

# Parabolic Equations with Nonlocal Conditions

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**Abstract.** In this paper, we study a boundary value problem for a class of parabolic equations with nonlocal conditions. We prove the existence and uniqueness of the solution, using a *priori estimate* and Fourier's method.

**Keywords:** Parabolic equation, Nonlocal condition, a Priori estimate

## 1. INTRODUCTION

Parabolic equations with integral conditions have been investigated in [1, 2, 3, 4, 5, 6, 7, 8, 9]. Such problems constitute a very interesting and important class of problems. These integrals may appear in boundary conditions where the boundary conditions are called nonlocal conditions or integral conditions. In the present paper, a class of parabolic equations with nonlocal conditions is considered.

In the rectangular domain  $\Omega = (0, T) \times (0, 1)$ , consider the linear parabolic equation

$$(1.1) \quad \frac{\partial u}{\partial t} - \frac{1}{x} \frac{\partial}{\partial x} \left( x^2 \frac{\partial u}{\partial x} \right) + ku = F(t, x), \quad k \geq 0.$$

To equation (1.1) we attach the initial condition

$$(1.2) \quad u(0, x) = \varphi(x),$$

and the nonlocal boundary conditions

$$(1.3) \quad \int_0^1 u(t, x) dx = 0,$$

$$(1.4) \quad \int_0^1 xu(t, x) dx = 0,$$

where  $\varphi(x) \in L_2(0, 1)$  is a known function and satisfies the compatibility conditions

$$\int_0^1 \varphi(x)dx = \int_0^1 x\varphi(x)dx = 0.$$

The general difficult which arises to us is the presence of integral conditions which complicates the application of standard methods. It may, however, be worth while if this type of problems can be transformed into another equivalent problem which involves no integral conditions. For this, we convert (1.1)-(1.4) to the following classical problem.

**Lemma 1.** *Problem (1.1)-(1.4) is equivalent to the following problem*

$$(\text{Pr})_1 \left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \frac{1}{x} \frac{\partial}{\partial x} \left( x^2 \frac{\partial u}{\partial x} \right) + ku = F(t, x), \\ u(0, x) = \varphi(x), \\ u(t, 1) - u(t, 0) = - \int_0^1 F(t, x)dx + \int_0^1 xF(t, x)dx, \\ u_x(t, 1) = - \int_0^1 xF(t, x)dx. \end{array} \right.$$

*Proof.* Let  $u(t, x)$  be a solution of (1.1)-(1.4). Integrating Eq.(1.1) with respect to  $x$  over  $(0, 1)$ , and taking in account of (1.3), we obtain

$$u_x(t, 1) + (u(t, 1) - u(t, 0)) = - \int_0^1 F(t, x)dx.$$

To eliminate the second nonlocal condition  $\int_0^1 xu(t, x)dx = 0$ , multiplying both sides of (1.1) by  $x$  and integrating the resulting over  $(0, 1)$ , and taking in account of (1.4), we obtain

$$u_x(t, 1) = - \int_0^1 xF(t, x)dx.$$

These may also be written

$$u(t, 1) - u(t, 0) = - \int_0^1 F(t, x)dx + \int_0^1 xF(t, x)dx,$$

and

$$u_x(t, 1) = - \int_0^1 xF(t, x)dx.$$

Let now  $u(t, x)$  be a solution of  $(\text{Pr})_1$ , it remains to prove that

$$\int_0^1 u(t, x)dx = 0,$$

and

$$\int_0^1 xu(t, x)dx = 0.$$

We integrate Eq.(1.1) with respect to  $x$ , we obtain

$$\frac{d}{dt} \int_0^1 u(t, x)dx + k \int_0^1 u(t, x)dx = 0, \quad t \in (0, T),$$

and it also follows that

$$\frac{d}{dt} \int_0^1 xu(t, x)dx + k \int_0^1 xu(t, x)dx = 0, \quad t \in (0, T).$$

By virtue of the compatibility conditions, we get

$$\int_0^1 u(t, x)dx = 0, \quad \int_0^1 xu(t, x)dx = 0.$$

□

Introduce now the new unknown function  $v = u - \alpha(x) \int_0^1 xF(t, x)dx - \beta(x) \int_0^1 F(t, x)dx$ , where  $\alpha(x) = -2x^2 + 3x$  and  $\beta(x) = x^2 - 2x$ . Then  $(Pr)_1$  is transformed into the following problem

$$(Pr)_2 \quad \begin{cases} \ell v \equiv \frac{\partial v}{\partial t} - \frac{1}{x} \frac{\partial}{\partial x} \left( x^2 \frac{\partial v}{\partial x} \right) + kv = f(t, x), \\ v(0, x) = \phi(x), \\ v(t, 1) = v(t, 0), \\ v_x(t, 1) = 0, \end{cases}$$

where

$$f(t, x) = F(t, x) - \alpha(x) \int_0^1 xF_t(t, x)dx - \beta(x) \int_0^1 F_t(t, x)dx + \gamma(t, x),$$

$$\gamma(t, x) = (12x - 6) \int_0^1 xF_t(t, x)dx + (4 - 6x) \int_0^1 F(t, x)dx -$$

$$k\alpha(x) \int_0^1 xF(t, x)dx - k\beta(x) \int_0^1 F(t, x)dx,$$

and

$$\phi(x) = \varphi(x) - \alpha(x) \int_0^1 xF(0, x)dx - \beta(x) \int_0^1 F(0, x)dx.$$

