

# Representation of Exact Solution for Singular Boundary Value Problems of System of Equations

Huanmin Yao <sup>a,b</sup> <sup>1</sup>, Yingzhen Lin <sup>a</sup> and Minggen Cui <sup>a</sup>

<sup>a</sup> Department of Mathematics  
Harbin Institute of Technology at Weihai  
Weihai, Shandong, 264209, P.R. China

<sup>b</sup> Department of Information Science  
Harbin Normal University  
Harbin, Heilongjiang, 150025, P.R. China

## Abstract

An approximation method with property of projection and interpolation for singular boundary value problems of system of equations under the reproducing kernel space  $W_2^3 \oplus W_2^3$  is considered. The exact solutions are given in the form of series. By truncating the series, we obtain the approximate solutions. The errors of the approximate solutions are monotone decreasing with the increasing of nodal points. The numerical results are displayed to demonstrate the validity of this method.

**Keywords:** System of equations; Singular boundary value problem; Reproducing kernel space

## 1 Introduction

There is considerable interest on numerical method for solving singular boundary value problems as follows:

$$p(x)u''(x) + \frac{1}{q(x)}u'(x) + \frac{1}{r(x)}u(x) = f(x), \quad 0 < x < 1 \quad (1.1)$$

with the boundary conditions

$$u(0) = \alpha, u(1) = \beta \quad (1.2)$$

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<sup>1</sup>Corresponding author  
E-mail address: cmgyfs@263.net

where function  $p(x), q(x) \in C(0, 1)$ ,  $q(0)q(1) = 0$  or  $r(0)r(1) = 0$ ,  $r(x), f(x) \in W_2^1[0, 1]$  and  $\alpha, \beta$  are finite constants. Singular boundary value problems for ordinary differential equations arise very frequently in the theory of thermal explosions and in the study of Electro-hydrodynamics. Such problems also occur in the study of generalized axially symmetric potentials after separation of variables.

Many authors are interested in the study of singular boundary value problems for equation. Jamet[1] discussed existence and uniqueness of solutions of second-order linear singular boundary value problems and presented finite difference method for numerically solving such problems and considered the usual three point finite difference scheme for singular boundary value problems and showed in the maximum norm that their scheme is  $O(h^{1-\alpha})$  convergent. The usual classical three-point finite difference discretization for singular boundary value problems was studied by Russell and Shampine[2]. Erikson and Thomee[3] studied the Garlekin type piece wise polynomial procedure for the problems and provide fourth order methods for the singular boundary value problems. A.S.V.Ravi Kanth and Y.N.Reddy[4] presented a numerical method for solving boundary value problems with regular singularity. The singular problem over the interval  $[0, 1]$  was first reduced to regular problem over  $[\delta, 1]$ ,  $\delta > 0$  near the singularity. It was done by making use of Chebshey economization in the vicinity of the singularity and obtaining a boundary condition at  $x = \delta$ . They also discussed a direct method for solving singular boundary value problem. The original differential equation was modified at the singular point. The fourth order finite difference method was then employed to solve the boundary value problem. By stabilizing the classical central difference method(CD), they developed a fourth order finite difference method. To obtain this method, they re-approximate the CD approximation by rewriting its error terms as a combination of first and second derivative terms and approximating them(see *Ref.*[5]). B-spline method for numerically solving singular two-point boundary value problems for certain ordinary differential equation having singular coefficients was presented by Mohan K.Kadalbajoo and Vivek K.Aggarwal[6]. W.Auzinger,O.Koch and E.Weinmuller[7] described a mesh selection strategy for the numerical solution of boundary value problems for singular ordinary differential equations.This mesh adaptation procedure was implemented in their MATLAB code sbvp which was based on polynomial collocation. They proved that under realistic assumptions their mesh selection strategy serves to approximately equidistribute the global error of the collocation solution, thus enabling to reach prescribed tolerances efficiently. J.R. Cash,F.Mazzia,N.Sumarti and D.Trigiant[8] provided an estimate of the condition number of the problem as well as the numerical solution. They considered some algorithms for estimating the condition number of boundary value problems and showed how this estimate can be used in the grid refine-

ment algorithm. However, few thesis about singular boundary value problems of the system of equations in recent years.

