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Modified Decomposition Method for Solving the Cauchy Problem for Nonlinear Parabolic-Hyperbolic Equations

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Abstract. In this paper Modified decomposition method is applied to the solvability of nonlinear parabolic-hyperbolic equations and illustrated with a few simple examples.

Keywords: Adomian decomposition method, nonlinear parabolic-hyperbolic equations

1. Introduction

We shall consider the Cauchy problem for the nonlinear parabolic-hyperbolic equation

$$\left(\frac{\partial}{\partial t} - \triangle\right)\left(\frac{\partial^2}{\partial t^2} - \triangle\right)u = F(u), \tag{1.1}$$

where the nonlinear term is represented by $F(u)$ and $\triangle$ is the Laplace operator in $\Re^n$.
To equation (1.1) we attach the initial conditions

$$\frac{\partial^k u}{\partial t^k}(0,x) = \varphi_k(x),\ \ k = 0,1,2. \tag{1.2}$$

The results concerning existence and uniqueness of solutions for this type of equations with initial and boundary conditions such that $F(u) = f(t,x) \in L_2(Q)$, where $Q = \Omega \times [0,T],\ \Omega \subset \Re^n$ have been proved in [1].
Here, we have to mention that many mathematicians devoted many papers for Adomian decomposition method [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] to some more general types of parabolic and hyperbolic equations, and various problems. On the other hand, such type of parabolic-hyperbolic equations with initial conditions has not solved analytically by previously known results.

In this paper we shall apply modified decomposition method [2, 3] to solve the Cauchy problem for the new type of nonlinear partial differential equations (1.1)-(1.2).

2. Modified decomposition method

2.1. **The one-dimensional equation.** In the one-dimensional case, Eq.(1.1) becomes

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)u=F(u), \tag{2.1}$$

subject to the initial conditions (1.2).
The modified decomposition method [2, 3] may be used to solve the linear problem given by (2.1) subject to the initial conditions (1.2).
Defining the differential operator $L_{ttt}=\frac{\partial^3}{\partial t^3}$.
We rewrite Eq.(2.1) in the operator form

$$L_{ttt}u=\frac{\partial^3 u}{\partial t\partial x^2}+\frac{\partial^4 u}{\partial t^2\partial x^2}-\frac{\partial^4 u}{\partial x^4}+F(u). \tag{2.2}$$

Applying the inverse operator L_{ttt}^{-1}

$$L_{ttt}^{-1}=\int_0^t\int_0^t\int_0^t(.)dtdtdt, \tag{2.3}$$

to Eq.(2.2) and using the initial conditions (1.2) we obtain

$$u(t,x)=L_{ttt}^{-1}\left(\frac{\partial^3 u}{\partial t\partial x^2}+\frac{\partial^4 u}{\partial t^2\partial x^2}-\frac{\partial^4 u}{\partial x^4}\right) \tag{2.4}$$

$$+\sum_{i=0}^{2}\varphi_i(x)\frac{t^i}{i!}+L_{ttt}^{-1}F(u).$$

The unknown solution u is assumed to be given by series of the form

$$u=\sum_{n=0}^{\infty}a_n(x)t^n, \tag{2.5}$$

and write

$$F(u)=\sum_{n=0}^{\infty}A_n(x)t^n=\sum_{n=0}^{\infty}A_n(a_0,a_1,...,a_n,...)t^n,$$

where A_n are called Adomian polynomials and can be generated for all types of nonlinearities according to algorithm set by Adomian [2, 3]

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}F\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0},\ n=0,1,2,...$$

The substitution yields

(2.6)
$$\sum_{n=0}^{\infty} a_n(x)t^n = L_{ttt}^{-1}\left(\sum_{n=1}^{\infty} n a_n^{(2)}(x)t^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n^{(2)}(x)t^{n-2} - \sum_{n=0}^{\infty} a_n^{(4)}(x)t^n\right)$$

$$+\sum_{i=0}^{2}\varphi_i(x)\frac{t^i}{i!} + \; L_{ttt}^{-1}\sum_{n=0}^{\infty} A_n(a_0, a_1, ..., a_n, ...)t^n,$$

or

(2.7)
$$\sum_{n=0}^{\infty} a_n(x)t^n = L_{ttt}^{-1}\sum_{n=0}^{\infty} t^n\left((n+1)a_{n+1}^{(2)}(x) + (n+1)(n+2)a_{n+2}^{(2)}(x) - a_n^{(4)}(x)\right)$$

$$+\sum_{i=0}^{2}\varphi_i(x)\frac{t^i}{i!} + \; L_{ttt}^{-1}\sum_{n=0}^{\infty} A_n(a_0, a_1, ..., a_n, ...)t^n.$$

