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On a Class of Integrodifferential Equations

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Abstract

In this paper, we are interested in a class of integrodifferential equations. Inspired by a result of M.Artola, we use the technique of FAEDO-GALERKIN, to show the existence of the solution. With supplementary hypothesis on the kernel and on the second member of the considered equation, we show the uniqueness of the solution as well as some results of regularity.

Keywords: Integrodifferential, operators of local type, Faedo-Galerkin method, existence, uniqueness and regularity

1 Introduction

Used in the modelling of raising problems of visco-elastic physics and the mechanics of the fluids, the integrodifferential equations caused a big interest among the scientists. For references in this area, we refer the reader to [2], [5], [4], [10], [6], [7], [9], [11], [15] and the references therein.
To contribute in this context, we consider the integrodifferential system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i=1}^{N} A_i \partial_i u + Bu - \triangle_\alpha u - \int_0^t G(t-s) \triangle_\beta u(x,s)ds = f - \nabla p \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad in \;\; \Omega \times]0, T[\\ div\ u = 0 \end{cases}$$

with the initials and boundary conditions

$$u(x,t) = 0 \; on \; \Gamma \times]0, T[$$
$$u(x,0) = u_0$$

where $u = (u_1, u_2, ..., u_N)$; $\partial_i u = (\partial_i u_1, \partial_i u_2, ...; \partial_i u_N)$; $\Delta_\alpha u = \alpha \Delta u$, $\alpha > 0$; $\Delta_\beta u = \beta \Delta u$, $\beta > 0$; Ω is an open and bounded subset of $\mathbb{R}^N$ of boundary Γ;

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B and A_i $(1 \leq i \leq N)$ are matrices of order N; G is a kernel and f is given.

In the case where $\alpha = 0$, $\beta = 1$, $A_i = 0$ and $B = 0$, Slemrod [11] used the theory of the semi-groups to show the existence and the uniqueness of the solution of the system.
While using the FAEDO-GALERKIN approach and a theorem on the operators of local type due to M.Artola, we demonstrate the existence of the solution of the system. Under the hypothesis $ReF(\widetilde{G}) \geq 0$ where $\widetilde{G}$ is the extension of G by 0 outside $[0, T]$ and $F(\widetilde{G})$ is the Fourier transform of $\widetilde{G}$, we establish the uniqueness of the solution. Inspired by the works of Bardos [3], Temam [14] and Tartar [12], and with some hypotheses on the data of the problem, we demonstrate some results of the regularity of the solution.

2 Preliminary results

Throughout this paper, $N \in \mathbb{N}^*$, Ω is an open of $\mathbb{R}^N$ of generic point $x = (x_1, x_2, ..., x_N)$, Γ denotes the boundary of Ω, Q_T denotes the cylinder $\Omega \times]0, T[$ with lateral boundary $\Gamma \times]0, T[$, of generic point $x = (x_1, x_2, ..., x_N, t)$, t being the time variable.

Let $\mathcal{D}(\Omega)$ (resp. $\mathcal{D}(Q_T)$) denote the space of the indefinitely differentiable functions with compact support in Ω (resp. in $\Omega \times]0, T[$), and let V be the space

$$V = \{\varphi = (\varphi_1, \varphi_2, ..., \varphi_N) \in (\mathcal{D}(\Omega))^N / \; div\varphi = 0\}$$

where $div\varphi = \sum_{i=1}^{N} \partial_i^2 \varphi_i$, we denote by H the adherence of V in $(L^2(\Omega))^N$ (resp. V the adherence of V in $(H^1(\Omega))^N$) where $H^1(\Omega)$ is the sobolev space, then we have (see [13])

$$H = \{\varphi \in (L^2(\Omega))^N / \; div\varphi = 0 \; and \; \varphi.\overrightarrow{n} = 0 \; on \; \Gamma\}$$

where $\overrightarrow{n}$ denotes the external normal to Γ.
Here we list the basic results that will be used in this work.

Lemma 2.1 *[13] Let $f = (f_1, ..., f_N)$ be in $(\mathcal{D}(Q^T))^N$, the necessary and sufficient condition so that $f = grad\; p$, $p \in \mathcal{D}'(Q_T)$ is*

$$(f, \varphi) = 0 \; , \quad \forall \varphi \in V$$

where $gradp = (\partial_1 p, ..., \partial_N p)$.

Then the orthogonal of H in $(L^2(\Omega))^N$ is characterized by.

$$H^{\perp} \; = \; \{\varphi \in (L^2(\Omega))^N \; / \; \varphi = gradp, \; p \in H^1(\Omega)\}.$$

Lemma 2.2 *[12] If $u \in L^2(0,T;V)$ and $u' \in L^2(0,T;V')$ then $u \in \mathcal{C}(0,T,H)$.*

Lemma 2.3 *[12] Let X be a Banach space. If $f \in L^p(0,T;X)$ and $f' \in L^p(0,T;X)$ then f is continuous almost everywhere on [0 , T] with values in X.*

Lemma 2.4 ***inequality of Young***
If p and q are two reals such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, then for all a and b two positif reals, we have

$$ab \;<\; \epsilon a^p \;+\; \frac{p-1}{p^q}\,\frac{b^q}{\epsilon^{\frac{1}{p-1}}} \quad (\epsilon > 0)\ .$$

To establish the existence theorem in later sections, we shall require the following definition.
If $L^p(\Omega)$, $1 \leq p \leq \infty$, denote spaces of functions whose power p is integrable (essentially bounded for $p=+\infty$) endowed with norms

$$|u|_{L^p(\Omega)} = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}} \quad , \quad |u|_{L^\infty(\Omega)} = ess\sup_\Omega |u(x)| \ \ ;$$

then

Definition 2.1 *[1] Let X be a real or complex Banach space and let M be a linear continuous operator from $L^\infty(0,T;X)$ to $L^\infty(0,T;X)$; M is said to be of local type if there exists a constant μ_T depending only on T such that for all $t_0 \in [0,T]$ and all $u \in L^\infty(0,T;X)$*

$$|r_{t_0} Mu|_{L^\infty(0,T;X)} \;<\; \mu_T \;|r_{t_0} u|_{L^\infty(0,T;X)}$$

where $r_{t_0}v(s) = \begin{cases} v(s)\ a.e\ on\ [0,t_0] \\ 0 \quad elsewhere \end{cases}$ *for* $v \in L^\infty(0,T;X)$

