (15B_n(f; \theta) + G_n(f; 4, \theta) - 12C_n(f; \theta)). \quad (18)$$

On $D_n(f; \theta)$, the author obtained the following results:

- (i) $D_n(f; \theta)$ converges to arbitrary continuous functions with period 2π uniformly on $(-\infty, +\infty)$;
- (ii) $D_n(f; \theta)$ has the best convergence order if $f(\theta) \in C_{2\pi}^b$, ($0 \leq b \leq 5$);
- (iii) The highest convergence order cannot exceed $1/n^6$.

From the results given by the author, one can see that the convergence properties of $D_n(f; \theta)$ are superior to $C_n(f; \theta)$, $B_n(f; \theta)$ and $G_n(f; 4, \theta)$. In particular, the highest convergence order of $B_n(f; \theta)$, $G_n(f; 4, \theta)$ and $C_n(f; \theta)$ are all $1/n^2$, but that of $D_n(f; \theta)$ is $1/n^6$.

Naturally, what will be interesting is, for arbitrary continuous functions with period 2π , how to construct some operators to make their convergence

properties the best under the same conditions, in particular, make the convergence order the highest.

In this paper, we answer the above assumption satisfactorily. A new class of triangle summation operators, $U_n(f; \rho, \theta)$, based on the equidistant nodes is constructed through combining the operators of Bernstein type linearly, i.e., $H_n(f; 2e - 1, \theta)$ and $G_n(f; 2e, \theta)$.

The form of operator $U_n(f; \rho, \theta)$ is given by

$$U_n(f; \rho, \theta) = \sum_{e=1}^l a_{2e-1} H_n(f; 2e - 1, \theta) + \sum_{e=1}^{l+1} a_{2e} G_n(f; 2e, \theta), \tag{19}$$

where

$$a_\rho = \frac{1}{2} \binom{2\rho + 4}{1},$$

$$a_{\rho-2e} = \frac{(-1)^e}{2^{2e+1}} \left[\binom{2\rho + 4}{2e + 1} - \sum_{d=0}^{e-1} (-1)^{e-2d} 2^{2d+1} \alpha_{\rho-2d} \binom{\rho - 2d + 1}{e - d} \right],$$

$$(e = 1, 2, \dots, l - 1), \tag{20}$$

$$a_{2e} = -\frac{1}{2^{\rho+1}} \binom{2\rho + 4}{\rho + 3 - 2e}, \quad (e = 1, 2, \dots, l + 1). \tag{21}$$

From (20), one can see that the coefficients before $H_n(f; 2e - 1, \theta)$ are obtained by inverse deduction.

On $U_n(f; \rho, \theta)$, we obtain the following results

Theorem 2.1. If $f(\theta) \in C_{2\pi}$, then $\lim_{n \rightarrow \infty} U_n(f; \rho, \theta) = f(\theta)$ is valid uniformly on $(-\infty, +\infty)$.

Theorem 2.2. If $f(\theta) \in C_{2\pi}$, then

$$|U_n(f; \rho, \theta) - f(\theta)| = O(1) \left(E_n^*(f) + \omega_{2\rho+4}(f, \frac{1}{n}) \right), \tag{22}$$

where “ O ” is independent of n , $E_n^*(f)$ is the minimum deviation with $f(\theta)$, $\omega_{2\rho+4}(f, \delta)$ is the $(2\rho + 4)$ -th modulus of continuity of $f(\theta)$.

Theorem 2.3. For any $f(\theta) \in C_{2\pi}^b$, $(0 \leq b \leq 2\rho + 3)$, we have

$$|U_n(f; \rho, \theta) - f(\theta)| = O(1) \left(\frac{1}{n^b} \omega(f^{(b)}, \frac{1}{n}) \right), \tag{23}$$

where $\omega(f^{(b)}, \delta)$ is the modulus of continuity of $f^{(b)}(\theta)$.

Theorem 2.4. For arbitrary functions with any derivatives, the highest convergence order of $U_n(f; \rho, \theta)$ cannot exceed $1/n^{2\rho+4}$.

3. Lemma

Lemma 3.1. The following estimation expressions are valid:

$$\sum_{k=0}^{2n} |m_{k,2e-1}(\theta)| = \sum_{k=0}^{2n} \frac{1}{2^{2e}} \sum_{i=0}^{2e} \binom{2e}{i} \left| \mu_k^{(n)}(\theta) + (-1)^{e+1+i} \mu_{k+e-i}^{(n)}(\theta) \right| = O(1)^{[5]},$$

$$as \quad e = 1, 2, \dots, l \quad (24)$$

$$\sum_{k=0}^{2n} |q_{k,2e}(\theta)| = \sum_{k=0}^{2n} \left| \mu_k^{(n)}\left(\theta - \frac{(2e-1)\pi}{2n+1}\right) + \mu_k^{(n)}\left(\theta + \frac{(2e-1)\pi}{2n+1}\right) \right| = O(1)^{[1]},$$

$$as \quad e = 1, 2, \dots, l+1, \quad (25)$$

$$\sum_{k=0}^{2n} |\sigma_k(\theta)| = O(1), \quad (26)$$

where

$$\sigma_k(\theta) = \mu_k^{(n)}(\theta) + \frac{1}{2^{\rho+2}} \sum_{i=0}^{2\rho+4} \binom{2\rho+4}{i} (-1)^{2l+1+i} \mu_k^{(n)}\left(\theta + \frac{(2l+1-i)\pi}{2n+1}\right). \quad (27)$$