We now carry out the above integrations to write

(2.8)
$$\sum_{n=0}^{\infty} a_n(x)t^n = \sum_{n=0}^{\infty}\frac{t^{n+3}}{(n+1)(n+2)(n+3)}\left((n+1)a_{n+1}^{(2)} + (n+1)(n+2)a_{n+2}^{(2)} - a_n^{(4)}\right)$$

$$+\sum_{i=0}^{2}\varphi_i(x)\frac{t^i}{i!} + \sum_{n=0}^{\infty}\frac{t^{n+3}}{(n+1)(n+2)(n+3)}A_n(a_0, a_1, ..., a_n, ...).$$

In the summation on the right, n can be replaced by $n-3$ to write

(2.9)
$$\sum_{n=0}^{\infty} a_n(x)t^n = \sum_{n=3}^{\infty}\frac{t^n}{(n-2)(n-1)n}\left((n-2)a_{n-2}^{(2)}(x) + (n-2)(n-1)a_{n-1}^{(2)} - a_{n-3}^{(4)}\right).$$

$$+\sum_{i=0}^{2}\varphi_i(x)\frac{t^i}{i!} + \; \sum_{n=3}^{\infty}\frac{t^n}{n(n-1)(n-2)}A_{n-3}(a_0, a_1, ..., a_n, ...).$$

Finally, we can equate coefficients of like powers of t on the left side and on the right side to obtain the recurrence relations for the coefficients. Thus

$$\begin{cases} a_0(x) &= \varphi_0(x), \\ a_1(x) &= \varphi_1(x), \\ a_2(x) &= \dfrac{\varphi_2(x)}{2!}, \\ a_n(x) &= \dfrac{a^{(2)}_{n-2}(x)}{n(n-1)} + \dfrac{a^{(2)}_{n-1}(x)}{n} - \dfrac{a^{(4)}_{n-3}(x)}{n(n-1)(n-2)} + \dfrac{A_{n-3}(a_0, a_1, ...)}{n(n-1)(n-2)}, \ n \geqslant 3. \end{cases}$$

The final solution is now given by $u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n$.
This method is illustrated in the following examples.

Example 1. *Consider problem (2.1)-(1.2) with*

$$\varphi_0(x) = -x^4, \ \varphi_1(x) = 0, \ \varphi_2(x) = 0,$$

and

$$F(u) = 2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 144u.$$

Now we solve this problem by modified decomposition method. Direct computation of Adomian's polynomials for the nonlinear term $F(u) = 2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 144u$ *are given as follows:*

$$A_0 = 4a_1a_2 - (a_0^{(2)})^2 - 144a_0,$$

$$A_1 = 12a_1a_3 + 8a_2^{(2)} - 2a_0^{(2)}a_1^{(2)} - 144a_0 - 144a_1,$$

$$...$$

$$A_n = \sum_{k=0}^{n} \left(2(k+1)(n-k+1)(n-k+2)a_{k+1}a_{n-k+2} - a_k^{(2)}a_{n-k}^{(2)} - 144a_n\right).$$

We find

$$a_0(x) = -x^4, \ a_1(x) = 0, \ a_2(x) = 0, \ a_3(x) = 4, \ a_n(x) = 0, \ n \geqslant 4,$$

and the above recurrence relations for the coefficients give with only two terms

$$u(t,x) = a_0(x) + a_3(x)t^3 = -x^4 + 4t^3,$$

the exact solution of this problem.

Example 2. *Consider problem (2.1)-(1.2) with*

$$\varphi_k(x) = e^x, \ \ k = 0, 1, 2$$

and

$$F(u) = \left(\frac{\partial^2 u}{\partial t^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 2u^2,$$

which has the exact solution $u(t, x) = e^{x+t}$. *The Adomian's polynomials for* $F(u) = \left(\frac{\partial^2 u}{\partial t^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 2u^2$ *are*

$$A_0 = 4a_2^2 + (a_0^{(2)})^2 - 2a_0^2,$$

$$...$$

$$A_n = \sum_{k=0}^{n} \left((k+1)(k+2)(n-k+1)(n-k+2)a_{k+2}a_{n-k+2} + a_k^{(2)} a_{n-k}^{(2)} - 2a_k a_{n-k}\right).$$

Then by using the above recurrence relations for the coefficients and direct calculation produce

$$a_n(x) = \frac{e^x}{n!}, \ \ n \geqslant 0,$$

and

$$u(t, x) = \sum_{n=0}^{\infty} a_n(x) t^n = \sum_{n=0}^{\infty} \frac{e^x}{n!} t^n,$$

which is the partial sum of the Taylor series of the exact solution.