Examples of operators of local type

1. We consider the application

$$\begin{array}{rcccc} K & : & [0,T]\times[0,T] & \longrightarrow & \mathbb{C} \\ & & (s,t) & \longmapsto & K(s,t) \end{array}$$

   such that $K(.,t)$ belongs to $L^1(\mathbb{C})$.
   M is the integral operator defined by

$$(Mu)(t) = \int_0^t K(s,t)u(s)ds.$$

   M is a linear continuous operator from $L^\infty(0,T;\mathbb{C})$ in $L^\infty(0,T;\mathbb{C})$. It is of local type.

2. Let X be a Banach space. If we suppose that for all $t \in [0,T]$, $N(t) \in \mathcal{L}(X,X)$ and there exists a constant $k(t)$ depending only on T such that

$$\forall t \in [0,T], \quad \forall u \in X \quad |N(t).u|_X \leq k(t)|u|_X$$

then the operator N defined by

$$Nu(t) = N(t)u(t) \quad a.e \quad \forall u \in L^\infty(0,T;X)$$

is in $\mathcal{L}(L^\infty(0,T;X); L^\infty(0,T;X))$ since

$$\begin{aligned} |Nu|_{L^\infty(0,T;X)} &= \sup_{s\in[0,T]} |N(s)u(s)|_X \\ &\leq k(t) \sup_{s\in[0,T]} |u(s)|_X \end{aligned}$$

moreover, the operator N is of local type.

Theorem 2.1 *[1] Let X be a real or complex Banach space and let M be a linear continuous operator from $L^\infty(0,T;X)$ to $L^\infty(0,T;X)$ of local type. Let $f \in L^1(0,T;X)$ and $u_0 \in X$.*
Then there exists a unique function u, solution of the equation

$$\begin{cases} u' = Mu(t) + f(t) \ a.e \ on \ [0,T] \\ u(0) = u_0 \end{cases}$$

and such that $u \in \mathcal{C}(0,T;X)$ and $u' \in L^1(0,T;X)$.

3 Problem statement and reduction to a variational formulation

The problem being addressed in this paper can be formulated as follows.

Given $f \in L^2(0,T;V')$ and $u_0 \in H$, we investigate the existence of $u \in (\mathcal{D}'(Q_T))^N$, and $p \in \mathcal{D}'(Q_T)$ such that

$$\begin{cases} 3.1 \quad u' + \sum_{i=1}^{N} A_i \partial_i u + Bu - \triangle_\alpha u - \int_0^t G(t-s)\triangle_\beta u(x,s)ds \ = \ f - \nabla p \\ 3.2 \quad div \ u = 0 \\ 3.3 \quad u(x,t) = 0 \ sur \ \Gamma \times]0,T[\\ 3.4 \quad u(x,0) = u_0 \end{cases}$$

where $u = (u_1, u_2, ..., u_N)$; $\partial_i u = (\partial_i u_1, \partial_i u_2, ...; \partial_i u_N)$; $\triangle_\alpha u = \alpha \triangle u$, $\alpha > 0$, $\triangle u = (\Delta u_1, \Delta u_2, ..., \Delta u_N)$, $\Delta u_i = \sum_{i=1}^{N} \partial_i^2 u_i$ with Δu_i is the Laplacian of u_i.

Ω is an open and bounded subset of $\mathbb{R}^N$ of boundary Γ.
B and A_i $(1 \leq i \leq N)$ are matrices of order N independent of the time t; G and f are given.

By considering the variational formulation of the problem above and the using lemma 2.1, this problem is equivalent to the following one

Given $f \in L^2(0,T;V')$ and $u_0 \in H$, we investigate the existence of $u \in L^\infty(0,T;H) \bigcap L^2(0,T;V)$ such that
For all v in V

$$\begin{cases} 3.5 \quad (u',v) + a(u,v) + b(u,v) + \alpha c(u,v) - \beta \int_0^t G(t-s)c(u,v)ds \; = \; (f,v) \\ 3.6 \quad u(0) = u_0 \\ \qquad where\ a,\ b\ and\ c\ are\ bilinear\ forms\ defined\ on\ V \times V\ by \\ \qquad a(u,v) = \sum_{i=1}^{N} \int_\Omega A_i u \partial_i v dx \\ \qquad b(u,v) = \int_\Omega Bu.v dx \\ \qquad c(u,v) = \sum_{i,j=1}^{N} \int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} dx \end{cases}$$

We suppose that $a(u,u) \geq 0$, $b(u,u) \geq 0$ and $c(u,u) = \|u\|^2$.

4 Existence and uniqueness theorems

Now we state our main result concerning existence of solutions for Eqs. (3.5) and (3.6).

