Convex Functions whose Epigraphs are Semi-closed: Duality Theory

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Abstract

A classical duality formula in general Banach spaces, usually established for a convex proper lower semicontinuous perturbation under one of the familiar Rockafellar, Robinson, Attouch-Brézis conditions, is shown to hold in more general setting. We provide an application to accredit this extension.

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1 Introduction.

Let $X$ be a normed vector space and let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two convex functions. Finding sufficient conditions ensuring the following fundamental duality result

$$\inf_{x \in X} \{f(x) + g(x)\} + \min_{y^* \in Y^*} \{f^*(-y^*) + g^*(y^*)\} = 0 \quad (1.1)$$

is of crucial importance in convex analysis. Our main objective is to attempt to prove that the statement (1.1) holds for a broad class of convex functions whose epigraphs are semi-closed under some constraint qualification in the setting of Fréchet spaces. This class has been studied by Laghdir in his recent
paper [10] from the point of view of subdifferentiability. Let us point out that this large class of convex functions includes convex lower semicontinuous functions, cs-convex functions and cs-closed functions. We give an application dealing with the convex composite optimization.

2 Preliminaries and Notations.

In what follows, for a given function \( f : X \to \mathbb{R} \cup \{ +\infty \} \) we denote by
\[
\text{dom } f := \{ x \in X : f(x) < +\infty \}
\]
its effective domain, by
\[
\text{Epi } f := \{ (x, r) \in X \times \mathbb{R} : f(x) \leq r \}
\]
its epigraph and by
\[
[f \leq r] := \{ x \in X : f(x) \leq r \}
\]
its sublevel set at height \( r \). We say that \( f \) is proper whenever \( \text{dom } f \neq \emptyset \).

Throughout this paper, we denote commonly by \( \langle , \rangle \) the duality pairing between \( X \) and \( X^* \) and between \( X^* \) and \( X^{**} \). The subdifferentiale of \( f \) at a point \( \bar{x} \in X \) is by definition
\[
\partial f(\bar{x}) := \{ x^* \in X^* : f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle, \forall x \in X \}.
\]
The Legendre-Fenchel conjugate function of \( f \) is defined for any \( x^* \in X^* \) by
\[
f^*(x^*) := \sup_{x \in X} \{ \langle x^*, x \rangle - f(x) \}.
\]
Let \( C \) be a subset of \( X \). The cone that it generates is
\[
\mathbb{R}_+ C := \bigcup_{\lambda \geq 0} \lambda C,
\]
its indicator function is
\[
\delta_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}
\]
The normal cone of \( C \) at \( \bar{x} \) is defined by
\[
N_C(\bar{x}) := \partial \delta_C(\bar{x}) = \{ x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq 0, \forall x \in C \}. \]
Let $C$ be a subset of $X$. Following [8] we say that $C$ is cs-closed if whenever $(x_n)_{n \in \mathbb{N}}$ is a sequence in $C$ and $(\alpha_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{R}^+$ with $\sum_{n=0}^{\infty} \alpha_n = 1$ and $x = \sum_{n=0}^{\infty} \alpha_n x_n$ exists in $X$, then $x \in C$. It is easy to see that every cs-closed subset is convex. $C$ is said to be semi-closed if its closure $\overline{C}$ have the same interior. Also, if $X$ is a locally convex space, then $C$ is said to be lower cs-closed if there exists a Fréchet space $Y$ and a cs-closed subset $A$ of $X \times Y$ such that $C = A_X$ where $A_X$ denotes the projection of $A$ on the space $X$. There are plenty of sets that are cs-closed, lower cs-closed or semi-closed (see [2], [3], [6], [7], [8], [13]). The subdifferential calculus and duality theory associated with the class of cs-closed functions have been studied by Laghdir [9] and Zălinescu [14].

Now, following [13], [14] and [10] we set

**Definition 2.1** Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$.

1. We say that $f$ is semi-closed if it is proper and its epigraph is semi-closed.

2. We say that $f$ is cs-closed (resp. lower cs-closed) if it is proper and its epigraph is cs-closed (resp. lower cs-closed).

