

# Infeasible Interior Point Method for Semidefinite Programs

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## **Abstract**

In Semidefinite programming one minimizes a linear function subject to the constraint that an affine combination of symmetric matrices is positive Semidefinite. Such a constraint is nonlinear and no smooth but convex, so semidefinite programs are convex optimization problems. Semidefinite programming unifies several standard problems (e.g., linear and quadratic programming) and finds many applications in engineering and combinatorial optimization (e.g., Max-cut problem, Graph bisection, Maximum cliques in graphs, Min-Max Eigen value problems). Although Semidefinite programs are much more general than linear programs, they are not much harder to solve. Most interior point methods for linear programming have been generalized to Semidefinite programs. However to find a strictly feasible initial point is difficult in practice. The notion of infeasible interior point algorithms introduced by Zhang and al is essential for linear programming, there were some successes in generalizing this notion to the convex quadratic and complementarily problems. In this paper we generalize this notion to solving the primal Semidefinite programming (PSDP) and its dual (DSDP) and we propose an algorithm with polynomial complexity..

**Mathematics Subject Classification:** 65K05, 90C25, 90C30

**Keywords:** Semidefinite programming, primal-dual, infeasible interior point methods.

## 1 Introduction

Several authors have discussed generalizations of interior-point algorithms for linear programming (LP) to the context of semidefinite programming (SDP). Nesterov and Nemirovskii propose in their landmark works [24, 25] a general approach for using interior-point methods to solve convex programs based on the notion of self-concordant functions (See their book [27] for a comprehensive treatment of this subject). They show that the problem of minimizing a linear function over a convex set can be solved in "polynomial time" as long as a self-concordant barrier function for the convex set is known. In particular, they show that linear programs, convex quadratic programs with convex quadratic constraints, and semidefinite programs all have explicit and easily computable self-concordant barrier functions, and hence can be solved in "polynomial time." On the other hand, Alizadeh [1] extends Ye's projective potential reduction algorithm [37] for LP to SDP and argues that many known interior point algorithms for LP can also be transformed into algorithms for SDP in a mechanical way. Since then, many authors have proposed interior-point algorithms for solving SDP problems, including Alizadeh, Haeberly, and Overton [2, 3]; Freund [4]; Helmberg and al. [6]; Jarre [8]; Kojima, Shida, and Shindoh [11, 13]; Kojima, Shindoh and Hara [14]; Lin and Saigal [15]; Luo, Sturm, and Zhang [16]; Monteiro [18]; Monteiro and Zhang [23]; Nesterov and Nemirovskii [26]; Nesterov and Todd [28, 29]; Potra and Sheng [30]; Sturm and Zhang [32]; Tseng [33]; Vandenberghe and Boyd [34]; and Zhang [38]. Most of these more recent works concentrate on primal-dual methods.

The first SDP algorithms that are extensions of primal-dual LP algorithms, such as the long-step path-following algorithm of Kojima, Mizuno, and Yoshise [10], the short-step path-following algorithm of Kojima, Mizuno, and Yoshise [9] and Monteiro and Adler [19, 20], and the predictor-corrector algorithm of Mizuno, Todd, and Ye [17], use one of the following three search directions: (i) the Alizadeh, Haeberly, and Overton (AHO) direction proposed in [2]; (ii) a direction independently proposed by Kojima, Shindoh, and Hara [14] and Helmberg et al. [6], and later rediscovered by Monteiro [18], (iii) Nesterov and Todd (NT) search direction [28, 29]

This paper establishes the polynomial convergence of the class of primal-dual infeasible interior-point algorithms for semidefinite programming (SDP) which is extended for the linear programming, complementarity and convex quadratic programming [31, 39, 40], we establish for the first time the polynomial convergence of algorithm based on Nesterov and Todd (NT) search direction [28, 29] into the context of SDP.