Theorem 4.1 *Suppose that the following hold*

1. *The function G is measurable and bounded.*
2. $f \in L^2(0,T;V')$
3. $u_0 \in H$

then the problem (3.5)- (3.6) has a solution u such that

$$u \; \in \; L^\infty(0,T,H) \cap L^2(0,T,V).$$

Proof
In order to prove the existence of a solution of (3.5)- (3.6), we use Faedo-Galerkin method
First step

V being a separable Hilbertian space, there exists an orthonormal basis $\{w_1, w_2, ...\}$ of V whose linear combinations are dense in V .
Let V_m be the subspace of V generated by $\{w_1, w_2, ..., w_m\}$, our object in this first step is to determine a solution

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j \tag{1}$$

of equation

$$\begin{aligned}(u'_m, w_i) + a(u_m, w_i) + b(u_m, w_i) + \alpha c(u_m, w_i)\\ +\beta \int_0^t G(t-s) c(u_m, w_i) ds \ = \ (f, w_i) \ \ 1 \le i \le m\end{aligned} \tag{2}$$

with

$$\begin{aligned}&g_{im}(0) = \alpha_{im}\\ &u_0 = \lim_{m \to +\infty} \sum_{i=1}^{m} \alpha_{im} w_i \ \textit{strongly in } H\end{aligned} \tag{3}$$

If we denote $(f, w_i) = F_m^i(t)$, the substitution of (1) in (2) allows to obtain

$$\begin{aligned}&\sum_{j=1}^{m} g'_{jm}(t)(w_j, w_i) + \sum_{j=1}^{m} g_{jm}(t) a(w_j, w_i) + \sum_{j=1}^{m} g_{jm}(t) b(w_j, w_i)\\ + \ \ &\alpha \sum_{j=1}^{m} g_{jm}(t) c(w_j, w_i) + \beta \sum_{j=1}^{m} \int_0^t G(t-s) g_{jm}(t) c(w_j, w_i) ds = F_m^i(t)\end{aligned} \tag{4}$$

$$g_{im}(0) = \alpha_{im} \tag{5}$$

Let us denote by $g_m(t)$ the vector of $\mathbb{C}^m$ defined by

$$g_m(t) = (g_{1m}(t), \ g_{2m}(t), \ ..., \ g_{mm}(t))^T$$

Since $(w_j)_{1 \le j \le m}$ is an orthonormal basis, we can easily establish that (4) is equivalent to

$$\begin{aligned}&g'_m(t) + K_m g_m(t) + M_m(t) g_m(t) = F_m(t)\\ &g_m(t) = \alpha_m\end{aligned}$$

where $K_m \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)$ is the matrix given by $K_m = (\gamma_{ji})_{1 \le i,j \le m}$ with $\gamma_{ji} = a(w_j, w_i) + b(w_j, w_i) + \alpha c(w_j, w_i)$ and $M_m \in \mathcal{L}(\ L^\infty(0,T;\mathbb{C}^m), L^\infty(0,T;\mathbb{C}^m) \ )$ is the matrix $M_m = (\varepsilon_{ji}(t))_{1 \le i,j \le m}$ with $\varepsilon_{ji}(t) = \beta \displaystyle\int_0^t G(t-s) c(w_j, w_i) ds$.
$F_m(t) = (F_m^1, F_m^2, ..., F_m^m)^T$.
Using the examples of operators of local type given in section 2, it is obvious that $K_m + M_m \ \in \ \mathcal{L}(\ L^\infty(0,T;\mathbb{C}^m), L^\infty(0,T;\mathbb{C}^m) \ )$ is an operator of the local

type then according to theorem 4.1, we have
For each m, there is an unique solution g_m verifying (4) and (5) such that

$$g_m \in \mathcal{C}(0,T;\mathbb{C}^m) \quad \textit{and} \quad g'_m \in L^1(0,T;\mathbb{C}^m)$$

We deduce that there is a solution u_m of (2) and (3) such that

$$u_m \in \mathcal{C}(0,T;V_m) \quad \textit{and} \quad u'_m \in L^1(0,T;V_m).$$

Second step: estimate a priori
We multiplying (2) by $g_{im}(t)$and summing, we have

$$\begin{array}{rl} & \frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + a(u_m(t),u_m(t)) + b(u_m(t),u_m(t)) \\ + & \alpha c(u_m(t),u_m(t)) + \beta \int_0^t G(t-s)c(u_m(t),u_m(t))ds \\ = & (f(t),u_m(t)). \end{array} \tag{6}$$

Since $a(u,u) \geq 0$, $b(u,u) \geq 0$ and $c(u,u) = \|u\|^2$, the integration (6) gives

$$\begin{array}{rl} & \frac{1}{2}|u_m(t)|^2 - \frac{1}{2}|u_m(0)|^2 + \alpha \int_0^t \|u_m(s)\|^2 ds \\ \leq & \int_0^t (f(s),u_m(s))ds - \beta \int_0^t \left(\int_0^s G(s-\sigma)c(u_m(\sigma),u_m(\sigma))d\sigma\right) ds \end{array} \tag{7}$$

with $|.|$ the usual norm on H and $\|.\|$ the usual one on V.

We will estimate the second member of the inequality (7).
On the one hand, under the terms of the inequality of Young, we have

$$\int_0^t (f(s),u_m(s))ds \;<\; \frac{c}{\eta}\int_0^t \|f(s)\|_*^2 ds \;+\; \eta \int_0^t \|u_m(s)\|^2 ds \tag{8}$$

where η is a strictly positive constant and $\|.\|_{V'} = \|.\|_*$.
In the other hand, since for $f \in L^2_{loc}(\mathbb{R})$

$$\left(\int_0^t |f(s)|ds\right)^2 \;<\; t\int_0^t |f(s)|^2 ds$$

we have

$$\begin{array}{rl} & |\beta \int_0^t \left(\int_0^s G(s-\sigma)c(u_m(\sigma),u_m(\sigma))d\sigma\right) ds| \\ \leq & C\int_0^t \left(\int_0^s \|u_m(\sigma)\|\|u_m(s))d\sigma\right) ds \\ \leq & C(\int_0^t \|u_m(s)\|ds)^2 \\ \leq & Ct\ \int_0^t \|u_m(s)\|^2 ds \end{array} \tag{9}$$

Using (7), (8) and (9), we deduce

$$|u_m(t)|^2 + (2\alpha - 2Ct - \eta)\int_0^t \|u_m(s)\|^2 ds \;\leq\; \frac{1}{\eta}\int_0^t \|f(s)\|_*^2 ds + 2|u_m(0)|^2$$