3. We say that $f$ is cs-convex if $f$ is proper and

\[ f(x) \leq \liminf_{m \rightarrow +\infty} \sum_{n=0}^{m} \lambda_n f(x_n) \]

whenever, $\forall n \in \mathbb{N}$, $\lambda_n \geq 0$, $x_n \in X$, $\sum_{n=0}^{\infty} \lambda_n = 1$ and $\sum_{n=0}^{\infty} \lambda_n x_n$ is convergent to $x$ in $X$.

**Remark 2.1**

1) Let us note that if $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous then it is cs-convex.

2) If $f$ is cs-convex then it is cs-closed. Conversely, Zălinescu in [14] proved that when $f^*$ is proper and $f$ is cs-closed then $f$ is cs-convex.

3) Every cs-closed function is semi-closed.

4) The indicator function $\delta_C$ of every convex semi-closed subset of $X$ is semi-closed.

5) In [10], Laghdir studied the subdifferentiability of a convex semi-closed function, i.e. $\partial f(\bar{x}) \neq \emptyset$ whenever $\bar{x} \in \text{dom } f$, $\mathbb{R}_+[\text{dom } f - \bar{x}] = X$ and $X$ is a Fréchet space. It was proved in [10], that this result falses under the weakened condition: $\mathbb{R}_+[\text{dom } f - \bar{x}]$ is a closed vector subspace.

6) In [10], it was established a characterization for a semi-closed function by means of its level sets given by: $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is semi-closed if and only if its level sets are semi-closed.
3 The fundamental duality formula.

Our goal in this section is to setting up the well-known fundamental duality result (1.1) for the class of convex semi-closed functions. This can be obtained provided a certain constraint qualification. In order to derive this result we will use the approach based on the use of a perturbation function. For this let us consider the following condition

\[
\begin{cases}
X \text{ is a Fréchet space} \\
f : X \rightarrow \mathbb{R} \cup \{+\infty\} \text{ convex and proper} \\
(C.Q_1) \quad g : X \rightarrow \mathbb{R} \cup \{+\infty\} \text{ convex, proper and semi-closed} \\
\text{there exists } \bar{x} \in \text{dom } f \cap \text{dom } g \text{ such that} \\
\mathbb{R}_+[\text{dom } g - \bar{x}] = X.
\end{cases}
\]

and the marginal function

\[
p : X \rightarrow \mathbb{R} \cup \{+\infty\} \\
y \mapsto p(y) = \inf_{x \in X} \{f(x) + g(y + x)\}
\]

Obviously \(p\) is convex since it is a marginal function of a convex function.

**Lemma 3.1** If \(\inf_{x \in X} \{f(x) + g(x)\} \in \mathbb{R}\) and the condition \((C.Q_1)\) is satisfied, then \(\partial p(0) \neq \emptyset\).

**Proof.** Let us note that the equality

\[
\mathbb{R}_+[\text{dom } g] = \bigcup_{n,m \geq 0} m[g \leq n]
\]

is obtained simply by observing that

\[
\text{dom } g = \bigcup_{n \geq 1} [g \leq n].
\]

Following [10], it follows from Baire's Theorem and the fact that \(g\) is semi-closed, that there exists some neighbourhood of zero \(U\) and some integer \(n \geq 1\) such that

\[
g(y + \bar{x}) \leq n, \forall y \in U
\]

which yields

\[
p(y) \leq f(\bar{x}) + g(y + \bar{x}) \leq f(\bar{x}) + n, \forall y \in U.
\]
Therefore, it follows that \( p \) is bounded above on a neighbourhood of zero and since \( p(0) = \inf_{x \in X} \{ f(x) + g(x) \} \) is finite and \( p \) is convex we obtain from a classical convex analysis result (see [5]) that \( p \) is subdifferentiable at zero i.e. \( \partial p(0) \neq \emptyset \).

Now, we are ready to state our main result.