In this paper, we consider the following system of equations lead to singular boundary value problems given by

$$\begin{cases} p_1(x)u''(x) + a_1(x)v'(x) + b_1(x)u(x) + c_1(x)v(x) = f(x), \\ p_2(x)v''(x) + a_2(x)u'(x) + b_2(x)v(x) + c_2(x)u(x) = g(x), \\ u(0) = v(0) = u(1) = v(1) = 0 \end{cases} \quad (1.3)$$

Where  $p_i(x) \in C[0, 1]$ ,  $a_i(x)$ ,  $b_i(x)$  and  $c_i(x)$  may be singular at  $x = 0$  or  $1$  ( $i = 1, 2$ ), namely

$$a_i(0) = b_i(0) = c_i(0) = \infty, a_i(1) = b_i(1) = c_i(1) = \infty \quad (1.4)$$

The representation of exact solution of Eq.(1.3) is given in the form of series. By truncating the series, we obtain the approximate solutions. It is worth mentioning that, using the method of projection under the reproducing kernel space  $W_2^3[0, 1] \oplus W_2^3[0, 1]$ , we guarantee the approximate solutions satisfy accurately the system Eq.(1.3) at the discrete points. Our method is a new approach which combine projection with approximation by interpolation. Moreover, the errors of the approximate solutions are monotone decreasing with the increasing of nodal points. The numerical results are displayed to demonstrate the validity of this method.

## 2 Several reproducing kernel spaces

### 2.1 The reproducing kernel space $W_2^3[0, 1]$

The reproducing kernel space  $W_2^3[0, 1]$  is defined by

$W_2^3[0, 1] = \{u(x) | u, u', u'' \text{ are one-variable absolutely continuous real value functions in } [0, 1], u, u', u'', u^{(3)} \in L^2[0, 1], u(0) = u(1) = 0\}$

The inner product and norm are defined as follows

$$\langle u, v \rangle_{W_2^3} = \int_0^1 (uv + 3u'v' + 3u''v'' + u'''v''') dx \quad (2.1)$$

$$\|u\|_{W_2^3} = \sqrt{\langle u, u \rangle_{W_2^3}} \quad (2.2)$$

$W_2^3[0, 1]$  is a complete reproducing kernel space, that is, for any  $u(y) \in W_2^3[0, 1]$  and each fixed  $x \in [0, 1]$ , there exists  $R_x(y) \in W_2^3[0, 1], y \in [0, 1]$ , such that  $\langle u(y), R_x(y) \rangle_{W_2^3} = u(x)$ , the reproducing kernel  $R_x(y)$  can be denoted by

$$R_x(y) = \begin{cases} R_{1x}(y), & y \leq x, \\ R_{2x}(y), & y > x, \end{cases} \quad (2.3)$$

Where

$$R_{1x}(y) = (e^{-x-y}(49e^{6+2y}(3+x^2+x(3-2y)-3y+y^2)+49e^{2x}(3+x^2+3y+y^2-x(3+2y))+7e^{2(2+x+y)}(93-43y+5y^2+x(-43+32y-6y^2)+x^2(5-6y+2y^2))+7e^{2(x+y)}(39-15y+y^2+x(-15+16y-2y^2)+x^2(1-2y+2y^2))+7e^2(25-y-y^2+x(-1+16y-2y^2)+x^2(1-2y+2y^2))+14e^{4+2y}(-12+19y-9y^2-x^2(7+2y)+x(-17+14y+2y^2))+7e^6(39+15y+y^2+x^2(1+2y+2y^2)+x(15+16y+2y^2))+7e^{4+2x}(63+53y+15y^2+x^2(11+4y)-x(49+30y+4y^2))+e^{2+2y}(37-39y+53y^2+x(59-90y-36y^2)+x^2(25+20y+8y^2))+2e^{2+2x}(-310-299y-97y^2+x^2(-83-18y+4y^2)+x(309+174y+10y^2))-2e^{2(1+x+y)}(550-219y+17y^2+x(-219+204y-40y^2)+x^2(17-40y+18y^2))-2e^4(288+65y-5y^2+x(65+116y+4y^2)+x^2(-5+4y+18y^2)))/(16(-49+219e^2-203e^4+49e^6))$$