α being a strictly positive real, choesing $\eta = \alpha$, there exists $t_0 < \frac{\alpha}{2C}$ such that

$$\sup_{0<t<t_0} |u_m(t)| \leq C$$

$$\int_0^t \|u_m(s)\|^2 ds \leq C$$

where C is an independent constant of m.
By an iterative process, we have

$$(u_m)_m \text{ is bounded in } L^\infty(0,T;H) \tag{10}$$

$$(u_m)_m \text{ is bounded in } L^2(0,T;V). \tag{11}$$

Third step: passage to the limit
From 10 and 11, we can extract a subsequence that we call also u_m verifying

$$\begin{array}{lll} u_m & \rightarrow u & in \quad L^\infty(0,T;H) \text{ weak star} \\ u_m & \rightarrow u & in \quad L^2(0,T;V) \text{ weak} \end{array}$$

By a reasoning similar to that done for the equations of Navier-Stokes (see for instance [12], p. 47), we deduce that u is a solution of problem (3.5)-(3.6) such that

$$u' \in L^2(0,T,V').$$

and

$$u \in L^\infty(0,T,H) \cap L^2(0,T,V).$$

Thus by lemma (2.2)

$$u \in \mathcal{C}(0,T,H).$$

■

The following theorem establish the uniqueness of solution of Eqs. (3.5)-(3.6) under an additional assumption on Kernel G.

Theorem 4.2 *Under the hypothesis of theorem (4.1), if we suppose moreover that*

$$ReF(\widetilde{G}(t)) > 0 \tag{12}$$

where F is the Fourier transform on $\mathbb{R}$, Re denotes the real part and
$$\widetilde{G}(t) = \begin{cases} G(t) & on \ [0,T] \\ 0 & elsewhere \end{cases}.$$
Then the solution of problem (3.5)-(3.6) is unique.

To prove this theorem, we need the following technical result

Lemma 4.1 *Under the hypothesis (12), the function defined by*

$$\begin{array}{ccc} L^2(0,T;V) & \to & \mathbb{R} \\ u & \mapsto & \int_0^t \left(\int_0^s G(s-\sigma)C(u(\sigma),u(s))d\sigma\right) ds \end{array}$$

is positive.

Proof of Lemma 4.1
Consider $\widetilde{u}$ and $\widetilde{G}$ respective prolongations of u and G by 0 apart from $[0,T]$, then we have

$$\int_0^t G(t-\sigma)\ C(u(\sigma),u(t))\ d\sigma = \int_{-\infty}^{+\infty} \widetilde{G}(t-\sigma)\ C(\widetilde{u}(\sigma),\widetilde{u}(t))\ d\sigma$$

(Note that $\int_t^T \widetilde{G}(t-\sigma)\ C(\widetilde{u}(\sigma),\widetilde{u}(t))\ d\sigma = 0$) we have then

$$\begin{aligned} & \int_0^T \left(\int_0^t G(t-\sigma)\ C(u(\sigma),u(t))d\sigma\right) dt \\ = \ & \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \widetilde{G}(t-\sigma) \sum_{i=1}^{N} \left(\tfrac{\partial \widetilde{u}}{\partial x_i}(\sigma)\ ,\ \tfrac{\partial \widetilde{u}}{\partial x_i}(t)\right) d\sigma\right) dt \\ = \ & \sum_{i=1}^{N} \left(\widetilde{G} * \tfrac{\partial \widetilde{u}}{\partial x_i}(t)\ ,\ \tfrac{\partial \widetilde{u}}{\partial x_i}(t)\right) \quad (by\ Fubini\ theorem) \end{aligned}$$

Let F be the Fourier transform defined from $L_t^2(-\infty,+\infty;H)$ to $L_\tau^2(-\infty,+\infty;H)$ by

$$F(u)\ (\tau)\ =\ \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{+\infty} exp(-it\tau)u(t)dt.$$

Since F is an isometry in L^2

$$\int_{-\infty}^{+\infty} \left(\widetilde{G} * \frac{\partial \widetilde{u}}{\partial x_i}(t)\ ,\ \frac{\partial \widetilde{u}}{\partial x_i}(t)\right) dt\ =\ \int_{-\infty}^{+\infty} F(\widetilde{G})(\tau)\ \left(F(\frac{\partial \widetilde{u}}{\partial x_i})(\tau)\ ,\ \overline{F(\frac{\partial \widetilde{u}}{\partial x_i})(\tau)}\right) d\tau.$$

Thus

$$\int_0^T \left(\int_0^t G(t-\sigma)\ C(u(\sigma),u(t))d\sigma\right) dt\ =\ \int_{-\infty}^{+\infty} F(\widetilde{G})(\tau)\ \left(\sum_{i=1}^{N} |F \frac{\partial \widetilde{u}}{\partial x_i})(\tau)|\right)^2\ d\tau$$

In order to prove that

$$\int_0^T \left(\int_0^t G(t-\sigma)\ C(u(\sigma),u(t))d\sigma\right) dt\ =\ \int_{-\infty}^{+\infty} Re\ F(\widetilde{G})(\tau)\ \left(\sum_{i=1}^{N} |F \frac{\partial \widetilde{u}}{\partial x_i})(\tau)|\right)^2\ d\tau. \tag{13}$$

we must thus demonstrate that

$$\int_{-\infty}^{+\infty} Im(\ F(\widetilde{G})(\tau))\ (\sum_{i=1}^{N}|F\frac{\partial\widetilde{u}}{\partial x_i})(\tau)|)^2\ d\tau\ =\ 0.$$

Indeed

$$\begin{array}{lcl} Im\ F(G(-\tau)) & = & -Im\ F(G(\tau)) \\ F(\frac{\partial\widetilde{u}}{\partial x_i}(-\tau)) & = & \bar{F}(\frac{\partial\widetilde{u}}{\partial x_i}(\tau)) \qquad i=1,2,...,N \end{array}$$

Thus we have

$$\int_{-\infty}^{+\infty} Im(\ F(\widetilde{G})(\tau))\ (\sum_{i=1}^{N}|F\frac{\partial\widetilde{u}}{\partial x_i})(\tau)|)^2\ d\tau\ =$$
$$-\int_{-\infty}^{+\infty} Im(\ F(\widetilde{G})(\tau))\ (\sum_{i=1}^{N}|F\frac{\partial\widetilde{u}}{\partial x_i})(\tau)|)^2\ d\tau$$

then

$$\int_{-\infty}^{+\infty} Im(\ F(\widetilde{G})(\tau))\ (\sum_{i=1}^{N}|F\frac{\partial\widetilde{u}}{\partial x_i})(\tau)|)^2\ d\tau\ =\ 0.$$

Since $ReF\widetilde{G}(\tau)>0$, we deduce from (13) that

$$\begin{array}{lcl} L^2(0,T;V) & \to & \mathbb{R} \\ u & \mapsto & \int_0^t\ (\int_0^s G(s-\sigma)C(u(\sigma),u(s))d\sigma)\ ds \end{array}$$

is positive.