## 2 Preliminary Notes

The following notation is used throughout the paper: let  $S^n$  denote the vector space of real symmetric  $n \times n$  matrices.

The standard inner product on  $S^n$  is:  $A \bullet B = \text{tr}(AB) = \sum_{i,j} A_{ij} B_{ij}$ .

$$\|x\| = \sqrt{\sum_{i=1}^m (x_i)^2}, x \in IR^m$$

$$\|X\| = \sqrt{\text{tr}(XX^T)}, X \in S^n \text{ is called Frobenius norm.}$$

$X \succeq 0$  ( $X \succ 0$ ), that  $X$  is positive semidefinite (positive definite).

Consider the primal semidefinite program (PSDP):

$$\begin{cases} \min_{X \in S^n} C \bullet X \\ A_i \bullet X = b_i \quad i = 1, \dots, m \\ X \succeq 0 \end{cases} .$$

Where  $b \in IR^m$ ,  $C \in S^n$ , and  $A_i \in S^n, i = 1, \dots, m$

The dual of (PSDP) is the following program (DSDP):

$$\begin{cases} \max_{y \in IR^m, X \in S^n} b^T y \\ \sum_{i=1}^m y_i A_i + Z = C \\ Z \succeq 0 \end{cases} .$$

Where  $Z \in S^n, y = (y_1, y_2, \dots, y_m)^T$

We impose the following assumptions:

o **(H1)**:  $K_{int} = \{X \in S^n / A_i \bullet X = b_i, \quad i = 1, \dots, m, \quad X \succ 0\}$  a feasible point of (PSDP) is non-empty,

o **(H2)**:  $T_{int} = \{y \in IR^m, Z \in S^n, Z \succ 0 / \sum_{i=1}^m y_i A_i + Z = C\}$  a feasible point of (DSDP) is non-empty,

o **(H3)**: The vectors of matrices  $A_i, i = 1, \dots, m$  are linearly independent.

These assumptions are often used to develop the interior points methods.

## 3 The Methods in a general framework:

Most of the new interior points methods are motivated by the logarithmic barrier function technique of Frisch (1955), to problem (PSDP) we associates the problem gate nonlinear (PSDP $\mu$ ) next one:

$$\begin{cases} \min_{X \in S^n} C \bullet X - \mu \ln(\det X) \\ A_i \bullet X = b_i \quad i = 1, \dots, m \\ X \succ 0 \end{cases}$$

The resolution of (PSDP $\mu$ ) is equivalent at that of (PSDP) with that if  $x^*(\mu)$  is an optimal solution of (PSDP $\mu$ ) then  $x^* = \lim_{\mu \rightarrow 0} x^*(\mu)$  is an optimal solution of (PSDP).

The principle of these methods is to solve the nonlinear system (1) of Karuch-Kuhn-Tucker (KKT) partner to the problem (PSDP $\mu$ ):

$$(1) \quad \begin{cases} \sum_{i=1}^m y_i A_i + Z - C = 0 \\ A_i \bullet X - b_i = 0 \quad i = 1, \dots, m \\ XZ - \mu I = 0 \end{cases}$$

by the Newton's method, while leaving from any positive point which is not necessarily feasible ( $X > 0$ ,  $Z > 0$  and  $y \in IR^m$ ).

To achieve feasibility and optimality we introduce a merit function defined by:

$$\varphi(X, y, Z) = X \bullet Z + r(X, y, Z)$$

where  $r(X, y, Z) = \|A' \bullet X - b\| + \|\sum_{i=1}^m y_i A_i + Z - C\|$ ,  $A' \bullet X = (A_1 \bullet X, A_2 \bullet X, \dots, A_m \bullet X)^T$ ,  $b = (b_1, b_2, \dots, b_m)^T$ . It is clear that  $r$  measure feasibility and  $X \bullet Z$  (duality gap) control the optimality.

The idea is to make the value of this function towards zero during iterations.