Proof. In fact, we need only to prove (26). From (4), it is easy to show that

$$\mu_k^{(n)}\left(\theta \pm \frac{2i\pi}{2n+1}\right) = \mu_{k\pm i}^{(n)}(\theta), \quad (i = 0, 1, 2, \dots, l). \quad (28)$$

Further, $\sigma_k(\theta)$ can be written as the linear combination of $m_{k,2e-1}(\theta)$ and $q_{k,2e}(\theta)$, namely,

$$\sigma_k(\theta) = \sum_{j=1}^l \beta_{2e-1} m_{k,2e-1}(\theta) + \sum_{j=1}^{l+1} \beta_{2e} q_{k,2e}(\theta), \quad (29)$$

and then we obtain a linear equations of lower triangle type of degree $\rho+1$ with regard to β_{2e-1} , and $\beta_{2e} = a_{2e}$ by using the method of unknown coefficients. Finally, by solving the equations, we get that $\beta_{2e-1} = a_{2e-1}$, where a_{2e-1} and a_{2e} are given by (20) and (21), respectively. Moreover, one can know that Lemma 3.1 is valid by using (24) and (25).

4. Proofs of the theorems

4.1. Proof of Theorem 2.1

Proof. In fact, from the result (i) on $H_n(f; \rho, \theta)$, we know that $H_n(f; 2e-1, \theta)$ converges to arbitrary continuous functions with period 2π uniformly on $(-\infty, +\infty)$ as $e = 1, 2, \dots, l$. So as to $G_n(f; 2e, \theta)$ as $e = 1, 2, \dots, l+1$. Thus

Theorem 1 is correct.

4.2. Proof of Theorem 2.2

From (27), the operator $U_n(f; \rho, \theta)$ can be rewritten as

$$U_n(f; \rho, \theta) = \sum_{k=0}^{2n} f(\theta_k^{(n)}) \sigma_k(\theta). \tag{30}$$

Let $g(\theta)$ be a triangle operator, which possesses the least deviation with $f(\theta)$, and its degree is less than n , then we have $|g(\theta) - f(\theta)| \leq E_n^*(f)^{[1]}$.

Furthermore, from the property of the Lagrange interpolation operators, namely, $g(\theta)$ coincides with its own interpolation operator, so it is not difficult to show that

$$U_n(g; \rho, \theta) = \sum_{k=0}^{2n} g(\theta_k^{(n)}) \sigma_k(\theta) = \frac{1}{2^{\rho+2}} \Delta_h^{2\rho+4} g(\theta) + g(\theta), \tag{31}$$

where

$$\Delta_h^{2\rho+4} g(\theta) = \sum_{i=0}^{2\rho+4} \binom{2\rho+4}{i} (-1)^{2l+1+i} g(\theta + (2l+1-i)h), \quad h = \frac{\pi}{2n+1}. \tag{32}$$

Further, we have

$$\begin{aligned} U_n(f; \rho, \theta) &= U_n(f - g; \rho, \theta) + U_n(g; \rho, \theta) - f(\theta) \\ &= \sum_{k=0}^{2n} (f(\theta_k^{(n)}) - g(\theta_k^{(n)})) \sigma_k(\theta) + \frac{1}{2^{\rho+2}} \Delta_h^{2\rho+4} f(\theta) \\ &\quad + (g(\theta) - f(\theta)) + \frac{1}{2^{\rho+2}} \Delta_h^{2\rho+4} (g(\theta) - f(\theta)) \\ &= \sum_{v=1}^4 e_v. \end{aligned} \tag{33}$$

For e_1 , from (26) in Lemma 3.1, we obtain

$$|e_1| = O(1) (E_n^*(f)). \tag{34}$$

Obviously, the following estimation is valid for e_2 , namely,

$$|e_2| = O(1) \left(\omega_{2\rho+4} \left(f, \frac{1}{n} \right) \right). \tag{35}$$

On the other hand, we also have

$$|e_v| = O(1) (E_n^*(f)), \quad v = 3, 4. \quad (36)$$

Thus, from (34)~(36), we know that Theorem 2.2 is correct.

4.3. Proof of Theorem 2.3

Let's reconsider the estimation of e_i , ($i = 1, 2, 3, 4$) in (33).