$$R_{2x}(y) = (e^{-x-y}(49e^{2y}(3+x^2+x(3-2y)-3y+y^2)+49e^{6+2x}(3+x^2+3y+y^2-x(3+2y))+14e^{4+2y}(-12-17y-7y^2+x^2(-9+2y)+x(19+14y-2y^2))+7e^{2(2+x+y)}(93-43y+5y^2+x(-43+32y-6y^2)+x^2(5-6y+2y^2))+7e^{2(x+y)}(39-15y+y^2+x(-15+16y-2y^2)+x^2(1-2y+2y^2))+7e^{2(x+y)}(39-15y+y^2+x(-15+16y-2y^2)+x^2(1-2y+2y^2))+7e^6(39+15y+y^2+x^2(1+2y+2y^2)+x(15+16y+2y^2))+7e^{4+2x}(63+x^2(15-4y)-49y+11y^2+x(53-30y+4y^2))+2e^{2+2x}(-310+309y-83y^2+x(-299+174y-18y^2)+x^2(-97+10y+4y^2))-2e^{2(1+x+y)}(550-219y+17y^2+x(-219+204y-40y^2)+x^2(17-40y+18y^2))-2e^4(288+65y-5y^2+x(65+116y+4y^2)+x^2(-5+4y+18y^2))+e^{2+2x}(37+59y+25y^2+x^2(53-36y+8y^2)+x(-39-90y+20y^2)))/(16(-49+219e^2-203e^4+49e^6))$$

We can prove the  $R_x(y)$  given by (2.3) is reproducing kernel in complete space  $W_2^3[0, 1]$ . (See *Ref.*[9])

## 2.2 The reproducing kernel space $W_2^1[0, 1]$

The inner product space  $W_2^1[0, 1]$  is defined by

$$W_2^1[0, 1] = \{u(x)|u \text{ is absolutely continuous real value function in } [0, 1], u' \in L^2[0, 1]\}$$

The inner product is given by

$$\langle u, v \rangle_{W_2^1} = \int_0^1 (uv + u'v') dx, \quad (2.4)$$

The norm is denoted by

$$\|u\|_{W_2^1} = \sqrt{\langle u, u \rangle_{W_2^1}} \quad (2.5)$$

The reproducing kernel function  $Q_x(y)$  is (See *Ref.*[10])

$$Q_x(y) = \frac{1}{2\sinh(1)} [\cosh(x+y-1) + \cosh(|x-y|-1)] \quad (2.6)$$

It follows that for arbitrary fixed  $x \in [0, 1]$ ,  $Q_x(y) \in W_2^1[0, 1]$ , and for any  $u(x) \in W_2^1[0, 1]$ ,

$$\langle u(y), Q_x(y) \rangle_{W_2^1} = u(x) \quad (2.7)$$

### 3 Main results

In this section, a kind of singular boundary value problems Eq.(1.3) of system of equations are considered on the assumption that the solution is existent and unique.