Example 3. *Consider problem (2.1)-(1.2) with*

$$\varphi_0(x) = \cos x, \ \ \varphi_1(x) = -\sin x, \ \ \varphi_2(x) = -\cos x,$$

and

$$F(u) = u\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t^2}.$$

The Adomian polynomials can be determined for $F(u) = u\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t^2}$ *as*

$$A_0 = a_0 a_1 + 2a_2 a_0^{(1)},$$

$$...$$

$$A_n = \sum_{k=0}^{n} \left((n-k+1)a_k a_{n-k+1} + (k+1)(k+2)a_{k+2}a_{n-k}^{(1)}\right).$$

By Modified decomposition method we can find

$$a_0(x) = \cos x, \ \ a_1(x) = -\sin x, \ \ a_2(x) = -\frac{\cos x}{2!}, a_3(x) = -\frac{\sin x}{5!}, ...,$$

and

$$u(t,x)=\sum_{n=0}^{\infty}a_n(x)t^n=\cos x\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}-\sin x\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!}=\cos(x+t),$$

which can be verified through substitution to be the exact solution of this problem.

2.2. **The two-dimensional equation.** In the two-dimensional, Eq.(1.1) can be written as follows

$$\frac{\partial^3 u(t,x,y)}{\partial t^3}-\frac{\partial}{\partial t}\triangle u(t,x,y)-\frac{\partial^2}{\partial t^2}\triangle u(t,x,y)+\triangle^2 u(t,x,y)=F(u), \tag{2.10}$$

where $\triangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, with the following initial conditions

$$\frac{\partial^k u}{\partial t^k}(0,x,y)=\varphi_k(x,y),\ \ k=0,1,2. \tag{2.11}$$

We now assume

$$u=\sum_{n=0}^{\infty}a_n(x,y)t^n. \tag{2.12}$$

In a similar way, we can obtain the recurrence relations for the coefficients.

$$\begin{cases} a_0(x,y) &=& \varphi_0(x,y),\\ a_1(x,y) &=& \varphi_1(x,y),\\ a_2(x,y) &=& \dfrac{\varphi_2(x,y)}{2!},\\ a_n(x,y) &=& \dfrac{S_n\left(a_{n-2},a_{n-1},a_{n-3}\right)}{n(n-1)(n-2)}+\dfrac{A_{n-3}}{n(n-1)(n-2)},\ \ n\geqslant 3,\end{cases}$$

where

$$S_n=(n-2)(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})a_{n-2}+(n-1)(n-2)(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})a_{n-1}-(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})^2a_{n-3}.$$

The final solution is now given by $u(t,x,y)=\sum_{n=0}^{\infty}a_n(x,y)t^n$.

Example 4. *Consider equation (2.10) with*

$$\varphi_0(x,y)=\sinh(x+y),\ \ \varphi_1(x,y)=2\sinh(x+y),\ \ \varphi_2(x)=4\sinh(x+y),$$

and

$$F(u)=F(u)=\left(\frac{\partial u}{\partial t}\right)^2-4u^2.$$

The Adomian polynomials can be determined for $F(u) = \left(\frac{\partial u}{\partial t}\right)^2 - 4u^2$ *as*

$$A_0 = a_1^2 - 4a_0^2,$$

$$\dots$$

$$A_n = \sum_{k=0}^{n} \left((k+1)(n-k+1)a_{k+1}a_{n-k+1} - 4a_k a_{n-k}\right).$$

Using the above recurrence relations of modified decomposition method and straightforward computation yields

$$a_0(x,y) = \sinh(x+y),\ a_1(x,y) = 2,\ a_2(x,y) = 2\sinh(x+y),$$

$$a_3(x,y) = \frac{4}{3}\sinh(x+y),\ a_4(x,y) = \frac{2}{3}\sinh(x+y),$$

$$\dots$$

$$a_n(x,y) = \frac{2^n}{n!}\sinh(x+y),$$

and

$$u(t,x,y) = \sum_{n=0}^{\infty} a_n(x,y)t^n = \sinh(x+y)\sum_{n=0}^{\infty}\frac{2^n t^n}{n!} = \sinh(x+y)e^{2t},$$

which can be verified through substitution to be exact solution of the given problem.

3. Conclusion

The modified decomposition method has been proved to be reliable in handing the initial value problems for nonlinear parabolic-hyperbolic equations. Some examples with closed form solutions are studied carefully, and the results obtained are just the same as those given from applying the modified decomposition.

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