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Convex Functions whose Epigraphs are Semi-closed: Duality Theory

Mohamed Laghdir

Département de Mathématiques et Informatique
Faculté des Sciences
B.P 20, El-Jadida, Maroc

Rana Benabbou

Département de Mathématiques et Informatique
Faculté des Sciences
B.P 20, El-Jadida, Maroc

Abstract

A classical duality formula in general Banach spaces, usually established for a convex proper lower semicontinuous perturbation under one of the familiar Rockafellar, Robinson, Attouch-Brézis conditions, is shown to hold in more general setting. We provide an application to accredit this extension.

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1 Introduction.

Let X be a normed vector space and let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two convex functions. Finding sufficient conditions ensuring the following fundamental duality result

$$\inf_{x \in X} \{f(x) + g(x)\} + \min_{y^* \in Y^*} \{f^*(-y^*) + g^*(y^*)\} = 0 \tag{1.1}$$

is of crucial importance in convex analysis. Our main objective is to attempt to prove that the statement (1.1) holds for a broad class of convex functions whose epigraphs are semi-closed under some constraint qualification in the setting of Fréchet spaces. This class has been studied by Laghdir in his recent paper [10] from the point of view of subdifferentiability. Let us point out that this large class of convex functions includes convex lower semicontinuous functions, cs-convex functions and cs-closed functions. We give an application dealing with the convex composite optimization.

2 Preliminaries and Notations.

In what follows, for a given function $f : X \to \mathbb{R} \cup \{+\infty\}$ we denote by

$$\text{dom } f : \;=\; \{x \in X : f(x) < +\infty\}$$

its effective domain, by

$$\text{Epi } f : \;=\; \{(x,r) \in X \times \mathbb{R} : f(x) \leq r\}$$

its epigraph and by

$$[f \leq r] : \;=\; \{x \in X : f(x) \leq r\}$$

its sublevel set at height r. We say that f is proper whenever dom $f \neq \emptyset$. Throughout this paper, we denote commonly by $\langle , \rangle$ the duality pairing between X and X^* and between X^* and X^{**}. The subdifferentiale of f at a point $\bar{x} \in X$ is by definition

$$\partial f(\bar{x}) : \;=\; \{x^* \in X^* \;:\; f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x}\rangle, \;\forall x \in X\}.$$

The Legendre-Fenchel conjugate function of f is defined for any $x^* \in X^*$ by

$$f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}.$$

Let C be a subset of X. The cone that it generates is

$$\mathbb{R}_+ C : = \bigcup_{\lambda \geq 0} \lambda C,$$

its indicator function is

$$\delta_C(x) := \begin{cases} 0 & \text{if } \; x \in C \\ \\ +\infty & \text{otherwise.} \end{cases}$$

The normal cone of C at $\bar{x}$ is defined by

$$N_C(\bar{x}) := \; \partial \delta_C(\bar{x}) = \{x^* \in X^* : \; \langle x^*, x - \bar{x}\rangle \leq 0, \; \forall x \in C\}.$$

Let C be a subset of X. Following [8] we say that C is cs-closed if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in C and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^+$ with $\sum_{n=0}^{\infty}\alpha_n = 1$ and $x = \sum_{n=0}^{\infty}\alpha_n x_n$ exists in X, then $x \in C$. It is easy to see that every cs-closed subset is convex. C is said to be semi-closed if C and its closure $\overline{C}$ have the same interior. Also, if X is a locally convex space, then C is said to be lower cs-closed if there exists a Fréchet space Y and a cs-closed subset A of $X \times Y$ such that $C = A_X$ where A_X denotes the projection of A on the space X. There are plenty of sets that are cs-closed, lower cs-closed or semi-closed (see [2], [3], [6], [7], [8], [13]). The subdifferential calculus and duality theory associated with the class of cs-closed functions have been studied by Laghdir [9] and Zălinescu [14].