#### 3.1 Introduction to a linear operator

After multiplying  $a_i^{-1}(x)b_i^{-1}(x)c_i^{-1}(x)$  ( $i = 1, 2$ ) both sides of Eq.(3.5), we obtain

$$\begin{cases} a_1^{-1}(x)b_1^{-1}(x)c_1^{-1}(x)p_1(x)u''(x) + b_1^{-1}(x)c_1^{-1}(x)v'(x) + a_1^{-1}(x)c_1^{-1}(x)u(x) \\ \quad + a_1^{-1}(x)b_1^{-1}(x)v(x) = a_1^{-1}(x)b_1^{-1}(x)c_1^{-1}(x)f(x) \\ a_2^{-1}(x)b_2^{-1}(x)c_2^{-1}(x)p_2(x)v''(x) + b_2^{-1}(x)c_2^{-1}(x)u'(x) + a_2^{-1}(x)b_2^{-1}(x)u(x) \\ \quad + a_2^{-1}(x)c_2^{-1}(x)v(x) = a_2^{-1}(x)b_2^{-1}(x)c_2^{-1}(x)g(x) \\ u(0) = v(0) = 0, \quad u(1) = v(1) = 0 \end{cases} \quad (3.1)$$

In order to solve Eq.(3.1), we introduce linear operators  $A_{ij} : W_2^3[0, 1] \rightarrow W_2^1[0, 1]$  ( $i, j = 1, 2$ ) as follows

$$\begin{aligned} (A_{11}u)(x) &\stackrel{def}{=} a_1^{-1}(x)b_1^{-1}(x)c_1^{-1}(x)p_1(x)u''(x) + a_1^{-1}(x)c_1^{-1}(x)u(x) \\ (A_{12}v)(x) &\stackrel{def}{=} b_1^{-1}(x)c_1^{-1}(x)v'(x) + a_1^{-1}(x)b_1^{-1}(x)v(x) \\ (A_{21}u)(x) &\stackrel{def}{=} b_2^{-1}(x)c_2^{-1}(x)u'(x) + a_2^{-1}(x)b_2^{-1}(x)u(x) \\ (A_{22}v)(x) &\stackrel{def}{=} a_2^{-1}(x)b_2^{-1}(x)c_2^{-1}(x)p_2(x)v''(x) + a_2^{-1}(x)c_2^{-1}(x)v(x) \end{aligned} \quad (3.2)$$

The inner product space  $W$  is defined by

$$W \triangleq W_2^3 \oplus W_2^3 = \{U | U = (u, v)^\top, u, v \in W_2^3[0, 1]\} \quad (3.3)$$

The inner product is defined by

$$\langle U, V \rangle_W = \langle (u_1, v_1)^\top, (u_2, v_2)^\top \rangle_W \stackrel{def}{=} \langle u_1, u_2 \rangle_{W_2^3} + \langle v_1, v_2 \rangle_{W_2^3} \quad (3.4)$$

Thus Eq.(3.1) can be rewritten into operator equations

$$\begin{cases} AU = F \\ U(0) = U(1) = 0 \end{cases} \quad (3.5)$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, U = (u, v)^\top, F = (f, g)^\top \quad (3.6)$$

Obviously,  $A : W_2^3 \oplus W_2^3 \rightarrow W_2^1 \oplus W_2^1$  is a bounded linear operator.

**Lemma 3.1** The conjugate operator of  $A$  is  $A^* = \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix}$

It is easy to prove the correctness of Lemma 3.1.

Let  $\{x_i\}_{i=1}^{\infty}$  be a dense set in  $[0, 1]$ , without loss of generality, we denote

$$\varphi_{2i-1}(x) = (Q_{x_i}(x), 0)^{\top}, \varphi_{2i}(x) = (0, Q_{x_i}(x))^{\top} \quad (3.7)$$

where  $Q_y(x)$  is given by (2.5). Denote  $\psi_i(x) = A^* \varphi_i(x)$  ( $i = 1, 2, \dots$ ).