**Theorem 3.2** If \( \inf_{x \in X} \{ f(x) + g(x) \} \in \mathbb{R} \) and the condition \((C.Q_1)\) is satisfied, then

\[
\inf_{x \in X} \{ f(x) + g(x) \} + \min_{x^* \in X^*} \{ f^*(-x^*) + g^*(x^*) \} = 0.
\]

**Proof.** It is straightforward to see that for any \( x^* \in X^* \)

\[
p^*(x^*) = f^*(-x^*) + g^*(x^*),
\]

so from the Fenchel’s inequality we have

\[
p^*(x^*) + p(0) \geq 0, \ \forall x^* \in X^*
\]

i.e.

\[
\inf_{x \in X} \{ f(x) + g(x) \} + f^*(-x^*) + g^*(x^*) \geq 0, \ \forall x^* \in X^*, \quad (3.1)
\]

which yields

\[
\inf_{x^* \in X^*} \{ f^*(-x^*) + g^*(x^*) \} + \inf_{x \in X} \{ f(x) + g(x) \} \geq 0.
\]

Since \( \partial p(0) \neq \emptyset \), taking \( z^* \in \partial p(0) \) i.e.

\[
p^*(z^*) + p(0) = 0, \quad (3.2)
\]

it results by combining (3.1) and (3.2) that

\[
\inf_{x \in X} \{ f(x) + g(x) \} + \min_{x^* \in X^*} \{ f^*(-x^*) + g^*(x^*) \} = 0.
\]

□

**Corollary 3.3** Let \( x^* \in X^* \) such that \( \inf_{x \in X} \{ f(x) + g(x) - \langle x^*, x \rangle \} \in \mathbb{R} \) and assume that \((C.Q_1)\) holds, then we have

\[
(f + g)^*(x^*) = \min_{y^* \in X^*} \{ f^*(x^* - y^*) + g^*(y^*) \}.
\]
Proof. It suffices to apply Theorem 3.1 to the functions

\[ F : X \rightarrow \mathbb{R} \cup \{+\infty\} \]
\[ x \mapsto F(x) = f(x) - \langle x^*, x \rangle \]

\[ G : X \rightarrow \mathbb{R} \cup \{+\infty\} \]
\[ x \mapsto G(x) = g(x) \]

by observing that \( F \) and \( G \) verify together the condition \((C.Q_1)\). \(\square\)

Corollary 3.4 Under the condition \((C.Q_1)\) we have

\[ \partial(f + g)(\bar{x}) = \partial f(\bar{x}) + \partial g(\bar{x}). \]

Proof. The inclusion \( \partial f(\bar{x}) + \partial g(\bar{x}) \subset \partial(f + g)(\bar{x}) \) is immediate. Conversely, let \( x^* \in \partial(f + g)(\bar{x}) \), i.e.

\[ (f + g)(x) - \langle x^*, x \rangle \geq f(\bar{x}) + g(\bar{x}) - \langle x^*, \bar{x} \rangle, \quad \forall x \in X, \]

and since \( \bar{x} \in \text{dom } f \cap \text{dom } g \), it follows that \( \inf_{x \in X} \{f(x) + g(x) - \langle x^*, x \rangle\} \in \mathbb{R} \).

As

\[ (f + g)(\bar{x}) + (f + g)^*(x^*) - \langle x^*, \bar{x} \rangle = 0, \]

and using Corollary 3.1, we obtain for some \( z^* \in X^* \) that

\[ g^*(z^*) + f^*(x^* - z^*) + f(\bar{x}) + g(\bar{x}) - \langle x^*, \bar{x} \rangle = 0, \]

by setting \( y^* := x^* - z^* \), we have

\[ [g^*(z^*) + g(\bar{x}) - \langle z^*, \bar{x} \rangle] + [f^*(y^*) + f(\bar{x}) - \langle y^*, \bar{x} \rangle] = 0, \]

which yields, thanks to Fenchel's inequality, that

\[ \begin{cases} 
  g^*(z^*) + g(\bar{x}) - \langle z^*, \bar{x} \rangle = 0 \\
  f^*(y^*) + f(\bar{x}) - \langle y^*, \bar{x} \rangle = 0 
\end{cases} \]
i.e.