For $f(\theta) \in C_{2\pi}^b$, ($0 \leq b \leq 2\rho + 3$), from the Jackson's Theorem, we know that $|g(\theta) - f(\theta)| = O(1) \left(\frac{1}{n^b} \omega(f^{(b)}, \frac{1}{n})\right)^{[1]}$.

Thus we have

$$|e_v| = O(1) \left(\frac{1}{n^b} \omega(f^{(b)}, \frac{1}{n})\right), \quad (v = 1, 3, 4). \quad (37)$$

For e_2 , from the relation between the derivative and the difference, as $f(\theta) \in C_{2\pi}^b$, ($0 \leq b \leq 2\rho + 3$), we have

$$\Delta_h^b f(\theta) = - \left(-\frac{\pi}{2n+1}\right)^b f^{(b)}(\xi_\theta), \quad (38)$$

where ξ_θ lies between $\theta - (\rho+2)\frac{\pi}{2n+1}$ and $\theta + (b-\rho-2)\frac{\pi}{2n+1}$. Further, we have

$$\begin{aligned} |e_2| &= \frac{1}{2^{\rho+2}} \left| \sum_{c=0}^{2\rho+4-b} (-1)^c \binom{2\rho+4-b}{c} \Delta_h^b f\left(\theta - (c-\rho-2)\frac{\pi}{2n+1}\right) \right| \\ &= \frac{1}{2^{\rho+2}} \left(\frac{\pi}{2n+1}\right)^b \left| \sum_{c=0}^{2\rho+4-b} (-1)^c \binom{2\rho+4-b}{c} f^{(b)}(\xi_{\theta+(c-\rho-2)h}) \right| \\ &= \frac{1}{2^{\rho+2}} \left(\frac{\pi}{2n+1}\right)^b * \\ &\quad \left| \sum_{c=0}^{2\rho+3-b} (-1)^c \binom{2\rho+3-b}{c} (f^{(b)}(\xi_{\theta+(c-\rho-2)h}) - f^{(b)}(\xi_{\theta+(c-\rho-1)h})) \right| \\ &= O(1) \left(\frac{1}{n^b} \omega(f^{(b)}, \frac{1}{n})\right). \end{aligned} \quad (39)$$

From (37) and (39), Theorem 2.3 is correct.

4.4. Proof of Theorem 2.4

In fact, from (33), we need only to show that the highest convergence order of e_4 is $1/n^{2\rho+4}$. For arbitrary functions with any derivatives, we have

$$|e_4| = \frac{1}{2^{\rho+2}} \left(\frac{\pi}{2n+1} \right)^{2\rho+4} |f^{(2\rho+4)}(\xi)|, \quad (40)$$

where ξ lies between $\theta - (\rho+2)\frac{\pi}{2n+1}$ and $\theta + (\rho+2)\frac{\pi}{2n+1}$.

Let $f(x) = \sin x$, $x = \frac{\pi}{4}$, we obtain

$$|e_4| \approx \frac{\sqrt{2}}{2} \frac{1}{2^{\rho+2}} \left(\frac{\pi}{2n+1} \right)^{2\rho+4}, \quad (41)$$

as n is sufficient large.

Remarks. For $0 \leq b \leq 2\rho+3$, we have

$$|U_n(f; \rho, \theta) - f(\theta)| \leq \frac{A}{n^{b+\alpha}} \Leftrightarrow f(\theta) \in W^b H^\alpha, \quad 0 < \alpha < 1,$$

$$|U_n(f; \rho, \theta) - f(\theta)| \leq \frac{A}{n^{b+1}} \Leftrightarrow f(\theta) \in W^b Z,$$

where A is a constant, $W^b H^\alpha$ is the Hölder class, while $W^b Z$ is the Zygmund class.

5. Conclusions

In this paper, a new family of triangle summation operators $U_n(f; \rho, \theta)$ is constructed by combining two classes of operators of Bernstein type, i.e., $H_n(f; 2e-1, \theta)$, ($e = 1, 2, \dots, l$) and $G_n(f; 2e, \theta)$, ($e = 1, 2, \dots, l+1$). From Theorem 2.1, we can see that $U_n(f; \rho, \theta)$ converges to arbitrary continuous functions with period 2π uniformly on $(-\infty, +\infty)$ as $n \rightarrow \infty$. From Theorem 2.2 and 2.3, we know that $U_n(f; \rho, \theta)$ has the best convergence order if $f(\theta) \in C_{2\pi}^b$, ($0 \leq b \leq 2r+3$). On the other hand, since the highest convergence order of the summation operator $G_n(f; 2e, \theta)$ is only $1/n^2$, and the highest convergence order of $H_n(f; 2e-1, \theta)$ is $1/n^{2e}$, thus from Theorem 2.4, we know that the highest convergence order of $U_n(f; \rho, \theta)$ is superior to other operators.

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