**Lemma 3.2**  $\{\psi_i(x)\}_{i=1}^{\infty}$  is the complete system of  $W$ .

*Proof.* For  $U \in W$ , let  $\langle U, \psi_j(x) \rangle = 0$  ( $j = 1, 2, \dots$ )

$$\begin{aligned} \langle U, \psi_{2j-1}(x) \rangle &= \langle A(u, v)^{\top}, (Q_{x_j}(x), 0)^{\top} \rangle \\ &= \langle (A_{11}u + A_{12}v, A_{21}u + A_{22}v)^{\top}, (Q_{x_j}(x), 0)^{\top} \rangle \\ &= (A_{11}u + A_{12}v)(x_j) \\ &= 0 \\ \langle U, \psi_{2j}(x) \rangle &= \langle A(u, v)^{\top}, (0, Q_{x_j}(x))^{\top} \rangle \\ &= \langle (A_{11}u + A_{12}v, A_{21}u + A_{22}v)^{\top}, (0, Q_{x_j}(x))^{\top} \rangle \\ &= (A_{21}u + A_{22}v)(x_j) \\ &= 0 \end{aligned}$$

Therefore  $A(x_j) = 0$ . Since  $\{x_j\}_{j=1}^{\infty}$  is a dense set in  $[0, 1]$ , we have  $AU = 0$ . It follows that  $U = (0, 0)^{\top}$  from the existence of  $A^{-1}$ .  $\square$

Using Gram-Schmidt process of  $\{\psi_i(x)\}_{i=1}^{\infty}$ , we obtain an normal orthogonal functions system  $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$

$$\bar{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x), \quad (3.8)$$

where  $\beta_{ik}$  are orthogonalization coefficients,  $i = 1, 2, \dots, k \leq i$ .

### 3.2 The representation of solution to Eq.(3.4)

**Theorem 3.3** If  $U$  are the solutions of Eq.(3.4), then  $U$  can be expressed as follows

$$U = (u, v)^{\top} = \sum_{i=1}^{\infty} \left[ \sum_{2k-1 \leq i} \beta_{i2k-1} f(x_k) + \sum_{2k \leq i} \beta_{i2k} g(x_k) \right] \bar{\psi}_i(x)$$

*Proof.* Let  $U$  be the solution of Eq(3.4), thus

$$\begin{aligned} U(x) &= \sum_{i=1}^{\infty} \langle U, \bar{\psi}_i(x) \rangle_W \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \langle U, \sum_{k=1}^i \beta_{ik} \psi_k(x) \rangle_W \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i \beta_{ik} \langle U, \psi_k(x) \rangle_W \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i \langle (AU, \varphi_k) \rangle_W \bar{\psi}_i(x) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} [\sum_{2k-1 \leq i} \beta_{i2k-1} \langle (AU, \varphi_{2k-1})_W \rangle + \sum_{2k \leq i} \beta_{i2k} \langle (AU, \varphi_{2k})_W \rangle] \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} [\sum_{2k-1 \leq i} \beta_{i2k-1} \langle (A_{11}u + A_{12}v, A_{21}u + A_{22}v)^\top, (Q_{x_k}, 0)^\top \rangle \\
&\quad + \sum_{2k \leq i} \beta_{i2k} \langle (A_{11}u + A_{12}v, A_{21}u + A_{22}v)^\top, (0, Q_{x_k})^\top \rangle] \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} [\sum_{2k-1 \leq i} \beta_{i2k-1} (A_{11}u + A_{12}v)(x_k) + \sum_{2k \leq i} \beta_{i2k} (A_{21}u + A_{22}v)(x_k)] \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} [\sum_{2k-1 \leq i} \beta_{i2k-1} f(x_k) + \sum_{2k \leq i} \beta_{i2k} g(x_k)] \bar{\psi}_i(x)
\end{aligned}$$

□

Note  $\Psi_n = \text{span}\{\psi_1, \psi_2, \dots, \psi_n\}$ ,  $P_n : W_2^3 \oplus W_2^3 \rightarrow \Psi_n$  is the projector. We denote the approximate solution of  $U$  by

$$U_n = P_n U = \sum_{i=1}^n [\sum_{2k-1 \leq i} \beta_{i2k-1} f(x_k) + \sum_{2k \leq i} \beta_{i2k} g(x_k)] \bar{\psi}_i(x)$$