■

Proof of theorem 4.2

Let u_1 and u_2 be two solutions of (3.5)-(3.6), and consider $W=u_1-u_2$.
In (3.5), let us take u = v = w and integrate in t. We deduce that

$$\begin{array}{ll} & |w(t)|^2+A(w,w)+B(w,w)+2\alpha\int_0^t\|w(s)\|^2ds \\ + & 2\beta\int_0^t\ (\int_0^s G(s,\sigma)C(w(\sigma),w(s))d\sigma)\ ds \\ = & 0 \end{array}$$

To have the uniqueness, it is sufficient to prove that

$$\int_0^t\ (\int_0^s G(s,\sigma)C(w(\sigma),w(s))d\sigma)\ ds\ \geq\ 0.$$

which is established in Lemma (4.1)
To motivate the hypothesis $ReF(\widetilde{G}(t))\ >\ 0$, one can see (Maccamy and J.S.W. Wong [**?**]).

5 Regularity of solutions

Theorem 5.1 *Under the assumptions of theorem 4.1, if moreover we suppose that the following hold*

1. *The function G is derivable and its derivative is bounded.*
2. $f' \in L^2(0,T;V')$
3. $u_0 \in V$
4. $f(0) \in H$

then the solution obtained in theorem 4.1 verify

$$u' \in L^\infty(0,T,H) \cap L^2(0,T,V).$$

Proof
For each i, let us derive (2) in t and multiply by $g'^i_m(t)$, the summation gives

$$\begin{array}{rl} & \frac{1}{2}\frac{d}{dt}|u'_m(t)| + \alpha\|u'_m(t)\|^2 + a(u'_m(t),u'_m(t)) + b(u'_m(t),u'_m(t)) + \alpha\|u'_m(t)\|^2 \\ = & (f'(t),u'_m(t)) - \int_0^t G'(t-s)C(u_m(s),u_m(t))ds - G(0)C(u_m(t),u'_m(t)) \end{array} \quad (14)$$

Since $a(u'_m(t),u'_m(t)) \geq 0$ and $b(u'_m(t),u'_m(t)) \geq 0$, we have

$$\begin{array}{rl} & \frac{1}{2}\frac{d}{dt}|u'_m(t)| + \alpha\|u'_m(t)\|^2 \\ \leq & (f'(t),u'_m(t)) - \int_0^t G'(t-s)C(u_m(s),u'_m(t))ds - G(0)C(u_m(t),u'_m(t)). \end{array} \quad (15)$$

We will estimate each term of the second member of (15) by using Young inequality

$$\begin{array}{rl} & \int_0^t G'(t-s)C(u_m(s),u'_m(t))ds \\ \leq & C\|u'_m(t)\|\int_0^t \|u_m(s)\|ds - G(0)C(u_m(t),u'_m(t)) \\ \leq & \frac{C}{2}\gamma\|u'_m(t)\|^2 + \frac{C}{2\gamma}(\|\int_0^t \|u_m(s)\|ds)^2 \end{array} \quad (16)$$

We deduce from (10) and (16) that

$$\int_0^t G'(t-s)C(u_m(s),u'_m(t))ds \leq \frac{C}{2}\gamma\|u'_m(t)\|^2 + C_1$$

where C_1 is a strictly positive constant.
Moreover, if k_1 and k_2 are two positif constants

$$G(0)C(u_m(t),u'_m(t)) \leq \frac{G(0)}{2}k_1\|u'_m(t)\|^2 + \frac{G(0)}{2k_1}\|u_m(t)\|^2 \quad (17)$$

$$|(f'(t), u'_m(t))| \leq \frac{2}{k_2}\|f'(t)\|_*^2 + \frac{k_2}{2}\|u'_m(t)\|^2 \tag{18}$$

By integration of (15) and using (16), (17) and (18), we obtain

$$\begin{array}{ll} & |u'_m(t)|^2 + 2\alpha \int_0^t \|u'_m(s)\|^2 ds \\ < & C\gamma \int_0^t \|u'_m(s)\|^2 ds + G(0)k_1 \int_0^t \|u'_m(s)\|^2 ds + k_2 \int_0^t \|u'_m(s)\|^2 ds + |u'_m(0)| + C \end{array}$$

We take $\gamma = \frac{\alpha}{3C}$, $k_1 = \frac{\alpha}{3G(0)}$ and $k_2 = \frac{\alpha}{3}$, hence

$$|u'_m(t)|^2 + \alpha \int_0^t \|u'_m(s)\|^2 ds \ < \ \lambda + |u'_m(0)| \tag{19}$$

where λ is a constant depending only on T, G, G' and α.