Now, following [13], [14] and [10] we set

Definition 2.1 *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$.
1. We say that f *is semi-closed if it is proper and its epigraph is semi-closed.*
2. We say that f *is cs-closed (resp. lower cs-closed) if it is proper and its epigraph is cs-closed (resp. lower cs-closed).*
3. We say that f *is cs-convex if* f *is proper and*

$$f(x) \leq \liminf_{m\to+\infty} \sum_{n=0}^{m} \lambda_n f(x_n)$$

whenever, $\forall n \in \mathbb{N},\ \lambda_n \geq 0,\ x_n \in X,\ \sum_{n=0}^{\infty}\lambda_n = 1$ *and* $\sum_{n=0}^{\infty}\lambda_n x_n$ *is convergent to* x *in* X.

Remark 2.1 1) Let us note that if $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous then it is cs-convex.
2) If f is cs-convex then it is cs-closed. Conversely, Zălinescu in [14] proved that when f^* is proper and f is cs-closed then f is cs-convex.
3) Every cs-closed function is semi-closed.
4) The indicator function δ_C of every convex semi-closed subset of X is semi-closed.
5) In [10], Laghdir studied the subdifferentiability of a convex semi-closed function, i.e. $\partial f(\bar{x}) \neq \emptyset$ whenever $\bar{x} \in \text{dom } f$, $\mathbb{R}_+[\text{dom } f - \bar{x}] = X$ and X is a Féchet space. It was proved in [10], that this result falses under the weakened condition: $\mathbb{R}_+[\text{dom } f - \bar{x}]$ is a closed vector subspace.
6) In [10], it was established a characterization for a semi-closed function by means of its level sets given by: $f : X \to \mathbb{R} \cup \{+\infty\}$ is semi-closed if and only if its level sets are semi-closed.

3 The fundamental duality formula.

Our goal in this section is to setting up the well-known fundamental duality result (1.1) for the class of convex semi-closed functions. This can be obtained provided a certain constraint qualification. In order to derive this result we will use the approach based on the use of a perturbation function. For this let us consider the following condition

$$(C.Q_1)\quad \begin{cases} X \text{ is a Fréchet space} \\ \\ f : X \to \mathbb{R} \cup \{+\infty\} \text{ convex and proper} \\ \\ g : X \to \mathbb{R} \cup \{+\infty\} \text{ convex, proper and semi-closed} \\ \\ \text{there exists } \bar{x} \in \text{dom } f \cap \text{dom } g \text{ such that} \\ \\ \mathbb{R}_+[\text{dom } g - \bar{x}] = X. \end{cases}$$

and the marginal function

$$\begin{array}{rll} p: & X & \longrightarrow \mathbb{R} \cup \{+\infty\} \\ & y & \longmapsto p(y) = \inf\limits_{x \in X} \{f(x) + g(y + x)\} \end{array}$$

Obviously p is convex since it is a marginal function of a convex function.

Lemma 3.1 *If* $\inf\limits_{x \in X} \{f(x) + g(x)\} \in \mathbb{R}$ *and the condition* $(C.Q_1)$ *is satisfied, then* $\partial p(0) \neq \emptyset$.

Proof. Let us note that the equality

$$\mathbb{R}_+[\text{dom } g] = \bigcup_{n,m \geq 0} m[g \leq n]$$

is obtained simply by observing that

$$\text{dom } g = \bigcup_{n \geq 1} [g \leq n].$$

Following [10], it follows from Baire's Theorem and the fact that g is semi-closed, that there exists some neighbourhood of zero U and some integer $n \geq 1$ such that

$$g(y + \bar{x}) \leq n, \ \forall y \in U$$

which yields

$$p(y) \leq f(\bar{x}) + g(y + \bar{x}) \leq f(\bar{x}) + n, \ \forall y \in U.$$

Therefore, it follows that p is bounded above on a neighbourhood of zero and since $p(0) = \inf_{x \in X}\{f(x) + g(x)\}$ is finite and p is convex we obtain from a classical convex analysis result (see [5]) that p is subdifferentiable at zero i.e. $\partial p(0) \neq \emptyset$. □

Now, we are ready to state our main result.