Resolution of the system (1):

A direct application of Newton's method to the nonlinear system (1) from an infeasible starting point ( $X > 0$ ,  $Z > 0$ ) and  $y \in IR^m$  and  $\mu = (X \bullet Z)/n > 0$  produces the following equations for the search direction  $\Delta X$ ,  $\Delta y$  and  $\Delta Z$ :

$$(2) \quad \begin{cases} X\Delta Z + Z\Delta X = -XZ + \sigma\mu I \quad 0 < \sigma < 1 \\ A_i \bullet \Delta X = b_i - A_i \bullet X \quad i = 1, \dots, m \\ \sum_{i=1}^m \Delta y_i A_i + \Delta Z = C - \sum_{i=1}^m y_i A_i - Z \end{cases}$$

It can be showed that this system has a unique solution, see [35].

We can rewrite system (2) as follows:

$$(3) \quad \begin{cases} X\Delta Z Z^{-1} + \Delta X = \sigma\mu Z^{-1} - X \quad 0 < \sigma < 1 \\ A_i \bullet \Delta X = b_i - A_i \bullet X \quad i = 1, \dots, m \\ \sum_{i=1}^m \Delta y_i A_i + \Delta Z = C - \sum_{i=1}^m y_i A_i - Z \end{cases}$$

It is obvious that  $\Delta Z$  is symmetric due to the third equation in (3). However, a crucial observation is that  $\Delta X$  is not necessary symmetric because  $X\Delta Z Z^{-1}$  may be not symmetric. Many researchers have proposed methods for symmetrizing the first equation in the above Newton system such that the resulting new system has a unique symmetric solution. Among them, the following three directions are the most popular ones, the direction introduced by Alizadeh, Haerberly, Overton (AHO) in [2], by Helmberg, et al., Kojima et al., and Monteiro (HKM) in [6, 9, 23], Nesterov and Todd (NT) in [28, 29], respectively, called AHO, HKM, and NT directions. In this paper we use the symmetrization scheme from which the NT direction [28] is derived. One important reason for this is that the NT scaling technique transfers the primal variable  $X$  and the dual  $Z$  into the same space. If we apply the NT-symmetrized scheme, namely, the term  $X\Delta Z Z^{-1}$  in the first equation is replaced by  $P\Delta Z P^T$ , where  $P = X^{1/2}(X^{1/2} Z X^{1/2})^{-1/2} X^{1/2} = Z^{-1/2}(Z^{1/2} X Z^{1/2})^{1/2} Z^{-1/2}$

Then the above system is replaced by the system:

$$(4) \quad \begin{cases} P\Delta Z P^T + \Delta X = \sigma\mu Z^{-1} - X \quad 0 < \sigma < 1 \\ A_i \bullet \Delta X = b_i - A_i \bullet X \quad i = 1, \dots, m \\ \sum_{i=1}^m \Delta y_i A_i + \Delta Z = C - \sum_{i=1}^m y_i A_i - Z \end{cases}$$

Obviously. Now  $\Delta X$  is a symmetric matrix and (4) still has a unique solution (See [35]). The new iterate is then:  $(X, y, Z) = (X, y, Z) + \alpha(\Delta X, \Delta y, \Delta Z)$  with  $\alpha > 0$  is the displacement step chosen such a way that  $X > 0$ ,  $Z > 0$  and  $\varphi$  decreases. If the stop test is not satisfied  $\varphi > \varepsilon$  ( $\varepsilon$  parameter of precision) we replaces  $\mu$  by  $\mu_1$  ( $\mu_1 < \mu$ ) and reiterate.

Our infeasible interior point algorithm is described as follows:

**Basic Algorithm**

**Beginning:**

**Initialization:**

Start with  $(X, Z) > 0$ ,  $y \in IR^m$  (arbitrary) and calculate  $\varphi$ .  $\varepsilon > 0$  a parameter of precision.