\[ \begin{cases} 
  z^* \in \partial g(\bar{x}) \\
  y^* \in \partial f(\bar{x}) 
\end{cases} \]

and therefore we get \( \partial(f + g)(\bar{x}) \subset \partial f(\bar{x}) + \partial g(\bar{x}) \). \(\square\)
**Corollary 3.5** Let $C$ and $D$ be two convex sets of $X$ and $\bar{x} \in C \cap D$. Suppose that $C$ is semi-closed and $\mathbb{R}_+(C - \bar{x}) = X$, then
\[ N_{C \cap D}(\bar{x}) = N_C(\bar{x}) + N_D(\bar{x}). \]

**Proof.** It suffices to apply Corollary 3.2 to the indicator functions $\delta_C$ and $\delta_D$. □

**Corollary 3.6** Let $C$ be a semi-closed convex cone with $0 \in C$ and let $\bar{x} \in X$ such that $\mathbb{R}_+(C - \bar{x}) = X$. If $\inf_{x \in C} f(x) \in \mathbb{R}$ then one has
\[ \inf_{x \in C} f(x) + \min_{x^* \in C^0} f^*(-x^*) = 0. \]

**Proof.** Applying Theorem 3.1 to $f$ and $\delta_C$ we obtain
\[ \inf_{x \in C} f(x) + \min_{x^* \in C^0} \{ f^*(-x^*) + \sup_{x \in C} \langle x^*, x \rangle \} = 0. \]
Since $C$ is a cone it is easy to check that
\[ C^0 = \{ x^* \in X^* : \sup_{x \in C} \langle x^*, x \rangle \leq 0 \}, \]
hence
\[ \inf_{x \in C} f(x) + \min_{x^* \in C^0} \{ f^*(-x^*) + \sup_{x \in C} \langle x^*, x \rangle \} = 0. \]
We have $0 \in C$, so for every $x^* \in C^0$ we get $\sup_{x \in C} \langle x^*, x \rangle = 0$, therefore
\[ \inf_{x \in C} f(x) + \min_{x^* \in C^0} f^*(-x^*) = 0. \]
□

**Corollary 3.7** Let $C$ be a semi-closed convex cone with $0 \in C$ and let $\bar{x} \in X$ such that $\mathbb{R}_+(C - \bar{x}) = X$, then one has
\[ \operatorname{dist}(\bar{x}, C) = \max_{x^* \in C^0, \|x^*\| \leq 1} \langle x^*, \bar{x} \rangle. \]

**Proof.** Since $\inf_{x \in C} \|x - \bar{x}\| \in \mathbb{R}$ then applying Corollary 3.4 to $f := \|x - \bar{x}\|$ we obtain
\[ \inf_{x \in C} \|x - \bar{x}\| = \max_{x^* \in C^0} -f^*(-x^*). \]
After computing the conjugate function of $f$ we get that for any $x^* \in X^*$ we have
\[ f^*(x^*) = \delta_{B_{X^*}}(x^*) + \langle x^*, \bar{x} \rangle, \]
where $B_{X^*}$ is the closed unit ball of $X^*$. Hence we obtain
\[ \operatorname{dist}(\bar{x}, C) = \max_{x^* \in C^0, \|x^*\| \leq 1} \langle x^*, \bar{x} \rangle. \]
□
Remark 3.1 One may ask a natural question if the fundamental duality formula (1.1) holds under the weakened condition: \(\mathbb{R}_+[\text{dom } g]\) is a closed vector subspace? The answer is no. Just take \(X\) an infinite dimensional Banach space, \(f : X \to \mathbb{R} \cup \{+\infty\}\) a convex proper function, \(g : X \to \mathbb{R} \cup \{+\infty\}\) a noncontinuous linear functional, \(Y := X \times \mathbb{R}\), \(F : Y \to \mathbb{R} \cup \{+\infty\}\) defined by \(F(x,t) = \delta_{\{0,0\}}(x,t)\) and \(G : Y \to \mathbb{R} \cup \{+\infty\}\) defined by \(G(x,t) = +\infty\) if \(t \neq 0\) and \(G(x,0) = g(x)\). It is easy to see that \(F\) and \(G\) are convex, proper, \(G\) is semi-closed, \(\mathbb{R}_+[\text{dom } G] = X \times \{0\}\) is a closed linear subspace and \(G\) is nowhere subdifferentiable. So

\[
\begin{cases}
\inf_{(x,t) \in Y} \{F(x,t) + G(x,t)\} = 0 \\
\inf_{(x^*,t^*) \in X^* \times \mathbb{R}} \{F^*(-x^*,-t^*) + G^*(x^*,t^*)\} = +\infty.
\end{cases}
\]

Hence the fundamental duality formula (1.1) falses.