For t = 0, it follows from [2] that

$$(u'_m(0), w_i) + a(u_m(0), w_i) + b(u_m(0), w_i) + \alpha c(u_m(0), w_i) + \ = \ (f(0), w_i) \tag{20}$$

Since $u_{0m} \to u_0$ in V, we deduce from (20) that $u'_m(0)$ remains bounded in H. Thus, it results from (19) and (20)

$$(u'_m)_m \textit{ is bounded in } L^\infty(0,T;H)$$

$$(u'_m)_m \textit{ is bounded in } L^2(0,T;V).$$

Hence

$$u' \in L^\infty(0,T;H) \cap L^2(0,T;V).$$

■

Theorem 5.2 *Under the assumptions of theorem 5.1, if moreover we suppose that $f \in L^2(0,T;H)$, then the solution obtained in theorem 4.1 verifies*

$$u \ \in \ L^2(0,T;H^2 \cap V).$$

Moreover, we have

$$u \ \in \ C(0,T;V).$$

Proof

In fact, we proved the existence of a solution u and p of the problem

$$\left\{ \begin{array}{l} u' + \sum_{i=1}^{N} A_i \partial_i u + Bu - \Delta_\alpha u - \int_0^t G(t-s)\Delta_\beta u(s) ds \ = \ f - gradp \\ div\ u = 0 \\ u(x,t) = 0 \ sur\ \Gamma \\ u(0) = 0 \end{array} \right. \tag{21}$$

We apply to this system the orthogonal projection P of $L^2(\Omega)$ to H and using the characterization of $H^{\perp}$, the system becomes

$$-\alpha P\Delta u - \int_0^t \beta G(t-s)P\Delta u(s)ds \;=\; P(g(t)) \tag{22}$$

where

$$g(t) = f(t) - \sum_{i=1}^{N} A_i\partial_i u - Bu - u' \tag{23}$$

Let us pose $V \;=\; -P\Delta u$, (22) becomes

$$-\alpha V + \int_0^t (\beta G(t-s))V(s)ds = P(g(t)) \tag{24}$$

(26) is an Volterra equation of second species ($\alpha > 0$) whose solution is given (see Yoshida [16]) by

$$V_\alpha(t) = \sum_{n\geq 0} V_{n,\alpha}(t)$$

where

$$V_0 = \frac{P(g(t))}{\alpha} \;\; and \;\; V_{n+1,\alpha}(t) = \int_0^t \frac{\beta G(t-s)}{\alpha} V_{n,\alpha}(s)ds.$$

Since $V_0 \in L^2(0,T;H)$, $(u'\ and\ f\ \in L^2(0,T;H))$ and that G is bounded; we deduce that V_α belongs to $L^2(0,T;H)$ and Δu belongs to $L^2(0,T;H)$, therefore according to the theorem of regularity of Nirenberg, $u \;\in\; L^2(0,T,H^2 \cap V)$. Besides, the theorem of the intermediate derivatives (see Lions-Magenes [8]) gives $u \;\in\; C(0,T;V)$.

■

The following proposition is a complementary result of regularity

Proposition 5.1 *Under the hypothesis of theorem 5.2, if moreover we suppose that* $u_0 \in H^2 \cap V$ *and* $f \in L^2(0,T;V)$ *then the gotten solution verifies*

$$u \;\in\; L^2(0,T;H^3 \cap V).$$

Proof

We take the same demonstration that the one of the theorem 5.2 with this time the projection operator $P : H^1(\Omega) \to H^1(\Omega) \cap H$, we deduce that $\Delta u \in L^2(0,T;V)$ and therefore $u \;\in\; L^2(0,T;H^3 \cap V)$.

■

The regularity of the solution passes by the regularity of the initial condition, the function f and the Kernel G. More precisely, we have the following theorem

Theorem 5.3 *We suppose for $m > 2$ that*

1. $u_0 \in H^m \cap V$
2. $f \in L^2(0,T;H^{m-1} \cap V)$
3. $\frac{\partial^l f}{\partial tl} \in L^2(0,T;H)$ *if* $m = 2l+1$
4. $\frac{\partial^l f}{\partial tl} \in L^2(0,T;V')$ *if* $m = 2l$
5. $\frac{\partial G^k}{\partial tk}$ *is bounded,* $0 \le k \le l-2$ *and* $ReF\widetilde{G}(\tau) \ge 0$

The necessary and sufficient condition so that

$$\begin{array}{lll} \frac{\partial^j u}{\partial tj}, \in L^2(0,T;H^{m+1-2j} \cap V) & 0 \le j \le l \\ \frac{\partial^{l+1} u}{\partial tl+1}, \in L^2(0,T;V') & if \;\; m = 2l \\ \frac{\partial^{l+1} u}{\partial tl+1}, \in L^2(0,T;H) & if \;\; m = 2l+1 \end{array}$$

is that

$$\begin{array}{l} \frac{\partial^j u}{\partial tj}(0) \in V, \quad 0 \le j \le l-1 \\ \frac{\partial^l u}{\partial tl}(0) \in V, \quad if \;\; m = 2l+1 \\ \frac{\partial^l u}{\partial tl}(0) \in H, \quad if \;\; m = 2l \end{array} \tag{25}$$

The expression of the $\frac{\partial^j u}{\partial tj}(0)$ *according to the initial data is given by*

$$u'(0) + Au(0) = f(0)$$

and

$$\begin{array}{rcl} \frac{\partial^j u}{\partial tj}(0) & = & (-A)^j u_0 \; + \; \{ \sum_{i=0}^{j-1} (\frac{d}{dt})^i (-A)^{j-i-1} f \\ & & + \; \sum_{r=1}^{j-1} (-1)^{j-1-r} \sum_{i=0}^{j-1-r} \frac{\partial^{r-1}}{\partial t^{r-1}} [(A)^{j-1-i-r} G(t) \Delta (A)^i u(0)] \; \}. \end{array} \tag{26}$$

For $j = 2, ..., l$

where $Au \; = \; P(\sum_{i=1}^{N} A_i \partial_i u + Bu - \Delta_\alpha u - \int_0^t G(t-s) \Delta_\beta u(s) ds \;).$

The equation (26) is established in appendix at the end of this work.]

Proof

The idea consists in deriving (21) relatively to the time as Bardos [3] and R.Temam. [14]

Necessary condition

Suppose that $\frac{\partial^j u}{\partial tj} \in L^2(0,T;H^{m+1-2j}\cap V)$, $(0 \leq j \leq l)$. From the theorem of the traces (theorem 3.1 [8]), we deduce that $\frac{\partial^j u}{\partial tj} \in C(0,T;H^{m-2j}\cap V)$, $(0 \leq j \leq l-1)$ and therefore $u^{(j)}(0) \in H^{m-2j}\cap V$, hence $u^{(j)}(0) \in V$ for $0 \leq j \leq l-1$.