**Theorem 3.4** The approximate solutions satisfy accurately the system at the discrete points, namely

$$(AU_n)(x_i) = F(x_i), i = 1, 2, \dots, n$$

Where

$$F(x_i) = (f(x_i), g(x_i))^\top, i = 1, 2, \dots, n$$

*Proof.*

$$\begin{aligned}
(A_{11}u_n + A_{12}v_n)(x_i) &= \langle AU_n, \varphi_{2i-1}(x) \rangle_{W_2^3} \\
&= \left\langle \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} (u_n, v_n)^\top, (Q_{x_i}(x), 0)^\top \right\rangle_{W_2^3} \\
&= \langle (A_{11}u_n + A_{12}v_n, A_{21}u_n + A_{22}v_n)^\top, (Q_{x_i}(x), 0)^\top \rangle_{W_2^3} \\
&= \langle AP_n U, (Q_{x_i}(x), 0)^\top \rangle_{W_2^3} \\
&= \langle P_n U, A^*(Q_{x_i}(x), 0)^\top \rangle_{W_2^3} \\
&= \langle P_n U, \psi_{2i-1} \rangle_{W_2^3} \\
&= \langle U, P_n^* \psi_{2i-1} \rangle_{W_2^3} \\
&= \langle U, P_n \psi_{2i-1} \rangle_{W_2^3} \\
&= \langle U, \psi_{2i-1} \rangle_{W_2^3} \\
&= \langle U, A\varphi_{2i-1} \rangle_{W_2^3} \\
&= \langle A^*U, \varphi_{2i-1} \rangle_{W_2^3} \\
&= \langle (A_{11}^*u_n + A_{21}^*v_n, A_{12}^*u_n + A_{22}^*v_n)^\top, (Q_{x_i}(x), 0)^\top \rangle_{W_2^3} \\
&= (A_{11}^*u + A_{21}^*v)(x_{2i-1}) \\
&= f(x_i)
\end{aligned}$$

Similarly, we can obtain  $(A_{21}u_n + A_{22}v_n)(x_i) = \langle AU_n, \varphi_{2i}(x) \rangle_{W_2^3} = g(x_i)$ , namely

$$(AU_n)(x_i) = F(x_i) = (f(x_i), g(x_i))^T, i = 1, 2, \dots, n$$

□

**Theorem 3.5** The errors of the approximate solution are monotone decreasing with the increasing of nodal points.

*Proof.* 
$$\begin{aligned} \|U - U_n\| &= \left\| \sum_{i=n+1}^{\infty} [\sum_{2k-1 \leq i} \beta_{i2k-1} f(x_k) + \sum_{2k \leq i} \beta_{i2k} g(x_k)] \overline{\psi}_i(x) \right\| \\ &= \sum_{i=n+1}^{\infty} [\sum_{2k-1 \leq i} \beta_{i2k-1} f(x_k) + \sum_{2k \leq i} \beta_{i2k} g(x_k)]^2 \\ &\geq \sum_{i=n+2}^{\infty} [\sum_{2k-1 \leq i} \beta_{i2k-1} f(x_k) + \sum_{2k \leq i} \beta_{i2k} g(x_k)]^2 \\ &= \|U - U_{n+1}\| \end{aligned}$$

□

## 4 Numerical examples

Consider the singular boundary values problem for system of equations as follows:

$$\begin{cases} u''(x) + e^{\frac{1}{x}} v'(x) + \frac{1}{x} u(x) + (x-1)v(x) = f(x), \\ v''(x) + \frac{\sin x}{x} u'(x) + \frac{1}{1-x} v(x) + e^{\frac{1}{x}} u(x) = g(x), \\ u(0) = v(0) = u(1) = v(1) = 0 \end{cases} \quad (4.1)$$