4 Application to convex composite optimization.

In this section, we assume that the space \(Y\) is equipped with a partial preorder induced by a convex cone \(Y_+\) i.e. for any \(y_1, y_2 \in Y\)

\[y_1 \leq_Y y_2 \iff y_2 - y_1 \in Y_+\]

and an abstract maximal element \(+\infty\) will be adjoined to \(Y\). A mapping \(h : X \to Y \cup \{+\infty\}\) is said to be \(Y_+\)-convex in the sense that for any \(x_0, x_1 \in \text{dom } h := \{x \in X : h(x) \in Y\}\) and for any \(\lambda \in [0,1]\) we have

\[h(\lambda x_0 + (1-\lambda)x_1) \leq_Y \lambda h(x_0) + (1-\lambda)h(x_1).\]

A function \(g : Y \to \mathbb{R} \cup \{+\infty\}\) is said to be \(Y_+\)-nondecreasing on a subset \(C\) of \(Y\) if for any \(y_1, y_2 \in C\) we have

\[y_1 \leq_Y y_2 \implies g(y_1) \leq g(y_2).\]

In what follows, we extend to \(Y \cup \{+\infty\}\) the composite function \((g \circ h)\) by setting \((g \circ h)(x) = \sup_{y \in Y} g(y)\) for any \(x \notin \text{dom } h\). Here \(Y_+^*\) denotes the positive polar cone defined by

\[Y_+^* = \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall y \in Y_+\}.\]
Many of the convex minimization problems arising in Applied Mathematics, Operations research and Mathematical problems can be formulated as the following convex composite problem

\[(P) : \inf_{x \in X} (f + g \circ h)(x),\]

where \(f : X \to \mathbb{R} \cup \{+\infty\}\) is convex and proper, \(g : Y \to \mathbb{R} \cup \{+\infty\}\) is convex proper and nondecreasing on an appropriate subset of \(Y\) and \(h : X \to Y \cup \{+\infty\}\) is \(Y_+\)-convex and proper.

The aim of this section is to formulate the dual problem \((P^*)\) associated to the primal problem \((P)\). For this, let us consider the following constraint qualification \((C.Q_2)\)

\[
(C.Q_2) \begin{cases}
\text{X and Y are Fréchet spaces} \\
f : X \to \mathbb{R} \cup \{+\infty\}\text{ convex and proper} \\
g : Y \to \mathbb{R} \cup \{+\infty\}\text{ convex, proper and semi-closed} \\
h : X \to Y \cup \{+\infty\}\text{ } Y_+\text{-convex and proper} \\
\text{there exists } \bar{x} \in \text{dom } h \cap \text{dom } f \cap h^{-1}(\text{dom } g) \text{ such that} \\
\mathbb{R}^+[\text{dom } g - h(\bar{x})] = Y.
\end{cases}
\]

and the following auxiliary functions

\[
\tilde{f} : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\} \\
(x, y) \longmapsto \tilde{f}(x, y) = f(x) + \delta_{\text{Epi } h}(x, y)
\]

\[
\tilde{g} : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\} \\
(x, y) \longmapsto \tilde{g}(x, y) = g(y),
\]

where \(\text{Epi } h := \{(x, y) \in X \times Y : h(x) \leq_Y y\}\) is the epigraph of \(h\). Obviously \(\tilde{f}\) and \(\tilde{g}\) are both convex and proper.