Theorem 3.2 *If* $\inf_{x \in X}\{f(x) + g(x)\} \in \mathbb{R}$ *and the condition* $(C.Q_1)$ *is satisfied, then*

$$\inf_{x \in X}\{f(x) + g(x)\} + \min_{x^* \in X^*}\{f^*(-x^*) + g^*(x^*)\} = 0.$$

Proof. It is straightforward to see that for any $x^* \in X^*$

$$p^*(x^*) = f^*(-x^*) + g^*(x^*),$$

so from the Fenchel's inequality we have

$$p^*(x^*) + p(0) \geq 0, \ \forall x^* \in X^*$$

i.e.

$$\inf_{x \in X}\{f(x) + g(x)\} + f^*(-x^*) + g^*(x^*) \geq 0, \ \forall x^* \in X^*, \tag{3.1}$$

which yields

$$\inf_{x^* \in X^*}\{f^*(-x^*) + g^*(x^*)\} + \inf_{x \in X}\{f(x) + g(x)\} \geq 0.$$

Since $\partial p(0) \neq \emptyset$, taking $z^* \in \partial p(0)$ i.e.

$$p^*(z^*) + p(0) = 0, \tag{3.2}$$

it results by combining (3.1) and (3.2) that

$$\inf_{x \in X}\{f(x) + g(x)\} + \min_{x^* \in X^*}\{f^*(-x^*) + g^*(x^*)\} = 0.$$

□

Corollary 3.3 *Let* $x^* \in X^*$ *such that* $\inf_{x \in X}\{f(x) + g(x) - \langle x^*, x\rangle\} \in \mathbb{R}$ *and assume that* $(C.Q_1)$ *holds, then we have*

$$(f+g)^*(x^*) = \min_{y^* \in X^*}\{f^*(x^* - y^*) + g^*(y^*)\}.$$

Proof. It suffices to apply Theorem 3.1 to the functions

$$\begin{array}{rll} F: & X & \longrightarrow \mathbb{R} \cup \{+\infty\} \\ & x & \longmapsto F(x) = f(x) - \langle x^*, x \rangle \end{array}$$

$$\begin{array}{rll} G: & X & \longrightarrow \mathbb{R} \cup \{+\infty\} \\ & x & \longmapsto G(x) = g(x) \end{array}$$

by observing that F and G verify together the condition $(C.Q_1)$. □

Corollary 3.4 *Under the condition* $(C.Q_1)$ *we have*

$$\partial(f+g)(\bar{x}) = \partial f(\bar{x}) + \partial g(\bar{x}).$$

Proof. The inclusion $\partial f(\bar{x}) + \partial g(\bar{x}) \subset \partial(f+g)(\bar{x})$ is immediate. Conversely, let $x^* \in \partial(f+g)(\bar{x})$, i.e.

$$(f+g)(x) - \langle x^*, x \rangle \geq f(\bar{x}) + g(\bar{x}) - \langle x^*, \bar{x} \rangle, \quad \forall x \in X,$$

and since $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$, it follows that $\inf\limits_{x \in X} \{f(x) + g(x) - \langle x^*, x \rangle\} \in \mathbb{R}$. As

$$(f+g)(\bar{x}) + (f+g)^*(x^*) - \langle x^*, \bar{x} \rangle = 0,$$

and using Corollary 3.1, we obtain for some $z^* \in X^*$ that

$$g^*(z^*) + f^*(x^* - z^*) + f(\bar{x}) + g(\bar{x}) - \langle x^*, \bar{x} \rangle = 0,$$

by setting $y^* := x^* - z^*$, we have

$$[g^*(z^*) + g(\bar{x}) - \langle z^*, \bar{x} \rangle] + [f^*(y^*) + f(\bar{x}) - \langle y^*, \bar{x} \rangle] = 0,$$

which yields, thanks to Fenchel's inequality, that

$$\begin{cases} g^*(z^*) + g(\bar{x}) - \langle z^*, \bar{x} \rangle = 0 \\ \\ f^*(y^*) + f(\bar{x}) - \langle y^*, \bar{x} \rangle = 0 \end{cases}$$

i.e.

$$\begin{cases} z^* \in \partial g(\bar{x}) \\ \\ y^* \in \partial f(\bar{x}) \end{cases}$$

and therefore we get $\partial(f+g)(\bar{x}) \subset \partial f(\bar{x}) + \partial g(\bar{x})$. □