If $m = 2l$, $\frac{\partial^{(l+1)} u}{\partial t^{(l+1)}} \in L^2(0,T;V')$ and $\frac{\partial^l u}{\partial t^l} \in L^2(0,T;V)$. then $\frac{\partial^l u}{\partial t^l} \in C(0,T;H)$ and therefore $\frac{\partial^l u}{\partial t^l}(0) \in H$.

If $m = 2l+1$, $\frac{\partial^l u}{\partial t^l} \in L^2(0,T;H^2\cap V)$ and $\frac{\partial^{(l+1)} u}{\partial t^{(l+1)}} \in L^2(0,T;H)$ then $u \in C(0,T;H)$ and therefore $u_0 \in V$.

Sufficient condition

First case: m = 2l

We differentiate l-time the equation (21), using $\frac{\partial}{\partial t}G(t-s) = -\frac{\partial}{\partial t}G(t-s)$, we get after an integration by parts, the following problem with initial condition

$$\begin{aligned} & u^{(l+1)}(t) + \sum_{i=1}^{N} A_i \partial_i u^{(l)}(t) + Bu^{(l)}(t) + \Delta_\alpha u^{(l)}(t) - \textstyle\int_0^t G(t-s)\Delta_\beta u^{(l)}(s)ds \\ = \quad & f^{(l)(t)} - \sum_{k=0}^{l-1} G^{l-1-k}(t)\Delta_\alpha u(0) - gradp^{(l)}(t) \end{aligned} \tag{27}$$

$u^l(0)$ is given by (26).

Let's apply P to (26) and (27). Since $f^{(l)} \in L^2(0,T;V')$ and $u^{(l)}(0) \in H$ we deduce that $u^{(l)} \in L^\infty(0,T;H)\cap L^2(0,T;V)$ and $u^{(l+1)} \in L^2(0,T;V')$ (theorem 4.1).

We differentiate (21) (l - 1)times. Since $f^{(l-1)} \in L^2(0,T;V)$, $u^{(l-1)}(0) \in V$ and $u^{(l)} \in L^2(0,T;V)$, we deduce that $u^{(l-1)} \in L^2(0,T;H^3\cap V)$ (proposition 5.1). Let's suppose that the result is true until the order l - (j - 1), i.e., $u^{(l-j+1)} \in L^2(0,T;H^{2j-1}\cap V)$, we differentiate (21) (l-j)times. Since $f^{(l-j)} \in L^2(0,T;H^{2j-1}\cap V)$ and $u^{(l-j+1)} \in L^2(0,T;H^{2j-1}\cap V)$, we deduce that $u^{(l-j)} \in L^2(0,T;H^{2j+1}\cap V)$ and therefore $u^{(j)} \in L^2(0,T;H^{m+1-2j}\cap V)$.

Second case: m = 2l + 1

Let us derive the equation (21). We have this time an initial data $u_0^{(l)} \in V$ and $f^{(l)} \in L^2(0,T;H)$, then $u^{(l+1)} \in L^\infty(0,T;H)\cap L^2(0,T;V)$ ((theorem 5.1).

We derive again the equation (21) (l - 1)-times, according to $f^{(l-1)} \in L^2(0,T;H^2\cap V)$ and $u^{(l)} \in L^2(0,T;H^2 \cap V)$, we deduce that $u^{l-1} \in L^2(0,T;H^4 \cap V)$.
Let's suppose that the result is true until the order l - (j - 1), i.e., $u^{(l-j+1)} \in L^2(0,T;H^{2j}\cap V)$. Since $f^{(l-j)} \in L^2(0,T;H^{2j}\cap V)$ and i.e., $u^{(l-j+1)} \in L^2(0,T;H^{2j-1}\cap V)$, we deduce that $u^{(l-j)} \in L^2(0,T;H^{2+2j}\cap V)$, and therefore $u^{(j)} \in L^2(0,T;H^{m+1-2j}\cap V)$, $0 \leq j \leq l$.

■

Remark 5.1 1. *In the proof of theorem 5.3, we must look at the regularity of* $f^{(j)} - \sum_{k=0}^{j-1} G^{(j-1-k)}(t)\Delta u_0$ *and not only the one of* $f^{(j)}$*; in fact we show easily that* $\sum_{k=0}^{j-1} G^{(j-1-k)}(t)\Delta u_0$ *belongs to* $L^2(0,T;H^{2l-2j-1} \cap V)$*,* ($u_0 \in H^m \cap V$)*.*

2. *We have the right to use in the step (l - j) the regularity of* $u^{(l-j-1)}$ *because* $(u^{(l-j)})'$ *is solution of the step (l - j + 1) and therefore* $(u^{(l-j)})' = u^{(l-j+1)}$ *according to the uniqueness of the solution*

3. *The disappearance of p in the demonstration results from the application of the orthogonal projection of* L^2 *on H that is linear continuous of* H^j *in* $H^j \cap H$ *for all positive j.*

4. *For m lower or equal to 2, we don't have relation of compatibility of the kind (26) (see Proposition (5.1)).*

corollaire 5.1 *For m superior or equal to 3, the necessary and sufficient condition so that* $u \in C^j(0,T;H^{2m-2j} \cap V)$*,* $u^{(l)} \in C(0,T;H)$ *if* $m = 2l$ *and* $u \in C(0,T;V)$ *if* $m = 2l+1$ *is the one given by (27) and (26).*

Proof
It is a consequence of the theorem 5.3 by application of the theorems of the traces (theorem 3.1, page 23, Lions-Magnes [8]).