**Proposition 4.1** If \(\inf_{x \in X} (f + g \circ h)(x) \in \mathbb{R}\), \(g\) nondecreasing on \(\text{Im } h + Y_+\) and the condition \((C.Q_2)\) is satisfied one has

\[
\inf_{x \in X} (f + g \circ h)(x) = \max_{y^* \in Y_+^*} \{- (f + y^* \circ h)^*(0) - g^*(y^*)\}
\]
Proof. Let us note that for any \( x \in X \)
\[
(f + g \circ h)(x) = \inf_{y \in Y} \{ \tilde{f}(x, y) + \tilde{g}(x, y) \}
\]
and dom \( \tilde{f} = (\text{dom } f \times Y) \cap \text{Epi } h \), dom \( \tilde{g} = X \times \text{dom } g \) and for any \( \lambda \in \mathbb{R} \) we have \([\tilde{g} \leq \lambda] = X \times [g \leq \lambda]\) and hence it is easy to check that the condition (C.Q₂) ensures that \( \tilde{f} \) and \( \tilde{g} \) satisfy together the qualification condition (C.Q₁).

Therefore by virtue of Theorem 3.1 we get
\[
\inf_{x \in X} (f + g \circ h)(x) = \inf_{y \in Y} \{ \tilde{f}(x, y) + \tilde{g}(x, y) \}
\]
= \[
\max_{y^* \in Y^*} \{-\tilde{f}^*(-x^*, -y^*) - \tilde{g}^*(x^*, y^*)\},
\]
by expliciting the conjugate functions \( \tilde{f}^* \) and \( \tilde{g}^* \) we have
\[
\tilde{f}^*(-x^*, -y^*) = (f + y^* \circ h)^*(-x^*) + \delta_{Y^*_+}(y^*)
\]
and thus we get
\[
\inf_{x \in X} (f + g \circ h)(x) = \max_{y^* \in Y^*_+} \{-f^*(-A^*y^*) - g^*(y^*)\},
\]
this completes the proof. \( \square \)

**Corollary 4.1** Let \( A : X \to Y \) be a continuous linear operator. If \( \inf_{x \in X} (f + g \circ A)(x) \in \mathbb{R} \) and the condition (C.Q₂) is satisfied one has
\[
\inf_{x \in X} (f + g \circ A)(x) = \max_{y^* \in Y^*_+} \{-f^*(-A^*y^*) - g^*(y^*)\},
\]
where \( A^* : Y^* \to X^* \) stand for the adjoint operator of \( A \).

**Proof.** By putting \( Y_+ = \{0\} \), it is obvious that \( g \) is nondecreasing on the whole space \( Y \) and \( Y^*_+ = Y^* \). Therefore, by applying Proposition 4.1, with \( h := A \), we obtain
\[
\inf_{x \in X} (f + g \circ A)(x) = \max_{y^* \in Y^*_+} \{-f^*(-A^*y^*) - g^*(y^*)\}.
\]
Since
\[
(f + y^* \circ A)^*(0) = \sup_{x \in X} \{-f(x) - \langle y^* \circ A, x \rangle \}
\]
= \[
\sup_{x \in X} \{-f(x) - \langle A^*y^*, x \rangle \}
\]
= \[
f^*(-A^*y^*),
\]
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hence we get
\[ \inf_{x \in X} (f + g \circ A)(x) = \max_{y^* \in Y^*} \{-f^*(-A^*y^*) - g^*(y^*)\}. \]

\[ \Box \]

**Corollary 4.2** Let \( x^* \in X^* \) such that \( \inf_{x \in X} \{ f(x) + (g \circ h)(x) - \langle x^*, x \rangle \} \in \mathbb{R} \), \( g \) nondecreasing on \( \text{Im } h + Y_+ \) and the condition \( (C.Q_2) \) is satisfied then
\[ (f + g \circ h)^*(x^*) = \min_{y^* \in Y_+^*} \{ g^*(y^*) + (f + y^* \circ h)^*(x^*) \}. \]

**Proof.** Let us consider the following function
\[ F : X \rightarrow \mathbb{R} \cup \{ +\infty \} \]
\[ x \mapsto F(x) = f(x) - \langle x^*, x \rangle, \]
by observing that \( F \) and \( g \) verify together the condition \( (C.Q_2) \) and hence by applying Proposition 4.1 we obtain
\[ (f + g \circ h)^*(x^*) = -\inf_{x \in X} \{ F(x) + (g \circ h)(x) \} \]
\[ = \min_{y^* \in Y_+^*} \{ g^*(-y^*) + (F - y^* \circ h)^*(0) \}, \]
and since \( (F - y^* \circ h)^*(0) = (f - y^* \circ h)^*(x^*) \), we get the desired result. \( \Box \)

The next corollary concerns the calculus of the subdifferential of composite convex functions using the preceding results.