Corollary 3.5 *Let C and D be two convex sets of X and $\bar{x} \in C \cap D$. Suppose that C is semi-closed and $\mathbb{R}_+(C-\bar{x}) = X$, then*

$$N_{C\cap D}(\bar{x}) = N_C(\bar{x}) + N_D(\bar{x}).$$

Proof. It sufficies to apply Corollary 3.2 to the indicator functions δ_C and δ_D. □

Corollary 3.6 *Let C be a semi-closed convex cone with $0 \in C$ and let $\bar{x} \in X$ such that $\mathbb{R}_+(C-\bar{x}) = X$. If $\inf_{x\in C} f(x) \in \mathbb{R}$ then one has*

$$\inf_{x\in C} f(x) + \min_{x^*\in C^0} f^*(-x^*) = 0.$$

Proof. Applying Theorem 3.1 to f and δ_C we obtain

$$\inf_{x\in C} f(x) + \min_{x^*\in X^*} \{f^*(-x^*) + \sup_{x\in C}\langle x^*, x\rangle\} = 0.$$

Since C is a cone it is easy to check that

$$C^0 = \{x^* \in X^* : \sup_{x\in C}\langle x^*, x\rangle \ \leq 0\},$$

hence

$$\inf_{x\in C} f(x) + \min_{x^*\in C^0} \{f^*(-x^*) + \sup_{x\in C}\langle x^*, x\rangle\} = 0.$$

We have $0 \in C$, so for every $x^* \in C^0$ we get $\sup_{x\in C}\langle x^*, x\rangle = 0$, therefore

$$\inf_{x\in C} f(x) + \min_{x^*\in C^0} f^*(-x^*) = 0.$$

□

Corollary 3.7 *Let C be a semi-closed convex cone with $0 \in C$ and let $\bar{x} \in X$ such that $\mathbb{R}_+(C-\bar{x}) = X$, then one has*

$$dist(\bar{x}, C) = \max_{x^*\in C^0, \|x^*\|\leq 1} \langle x^*, \bar{x}\rangle.$$

Proof. Since $\inf_{x\in C}\|x-\bar{x}\| \in \mathbb{R}$ then applying Corollary 3.4 to $f := \|.-\bar{x}\|$ we obtain

$$\inf_{x\in C}\|x-\bar{x}\| = \max_{x^*\in C^0} - f^*(-x^*).$$

After computing the conjugate function of f we get that for any $x^* \in X^*$ we have

$$f^*(x^*) = \delta_{\mathbb{B}_{X^*}}(x^*) + \langle x^*, \bar{x}\rangle,$$

where $\mathbb{B}_{X^*}$ is the closed unit ball of X^*. Hence we obtain

$$dist(\bar{x}, C) = \max_{x^*\in C^0, \|x^*\|\leq 1} \langle x^*, \bar{x}\rangle.$$

□

Remark 3.1 One may ask a natural question if the fundamental duality formula (1.1) holds under the weakened condition: $\mathbb{R}_+[\text{dom } g]$ is a closed vector subspace? The answer is no. Just take X an infinite dimensional Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ a convex proper function, $g : X \to \mathbb{R} \cup \{+\infty\}$ a noncontinuous linear functional, $Y := X \times \mathbb{R}$, $F : Y \to \mathbb{R} \cup \{+\infty\}$ defined by $F(x,t) = \delta_{\{(0,0)\}}(x,t)$ and $G : Y \to \mathbb{R} \cup \{+\infty\}$ defined by $G(x,t) = +\infty$ if $t \neq 0$ and $G(x,0) = g(x)$. It is easy to see that F and G are convex, proper, G is semi-closed, $\mathbb{R}_+[\text{dom } G] = X \times \{0\}$ is a closed linear subspace and G is nowhere subdifferentiable. So

$$\begin{cases} \inf\limits_{(x,t)\in Y} \{F(x,t) + G(x,t)\} = 0 \\ \\ \inf\limits_{(x^*,t^*)\in X^*\times\mathbb{R}} \{F^*(-x^*,-t^*) + G^*(x^*,t^*)\} = +\infty. \end{cases}$$

Hence the fundamental duality formula (1.1) falses.