By our method, after multiplying  $x$  and  $x(1-x)$  both side of Eq.(4.1), we obtain

$$\begin{cases} xu''(x) + xe^{\frac{1}{x}} v'(x) + u(x) + x(x-1)v(x) = F(x), \\ x(x-1)v''(x) + (1-x) \sin x u'(x) + xv(x) + x(x-1)e^{\frac{1}{x}} u(x) = G(x) \end{cases} \quad (4.2)$$

Where

$$\begin{cases} F(x) = 7(x+1) + 3e^{\frac{1}{x}} \pi \cos \pi x + 3(x-1) \sin \pi x, \\ G(x) = 7e^{\frac{1}{x}}(x-1)x + \frac{7(2x-1) \sin x}{x} - 3\pi^2 \sin \pi x - \frac{3 \sin \pi x}{x-1} \end{cases} \quad (4.3)$$

The exact solutions are given by  $u(x) = 7x(x-1)$  and  $v(x) = 3 \sin \pi x$ . We choose 600 points in  $[0, 1]$  and obtain approximate solutions  $u_{600}(x)$  and  $v_{600}(x)$ . The numerical results are presented in Table 1 and Table 2.

Table 1:

Node	True solution $u(x)$	Approximate solution $u_{600}(x)$	Absolute error	Relative error
1/600	-0.0116472	-0.0116472	1.67597E-12	1.43895E-10
41/600	-0.445647	-0.445647	1.47693E-12	3.31412E-12
81/600	-0.817425	-0.817425	1.51558E-09	1.85409E-09
121/600	-1.12698	-1.12698	2.4428E-08	2.16756E-08
161/600	-1.37431	-1.37431	4.7978E-08	3.49105E-08
201/600	-1.55943	-1.55942	1.98411E-07	1.27234E-07
241/600	-1.68231	-1.68231	4.80929E-07	2.85873E-07
281/600	-1.74298	-1.74298	8.39172E-07	4.81458E-07
321/600	-1.74143	-1.74142	1.20557E-06	6.9229E-07
361/600	-1.67765	-1.67765	1.51687E-06	9.04165E-07
401/600	-1.55165	-1.55165	1.71898E-06	1.10784E-06
441/600	-1.36343	-1.36342	1.76862E-06	1.29719E-06
481/600	-1.11298	-1.11298	1.63391E-06	1.46805E-06
521/600	-0.800314	-0.800313	1.29453E-06	1.61753E-06
561/600	-0.425425	-0.425424	7.41795E-07	1.74366E-06
599/600	-0.0116472	-0.0116472	2.15766E-08	1.85251E-06

Table 2:

Node	True solution $v(x)$	Approximate solution $v_{600}(x)$	Absolute error	Relative error
1/600	0.0157079	0.0157078	1.33923E-07	8.52579E-06
41/600	0.639091	0.639091	1.23564E-07	1.93343E-07
81/600	1.23454	1.23454	1.23441E-07	9.99889E-08
121/600	1.77604	1.77604	1.24312E-07	6.99937E-08
161/600	2.23991	2.23991	8.5726E-08	3.8272E-08
201/600	2.60589	2.60589	6.35518E-08	2.43877E-08
241/600	2.85798	2.85798	3.20043E-07	1.11982E-07
281/600	2.98517	2.98517	6.31855E-07	2.11665E-07
321/600	2.98188	2.98188	9.32303E-07	3.12656E-07
361/600	2.84828	2.84828	1.15945E-06	4.07072E-07
401/600	2.59019	2.59019	1.26699E-06	4.89151E-07
441/600	2.21889	2.21889	1.22983E-06	5.54254E-07
481/600	1.75062	1.75062	1.04611E-06	5.97563E-07
521/600	1.20584	1.20584	7.36501E-07	6.10777E-07
561/600	0.608362	0.608362	3.41339E-07	5.61079E-07
599/600	0.0157079	0.0157078	6.15963E-08	3.92134E-06

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