**Corollary 4.3** Under the condition \( (C.Q_2) \) and \( g \) supposed to be nondecreasing on \( \text{Im } h + Y_+ \) one has
\[ \partial(f + g \circ h)(\bar{x}) = \bigcup_{y^* \in \partial g(h(\bar{x})) \cap Y_+^*} \partial(f + y^* \circ h)(\bar{x}). \]

**Proof.** Let \( x^* \in \partial(f + g \circ h)(\bar{x}) \) i.e.
\[ (f + g \circ h)(x) - \langle x^*, x \rangle \geq f(\bar{x}) + (g \circ h)(\bar{x}) - \langle x^*, \bar{x} \rangle, \quad \forall x \in X, \]
and since \( \bar{x} \in \text{dom } h \cap \text{dom } f \cap h^{-1}(\text{dom } g) \), it follows that
\[ \inf_{x \in X} \{ f(x) + (g \circ h)(x) - \langle x^*, x \rangle \} \in \mathbb{R}. \]
As
\[ (f + g \circ h)(\bar{x}) + (f + g \circ h)^*(x^*) - \langle x^*, \bar{x} \rangle = 0, \]
and using Corollary 4.2, we obtain for some $z^* \in Y_+^*$ that
\[ g^*(z^*) + (f + z^* \circ h)^*(x^*) + f(h(\bar{x})) - \langle x^*, \bar{x} \rangle = 0, \]
i.e.
\[ [g^*(z^*) + g(h(\bar{x})) - \langle z^*, h(\bar{x}) \rangle] + [(f + z^* \circ h)^*(x^*) + f(\bar{x}) + (z^* \circ h)(\bar{x}) - \langle x^*, \bar{x} \rangle] = 0, \]
which yields, thanks to Fenchel’s inequality, that
\[
\begin{cases}
  g^*(z^*) + g(h(\bar{x})) - \langle z^*, h(\bar{x}) \rangle = 0 \\
  (f + z^* \circ h)^*(x^*) + f(\bar{x}) + (z^* \circ h)(\bar{x}) - \langle x^*, \bar{x} \rangle = 0
\end{cases}
\]
i.e.
\[
\begin{cases}
  z^* \in \partial g(h(\bar{x})) \\
  x^* \in \partial (f + z^* \circ h)(\bar{x}),
\end{cases}
\]
and therefore we get
\[
\partial (f + g \circ h)(\bar{x}) \subseteq \bigcup_{y^* \in \partial g(h(\bar{x})) \cap Y_+^*} \partial (f + y^* \circ h)(\bar{x})
\]
Conversely, let $x^* \in \bigcup_{y^* \in \partial g(h(\bar{x})) \cap Y_+^*} \partial (f + y^* \circ h)(\bar{x})$, i.e. there exists $y^* \in Y_+^*$ such that
\[
\begin{cases}
  y^* \in \partial g(h(\bar{x})) \\
  x^* \in \partial (f + y^* \circ h)(\bar{x})
\end{cases}
\]
i.e.
\[
\begin{cases}
  \langle y^*, y - h(\bar{x}) \rangle + g(h(\bar{x})) \leq g(y), \forall y \in Y, \\
  \langle x^*, x - \bar{x} \rangle + (f + y^* \circ h)(\bar{x}) \leq (f + y^* \circ h)(x), \forall x \in X.
\end{cases}
\]
By putting $y := h(x)$, we have
\[
\langle x^*, x - \bar{x} \rangle + f(\bar{x}) + g(h(\bar{x})) \leq f(x) + g(h(x)), \forall x \in X,
\]
i.e. $x^* \in \partial (f + g \circ h)(\bar{x})$ and the converse inclusion is then proved. \qed