4 Application to convex composite optimization.

In this section, we assume that the space Y is equipped with a partial preorder induced by a convex cone Y_+ i.e. for any $y_1,\ y_2 \in Y$

$$y_1 \leq_Y y_2 \quad \Leftrightarrow \quad y_2 - y_1 \in Y_+$$

and an abstract maximal element $+\infty$ will be adjoined to Y. A mapping $h : X \to Y \cup \{+\infty\}$ is said to be Y_+-convex in the sense that for any $x_0,\ x_1 \in \text{dom } h := \{x \in X\ :\ h(x) \in Y\}$ and for any $\lambda \in [0,1]$ we have

$$h(\lambda x_0 + (1-\lambda)x_1) \leq_Y \lambda h(x_0) + (1-\lambda)h(x_1).$$

A function $g : Y \to \mathbb{R} \cup \{+\infty\}$ is said to be Y_+-nondecreasing on a subset C of Y if for any $y_1,\ y_2 \in C$ we have

$$y_1 \leq_Y y_2 \quad \Rightarrow \quad g(y_1) \leq g(y_2).$$

In what follows, we extend to $Y \cup \{+\infty\}$ the composite function $(g \circ h)$ by setting $(g \circ h)(x) = \sup\limits_{y\in Y} g(y)$ for any $x \notin \text{dom } h$. Here Y_+^* denotes the positive polar cone defined by

$$Y_+^* = \{y^* \in Y^* :\ \langle y^*, y\rangle \geq 0, \quad \forall y \in Y_+\}.$$

Many of the convex minimization problems arising in Applied Mathematics, Operations research and Mathematical problems can be formulated as the following convex composite problem

$$(P): \qquad \inf_{x\in X}(f+g\circ h)(x),$$

where $f : X \to \mathbb{R}\cup\{+\infty\}$ is convex and proper, $g : Y \to \mathbb{R}\cup\{+\infty\}$ is convex proper and nondecreasing on an appropriate subset of Y and $h : X \to Y\cup\{+\infty\}$ is Y_+-convex and proper.

The aim of this section is to formulate the dual problem (P^*) associated to the primal problem (P). For this, let us consider the following constraint qualification

$$(C.Q_2) \quad \begin{cases} X \text{ and } Y \text{ are Fréchet spaces} \\ f : X \to \mathbb{R}\cup\{+\infty\} \text{ convex and proper} \\ g : Y \to \mathbb{R}\cup\{+\infty\} \text{ convex, proper and semi-closed} \\ h : X \to Y\cup\{+\infty\}\ Y_+\text{-convex and proper} \\ \text{there exists } \bar{x}\in \text{dom } h\cap \text{dom } f\cap h^{-1}(\text{dom } g) \text{ such that} \\ \mathbb{R}_+[\text{dom } g - h(\bar{x})] = Y. \end{cases}$$

and the following auxiliary functions

$$\begin{aligned} \tilde{f}: \quad X\times Y &\longrightarrow \mathbb{R}\cup\{+\infty\} \\ (x,y) &\longmapsto \tilde{f}(x,y) = f(x) + \delta_{\text{Epi } h}(x,y) \end{aligned}$$

$$\begin{aligned} \tilde{g}: \quad X\times Y &\longrightarrow \mathbb{R}\cup\{+\infty\} \\ (x,y) &\longmapsto \tilde{g}(x,y) = g(y), \end{aligned}$$

where $\text{Epi } h := \{(x,y)\in X\times Y \ : \ h(x)\le_Y y\}$ is the epigraph of h. Obviously $\tilde{f}$ and $\tilde{g}$ are both convex and proper.