**K = 0**

when  $\varphi > \varepsilon$  do:

**step0:** Calculate  $\mu = (1/n)X \bullet Z$  and choose  $\sigma \in (0, 1)$

**step1:** Solve the following linear system:

$$\begin{cases} P\Delta ZP^T + \Delta X = \sigma\mu Z^{-1} - X & 0 < \sigma < 1 \\ A_i \bullet \Delta X = b_i - A_i \bullet X & i = 1, \dots, m \\ \sum_{i=1}^m \Delta y_i A_i + \Delta Z = C - \sum_{i=1}^m y_i A_i - Z \end{cases}$$

**step2:** finds a step of displacement  $\alpha > 0$  such as:

$X = X + \alpha\Delta X > 0$ ,  $Z = Z + \alpha\Delta Z > 0$  and  $\varphi$  decreasing

**Step3:**  $y = y + \alpha\Delta y$

**K = K+1**

**End.**

## 4 Main Results

Under hypotheses (H1) and (H2), the convergence of the algorithm is based on the following lemma.

**lemma 4.1 :** Let  $\{(X^k, y^k, Z^k)\}$  be the sequence of iterates generated by the algorithm, then we have:

$$1/ A_i X^{k+1} - b_i = A_i(X^k + \alpha^k \Delta X^k) - b_i = (1 - \alpha^k)(A_i X^k - b_i) = v^{k+1}(A_i X^0 - b_i) \quad i = 1, \dots, m$$

$$2/ \sum_{i=1}^m y_i^{k+1} A_i + Z^{k+1} - C = v^{k+1}(\sum_{i=1}^m y_i^0 A_i + Z^0 - C)$$

$$3/ X^{k+1} \bullet Z^{k+1} = (1 - \alpha^k + \sigma^k \alpha^k) X^k \bullet Z^k + (\alpha^k)^2 \Delta X^k \bullet \Delta Z^k$$

where  $v^{k+1} = (1 - \alpha^k)v^k = \prod_{i=0}^k (1 - \alpha^i)v^0$ ,  $v^0 = 1$ .

$$\alpha^k(1 - \sigma^k) X^k \bullet Z^k + \alpha^k v^k r^0 - (\alpha^k)^2 \Delta X^k \bullet \Delta Z^k.$$

**proposition 4.2 :** *The sequence  $\varphi^k$  generated by the algorithm satisfies:*  
 $\varphi^{k+1} = (1 - \delta^k)\varphi^k$

$$\text{Where } \delta^k = \frac{\alpha^k(1-\sigma^k)X^k \bullet Z^k + \alpha^k v^k r^0 - (\alpha^k)^2 \Delta X^k \bullet \Delta Z^k}{X^k \bullet Z^k + v^k r^0}$$

**corollary 4.3 :** *the sequence  $\varphi^k$  converge linearly if  $0 < \alpha^k < 1$  to which case we have  $0 < \delta^k < 1$  and if  $\delta^k$  offers toward 1 the convergence becomes superlinear.*

**proposition 4.4 :** *Let us suppose the initial point is given by:  $(X^0, y^0, Z^0) = \zeta(I, 0, I)$ , ( $\zeta > 0$ ,  $I[i, j] = 0$  if  $i \neq j$  and  $I[i, j] = 1$  if  $i = j$ ) then: the algorithm converges on at most  $O(n^2 |\log(\varepsilon)|)$  iterations ( $\varepsilon$  a parameter of precision).*

## 5 Conclusion

In this paper we generalize the notion of infeasible interior point methods to solving the primal Semidefinite programming (PSDP) and its dual (DSDP) and we propose an algorithm with polynomial complexity. which stimulates greatly the development of this methods for problems of optimization. We can conclude that these methods constitute a valid solution as to the initialization problem. This one deserves some supplementary efforts essentially when choosing the displacement step and to calculate the search direction. This, until now, is the object of researches aiming to reduce the iteration cost.

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**Received: September 9, 2006**