**Corollary 4.4** Let $A : X \to Y$ be a continuous linear operator. Under the condition $(C.Q_2)$ one has
\[
\partial (f + g \circ A)(\bar{x}) = \partial f(\bar{x}) + A^*(\partial g(A\bar{x})).
\]
Proof. Putting $Y_+ = \{0\}$ and using Corollary 4.3 with $h := A$, we get

$$\partial(f + g \circ A)(\bar{x}) = \bigcup_{y^* \in \partial g(A\bar{x})} \partial(f + y^* \circ A)(\bar{x}).$$

Let $y^* \in \partial g(A\bar{x})$, since $(y^* \circ A)$ is a continuous linear form so it is semi-closed and since $\text{dom } (y^* \circ A) = X$ then by applying Corollary 3.2 with $g := y^* \circ A$, we obtain

$$\partial(f + y^* \circ A)(\bar{x}) = \partial f(\bar{x}) + \partial(A^* y^*)(\bar{x}).$$

As $A^* y^*$ is a linear continuous form, thus $\partial(A^* y^*)(\bar{x}) = \{A^* y^*\}$ and therefore

$$\partial(f + g \circ A)(\bar{x}) = \partial f(\bar{x}) + A^* (\partial g(A\bar{x})).$$

\[\square\]

**Corollary 4.5** Let $A : X \to Y$ be a continuous linear operator and $C$ and $D$ be two convex subsets of $X$ and $\bar{x} \in C \cap A^{-1}(D) := B$. Suppose that $D$ is semi-closed and $\mathbb{R}_+(D - A(\bar{x})) = X$, then

$$N_B(\bar{x}) = N_C(\bar{x}) + A^* N_D(A\bar{x}).$$

**Proof.** It is easy to check that

$$\delta_B(\bar{x}) = \delta_C(\bar{x}) + (\delta_D \circ A)(\bar{x}),$$

and by applying Corollary 4.4 to $f := \delta_C$ and $g := \delta_D$ we obtain

$$\partial \delta_B(\bar{x}) = \partial \delta_C(\bar{x}) + A^* (\partial \delta_D(A\bar{x}));$$

i.e.

$$N_B(\bar{x}) = N_C(\bar{x}) + A^* N_D(A\bar{x}).$$

\[\square\]

As an application of this last corollary, we derive the optimality conditions, related to the following mathematical programming problem

$$(Q) \quad \begin{cases} \inf f(x), \\ h(x) \in -Y_+ \\ x \in C \end{cases}$$

where $X$ and $Y$ are Fréchet spaces, $f : X \to \mathbb{R} \cup \{+\infty\}$ is a convex proper function, $h : X \to Y \cup \{+\infty\}$ is a $Y_+$-convex proper operator and $C$ a nonempty subset of $X$ supposed to be convex. In the following we will assume that $Y_+$ is semi-closed.
Proposition 4.2 Let $\bar{x}$ be a feasible point for the problem $(Q)$ i.e. $\bar{x} \in C \cap h^{-1}(-Y_+)$. If $\mathbb{R}_+[Y_++h(\bar{x})] = Y$, then $\bar{x}$ is an optimal solution for the problem $(Q)$ if and only if there exists $y^* \in Y_+^*$ such that $\langle y^*, h(\bar{x}) \rangle = 0$ and $0 \in \partial(f + \delta_C + y^* \circ h)(\bar{x})$.

Proof. $\bar{x}$ is an optimal solution for the problem $(Q)$ if and only if $0 \in \partial(f + \delta_C + \delta_{-Y_+} \circ h)(\bar{x})$. On the other hand since the cone is nonempty convex closed and following [4] $\delta_{-Y_+}$ is $Y_+$-nondecreasing, convex, proper and semiclosed, hence all the hypothesis of Corollary 4.3 are satisfied and

$$\partial(f + \delta_C + \delta_{-Y_+} \circ h)(\bar{x}) = \bigcup_{y^* \in N_{-Y_+}(h(\bar{x})) \cap Y_+^*} \partial(f + \delta_C + y^* \circ h)(\bar{x}),$$

which means that $\bar{x}$ is an optimal solution of the problem $(Q)$ if and only if there exists $y^* \in Y_+^*$ such that $\langle y^*, h(\bar{x}) \rangle = 0$ and $0 \in \partial(f + \delta_C + y^* \circ h)(\bar{x})$. $\square$

References


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