Proposition 4.1 *If* $\inf_{x\in X}(f+g\circ h)(x)\in\mathbb{R}$, *$g$ nondecreasing on* $\text{Im } h + Y_+$ *and the condition* $(C.Q_2)$ *is satisfied one has*

$$\inf_{x\in X}(f+g\circ h)(x) = \max_{y^*\in Y_+^*}\{-(f+y^*\circ h)^*(0) - g^*(y^*)\}$$

Proof. Let us note that for any $x \in X$

$$(f + g \circ h)(x) = \inf_{y \in Y} \{\tilde{f}(x, y) + \tilde{g}(x, y)\}$$

and dom $\tilde{f} = (\text{dom } f \times Y) \cap \text{Epi } h$, dom $\tilde{g} = X \times \text{dom } g$ and for any $\lambda \in \mathbb{R}$ we have $[\tilde{g} \leq \lambda] = X \times [g \leq \lambda]$ and hence it is easy to check that the condition $(C.Q_2)$ ensures that $\tilde{f}$ and $\tilde{g}$ satisfy together the qualification condition $(C.Q_1)$. Therefore by virtue of Theorem 3.1 we get

$$\begin{aligned} \inf_{x \in X} (f + g \circ h)(x) &= \inf_{\substack{x \in X \\ y \in Y}} \{\tilde{f}(x, y) + \tilde{g}(x, y)\} \\ &= \max_{\substack{x^* \in X^* \\ y^* \in Y^*}} \{-\tilde{f}^*(-x^*, -y^*) - \tilde{g}^*(x^*, y^*)\}, \end{aligned}$$

by expliciting the conjugate functions $\tilde{f}^*$ and $\tilde{g}^*$ we have

$$\begin{aligned} \tilde{f}^*(-x^*, -y^*) &= (f + y^* \circ h)^*(-x^*) + \delta_{Y_+^*}(y^*) \\ \tilde{g}^*(x^*, y^*) &= g^*(y^*) + \delta_{\{0\}}(x^*) \end{aligned}$$

and thus we get

$$\inf_{x \in X} (f + g \circ h)(x) = \max_{y^* \in Y_+^*} \{-(f + y^* \circ h)^*(0) - g^*(y^*)\}$$

this completes the proof. □

Corollary 4.1 *Let $A : X \to Y$ be a continuous linear operator. If $\inf_{x \in X} (f + g \circ A)(x) \in \mathbb{R}$ and the condition $(C.Q_2)$ is satisfied one has*

$$\inf_{x \in X} (f + g \circ A)(x) = \max_{y^* \in Y^*} \{-f^*(-A^* y^*) - g^*(y^*)\},$$

where $A^* : Y^* \to X^*$ stand for the adjoint operator of A.

Proof. By putting $Y_+ = \{0\}$, it is obvious that g is nondecreasing on the whole space Y and $Y_+^* = Y^*$. Therefore, by applying Proposition 4.1, with $h := A$, we obtain

$$\inf_{x \in X} (f + g \circ A)(x) = \max_{y^* \in Y^*} \{-(f + y^* \circ A)^*(0) - g^*(y^*)\}.$$

Since

$$\begin{aligned} (f + y^* \circ A)^*(0) &= \sup_{x \in X} \{-f(x) - \langle y^* \circ A, x\rangle\} \\ &= \sup_{x \in X} \{-f(x) - \langle A^* y^*, x\rangle\} \\ &= f^*(-A^* y^*), \end{aligned}$$

hence we get

$$\inf_{x\in X}(f+g\circ A)(x)=\max_{y^*\in Y^*}\{-f^*(-A^*y^*)-g^*(y^*)\}.$$

□

Corollary 4.2 *Let* $x^*\in X^*$ *such that* $\inf_{x\in X}\{f(x)+(g\circ h)(x)-\langle x^*,x\rangle\}\in\mathbb{R}$, g *nondecreasing on* $\mathrm{Im}\ h+Y_+$ *and the condition* $(C.Q_2)$ *is satisfied then*

$$(f+g\circ h)^*(x^*)=\min_{y^*\in Y_+^*}\{g^*(y^*)+(f+y^*\circ h)^*(x^*)\}.$$

Proof. Let us consider the following function

$$\begin{array}{rll} F: & X & \longrightarrow \mathbb{R}\cup\{+\infty\} \\ & x & \longmapsto F(x)=f(x)-\langle x^*,x\rangle, \end{array}$$

by observing that F and g verify together the condition $(C.Q_2)$ and hence by applying Proposition 4.1 we obtain

$$\begin{aligned}(f+g\circ h)^*(x^*) &= -\inf_{x\in X}\{F(x)+(g\circ h)(x)\}\\ &= \min_{y^*\in -Y_+^*}\{g^*(-y^*)+(F-y^*\circ h)^*(0)\},\end{aligned}$$

and since $(F-y^*\circ h)^*(0)=(f-y^*\circ h)^*(x^*)$, we get the desired result. □

The next corollary concerns the calculus of the subdifferential of composite convex functions using the preceding results.