■

6 Conclusion

In this paperwork, we focused on a set of integrodifferential equations. Based on the variational formulation of the problem, it was proved that the solution

of this equation does exist thanks to the FAEDO-GALERKIN method.
Under some assumptions on the Kernel of integral operator, we have demonstrated the uniqueness of the solution and therefore results of regularity were established.
As a natural continuation of this work, we are in the process of studying the asymptotic behavior solution of the system

Appendix

To show the formula (26) giving the $\frac{\partial^j u}{\partial t}(0)$ according to the initial data. we first establish the lemma.

Lemma 6.1 *For* $j \geq 1$

$$\frac{\partial}{\partial t}(A)^j u \;=\; -(A)^{j+1}u + (A)^j f - \sum_{i=0}^{j-1}(A)^{j-i-1}G(t)\Delta_\beta((A)^j u_0)$$

where

$$Au \;=\; P(\sum_{i=1}^{N} A_i\partial_i u + Bu - \Delta_\alpha u - \int_0^t G(t-s)\Delta_\beta u(s)ds\ ).$$

Proof

We proceed by recurrence
j = 1
Since $\frac{\partial}{\partial t}G(t-s) = -\frac{\partial}{\partial t}G(t-s)$, it results from an integration by parts

$$\begin{aligned}\frac{\partial}{\partial t}(A)u &= Au' - G(t)\Delta_\beta u_0\\ &= A(f - Au) - G(t)\Delta_\beta u_0\\ &= -A^2u + Af - G(t)\Delta_\beta u_0\end{aligned}$$

Let's suppose that the formula of lemma (6.1) is true until the order j and let's calculate

$$\begin{aligned}\frac{\partial}{\partial t}(A)^{j+1}u &= \frac{\partial}{\partial t}A(A^j u))\\ &= A(\frac{\partial}{\partial t}(A)^j u) - G(t)\Delta_\beta(A^j u_0)\\ &= A(-(A)^{j+1}u + (A)^j f - \sum_{i=0}^{j-1}(A)^{j-i-1}(G(t)\Delta_\beta(A)^i u_0) - G(t)\Delta_\beta(A^j u_0)\\ &= -(A)^{j+2}u + (A)^{j+1}f - \sum_{i=0}^{j-1}(A)^{j-i}(G(t)\Delta_\beta(A)^i u_0) - G(t)\Delta_\beta(A^j u_0)\end{aligned}$$

Then

$$\frac{\partial}{\partial t}(A)^{j+1}u = -(A)^{j+2}u + (A)^j f + \sum_{i=0}^{j}(A)^{j-1}(G(t)\Delta_\beta(A)^i u_0)$$

The formula of lemma (6.1) is then true at the order j + 1.

■

Lemma 6.2 *For $j \geq 2$*

$$\begin{aligned}\frac{\partial^j u}{\partial t} &= (-A)^j u + \sum_{i=0}^{j-1} (\tfrac{d}{dt})^i (-A)^{j-i-1} f \\ &+ \sum_{r=1}^{j-1} (-1)^{j-1-r} \sum_{i=0}^{j-1-r} \tfrac{\partial^{r-1}}{\partial t^{r-1}} [(A)^{j-1-i-r} G(t) \Delta_\beta (A)^i u_0]\end{aligned}$$

Proof

We know that $\frac{\partial u}{\partial t} = -Au + f$

We proceed by recurrence

$j = 2$

$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial t}(Au) + f' = -(-A^2 u + Af - G(t)\Delta_\beta u_0) + f'$ (lemma 1).

The formula of lemma 2 is then true at the order j = 2.

Let's suppose that the formula of lemma 2 is true until the order j and let's calculate:

$$\begin{aligned}\frac{\partial^{j+1}}{\partial t^{j+1}} u &= \frac{\partial}{\partial t}[\frac{\partial^j}{\partial t^j} u] \\ &= \frac{\partial}{\partial t}(-A)^j u + \sum_{i=0}^{j-1} (\tfrac{d}{dt})^{i+1} (-A)^{j-i-1} f \\ &+ \sum_{r=1}^{j-1} (-1)^{j-1-r} \sum_{i=0}^{j-1-r} \tfrac{\partial^r}{\partial t^r} [A^{j-1-i-r} (G(t) \Delta_\beta (A^i) u_0)].\end{aligned}$$

We deduce from lemma 1 that

$$\begin{aligned}\frac{\partial^{j+1}}{\partial t^{j+1}} u &= (-1)^j [-(A)^{j+1} u + (A)^j f - \sum_{i=0}^{j-1} (-A)^{j-1-i} (G(t) \Delta_\beta (A)^i u_0)] \\ &+ \sum_{i=0}^{j-1} (\tfrac{d}{dt})^{i+1} (-A)^{j-i-1} f \\ &+ \sum_{r=1}^{j-1} (-1)^{j-1-r} \sum_{i=0}^{j-1-r} \tfrac{\partial^r}{\partial t^r} [A^{j-1-i-r} (G(t) \Delta_\beta (A^i) u_0)]\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^{j+1}}{\partial t^{j+1}}u &= (-A)^{j+1}u + \sum_{i=0}^{j}(\tfrac{d}{dt})^i(-A)^{j-i}f \\
&+ (-1)^{j+1}\sum_{i=0}^{j-1}(-A)^{j-1-i}(G(t)\Delta_\beta(A)^i u_0)] \\
&+ \sum_{r=1}^{j-1}(-1)^{j-1-r}\sum_{i=0}^{j-1-r}\tfrac{\partial^r}{\partial t^r}[A^{j-1-i-r}(G(t)\Delta_\beta(A^i)u_0)]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^{j+1}}{\partial t^{j+1}}u &= (-A)^{j+1}u + \sum_{i=0}^{j}(\tfrac{d}{dt})^i(-A)^{j-i}f \\
&+ \sum_{r=1}^{j}(-1)^{j-r}\sum_{i=0}^{j-r}\tfrac{\partial^{r-1}}{\partial t^{r-1}}[A^{j-i-r}(G(t)\Delta_\beta(A^i)u_0)]
\end{aligned}
$$

■

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