Corollary 4.3 *Under the condition* $(C.Q_2)$ *and* g *supposed to be nondecreasing on* $\mathrm{Im}\ h+Y_+$ *one has*

$$\partial(f+g\circ h)(\bar{x})=\bigcup_{y^*\in\partial g(h(\bar{x}))\cap Y_+^*}\partial(f+y^*\circ h)(\bar{x}).$$

Proof. Let $x^*\in\partial(f+g\circ h)(\bar{x})$ i.e.

$$(f+g\circ h)(x)-\langle x^*,x\rangle\geq f(\bar{x})+(g\circ h)(\bar{x})-\langle x^*,\bar{x}\rangle,\quad \forall x\in X,$$

and since $\bar{x}\in\mathrm{dom}\ h\cap\mathrm{dom}\ f\cap h^{-1}(\mathrm{dom}\ g)$, it follows that

$$\inf_{x\in X}\{f(x)+(g\circ h)(x)-\langle x^*,x\rangle\}\in\mathbb{R}.$$

As

$$(f+g\circ h)(\bar{x})+(f+g\circ h)^*(x^*)-\langle x^*,\bar{x}\rangle=0,$$

and using Corollary 4.2, we obtain for some $z^* \in Y_+^*$ that

$$g^*(z^*) + (f + z^* \circ h)^*(x^*) + f(\bar{x}) + g(h(\bar{x})) - \langle x^*, \bar{x} \rangle = 0,$$

i.e.

$$[g^*(z^*)+g(h(\bar{x}))-\langle z^*, h(\bar{x})\rangle]+[(f+z^*\circ h)^*(x^*)+f(\bar{x})+(z^*\circ h)(\bar{x})-\langle x^*, \bar{x}\rangle] = 0,$$

which yields, thanks to Fenchel's inequality, that

$$\begin{cases} g^*(z^*) + g(h(\bar{x})) - \langle z^*, h(\bar{x})\rangle = 0 \\ (f + z^* \circ h)^*(x^*) + f(\bar{x}) + (z^* \circ h)(\bar{x}) - \langle x^*, \bar{x}\rangle = 0 \end{cases}$$

i.e.

$$\begin{cases} z^* \in \partial g(h(\bar{x})) \\ x^* \in \partial(f + z^* \circ h)(\bar{x}), \end{cases}$$

and therefore we get

$$\partial(f + g \circ h)(\bar{x}) \subseteq \bigcup_{y^* \in \partial g(h(\bar{x})) \cap Y_+^*} \partial(f + y^* \circ h)(\bar{x})$$

Conversely, let $x^* \in \bigcup_{y^* \in \partial g(h(\bar{x})) \cap Y_+^*} \partial(f + y^* \circ h)(\bar{x})$, i.e. there exists $y^* \in Y_+^*$ such that

$$\begin{cases} y^* \in \partial g(h(\bar{x})) \\ x^* \in \partial(f + y^* \circ h)(\bar{x}) \end{cases}$$

i.e.

$$\begin{cases} \langle y^*, y - h(\bar{x})\rangle + g(h(\bar{x})) \le g(y), \ \forall y \in Y, \\ \langle x^*, x - \bar{x}\rangle + (f + y^* \circ h)(\bar{x}) \le (f + y^* \circ h)(x), \ \forall x \in X. \end{cases}$$

By putting $y := h(x)$, we have

$$\langle x^*, x - \bar{x}\rangle + f(\bar{x}) + g(h(\bar{x})) \le f(x) + g(h(x)), \ \forall x \in X,$$

i.e. $x^* \in \partial(f + g \circ h)(\bar{x})$ and the converse inclusion is then proved. □

Corollary 4.4 *Let $A : X \to Y$ be a continuous linear operator. Under the condition $(C.Q_2)$ one has*

$$\partial(f + g \circ A)(\bar{x}) = \partial f(\bar{x}) + A^*(\partial g(A\bar{x})).$$

Proof. Putting $Y_+ = \{0\}$ and using Corollary 4.3 with $h := A$, we get

$$\partial(f + g \circ A)(\bar{x}) = \bigcup_{y^* \in \partial g(A\bar{x})} \partial(f + y^* \circ A)(\bar{x}).$$

Let $y^* \in \partial g(A\bar{x})$, since $(y^* \circ A)$ is a continuous linear form so it is semi-closed and since dom $(y^* \circ A) = X$ then by applying Corollary 3.2 with $g := y^* \circ A$, we obtain

$$\partial(f + y^* \circ A)(\bar{x}) = \partial f(\bar{x}) + \partial(A^* y^*)(\bar{x}).$$

As $A^* y^*$ is a linear continuous form, thus $\partial(A^* y^*)(\bar{x}) = \{A^* y^*\}$ and therefore

$$\partial(f + g \circ A)(\bar{x}) = \partial f(\bar{x}) + A^*(\partial g(A\bar{x})).$$

□

Corollary 4.5 *Let $A : X \to Y$ be a continuous linear operator and C and D be two convex subsets of X and $\bar{x} \in C \cap A^{-1}(D) := B$. Suppose that D is semi-closed and $\mathbb{R}_+(D - A(\bar{x})) = X$, then*

$$N_B(\bar{x}) = N_C(\bar{x}) + A^* N_D(A\bar{x}).$$

Proof. It is easy to check that

$$\delta_B(\bar{x}) = \delta_C(\bar{x}) + (\delta_D \circ A)(\bar{x}),$$

and by applying Corollary 4.4 to $f := \delta_C$ and $g := \delta_D$ we obtain

$$\partial \delta_B(\bar{x}) = \partial \delta_C(\bar{x}) + A^*(\partial \delta_D(A\bar{x})),$$

i.e.

$$N_B(\bar{x}) = N_C(\bar{x}) + A^* N_D(A\bar{x}).$$

□

As an application of this last corollary, we derive the optimality conditions, related to the following mathematical programming problem

$$(Q) \quad \begin{cases} \inf f(x), \\ h(x) \in -Y_+ \\ x \in C \end{cases}$$

where X and Y are Fréchet spaces, $f : X \to \mathbb{R} \cup \{+\infty\}$ is a convex proper function, $h : X \to Y \cup \{+\infty\}$ is a Y_+-convex proper operator and C a nonempty subset of X supposed to be convex. In the following we will assume that Y_+ is semi-closed.

Proposition 4.2 *Let $\bar{x}$ be a feasible point for the problem (Q) i.e. $\bar{x} \in C \cap h^{-1}(-Y_+)$. If $\mathbb{R}_+[Y_+ + h(\bar{x})] = Y$, then $\bar{x}$ is an optimal solution for the problem (Q) if and only if there exists $y^* \in Y_+^*$ such that $\langle y^*, h(\bar{x})\rangle = 0$ and $0 \in \partial(f + \delta_C + y^* \circ h)(\bar{x})$.*

Proof. $\bar{x}$ is an optimal solution for the problem (Q) if and only if $0 \in \partial(f + \delta_C + \delta_{-Y_+} \circ h)(\bar{x})$. On the other hand since the cone is nonempty convex closed and following [4] δ_{-Y_+} is Y_+-nondecreasing, convex, proper and semi-closed, hence all the hypothesis of Corollary 4.3 are satisfied and

$$\partial(f + \delta_C + \delta_{-Y_+} \circ h)(\bar{x}) = \bigcup_{y^* \in N_{-Y_+}(h(\bar{x})) \cap Y_+^*} \partial(f + \delta_C + y^* \circ h)(\bar{x}),$$

which means that $\bar{x}$ is an optimal solution of the problem (Q) if and only if there exists $y^* \in Y_+^*$ such that $\langle y^*, h(\bar{x})\rangle = 0$ and $0 \in \partial(f + \delta_C + y^* \circ h)(\bar{x})$. □

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