## 2. A PRIORI ESTIMATE

We introduce the function space

$$E = \left\{ v : v, x^{\frac{1}{2}}v, x \frac{\partial v}{\partial x} \in L_2(0, 1) \text{ and } v, x^{\frac{1}{2}} \frac{\partial v}{\partial x}, x^{\frac{1}{2}} \frac{\partial v}{\partial t}, x \frac{\partial v}{\partial x} \frac{\partial}{\partial x} \left( x^2 \frac{\partial v}{\partial x} \right) \in L_2(\Omega^\tau) \right\},$$

with respect to the norm

$$\|v\|_E^2 = \sup_{\tau \in (0, T)} \int_0^1 \left[ (1+x)v^2(\tau, x) + x^2 \left( \frac{\partial v}{\partial x} \right)^2(\tau, x) \right] dx + \|v\|_2^2,$$

where

$$\|v\|_2^2 = \int_{\Omega} \left[ v^2 + x \left( \frac{\partial v}{\partial x} \right)^2 + x \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial}{\partial x} \left( x^2 \frac{\partial v}{\partial x} \right) \right)^2 \right] (t, x) dt dx,$$

$$\|f\|_{L_2}^2 = \int_{\Omega} f^2(t, x) dt dx,$$

and  $\Omega^\tau = (0, \tau) \times (0, 1)$ . Notice  $E$  is a Hilbert space.

Here we establish an energy inequality which ensures the uniqueness of the solution  $v \in E$  of (Pr)<sub>2</sub>.

We need the following Lemma

**Lemma 2.** *For problem (Pr)<sub>2</sub>, we have*

$$\int_{\Omega^\tau} \left[ \frac{\partial}{\partial x} \left( x^2 \frac{\partial v}{\partial x} \right) \right]^2 dt dx \leq c_1 \int_{\Omega^\tau} \left[ x^2 f^2 + x^2 v^2 + x^2 \left( \frac{\partial v}{\partial t} \right)^2 \right] (t, x) dt dx,$$

where  $c_1 = 3 \max(1, k^2) > 0$ .

*Proof.* We see that

$$\frac{\partial}{\partial x} \left( x^2 \frac{\partial v}{\partial x} \right) = xkv + x \frac{\partial v}{\partial t} - xf(t, x),$$

by integration and using the fact that  $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ , we obtain the desired inequality.  $\square$

**Theorem 1.** *For problem (Pr)<sub>2</sub>, we have*

$$(2.1) \quad \|v\|_E \leq c \left[ \|f\|_{L_2(\Omega)} + \|\phi\|_{L_2(\Omega)} + \|\phi\|_{H^1(0,1)} \right],$$

where  $c > 0$  is independent on  $v$ .

*Proof.* Firstly, consider the scalar product  $(\ell v, v)_{L_2(\Omega^\tau)}$ , and employing integration by parts, and taking account of initial and boundary conditions of (Pr)<sub>2</sub>, we obtain

$$(2.2) \quad (\ell v, v)_{L_2(\Omega^\tau)} = \frac{1}{2} \int_0^1 v^2 + \int_{\Omega^\tau} x \left( \frac{\partial v}{\partial x} \right)^2 + k \int_{\Omega^\tau} v^2 - \frac{1}{2} \int_0^1 \phi^2,$$

similarly, for the scalar product  $(\ell v, x \frac{\partial v}{\partial t})_{L_2(\Omega^\tau)}$ , we obtain,

$$(2.3) \quad \left( \ell v, x \frac{\partial v}{\partial t} \right)_{L_2(\Omega^\tau)} = \int_{\Omega^\tau} x \left( \frac{\partial v}{\partial t} \right)^2 + \frac{1}{2} \int_0^1 x^2 \left( \frac{\partial v}{\partial x} \right)^2 + \frac{k}{2} \int_0^1 x v^2 - \frac{k}{2} \int_0^1 x \phi^2 - \frac{1}{2} \int_0^1 x^2 \left( \frac{d\phi}{dx} \right)^2.$$

Adding (2.2) and (2.3) we obtain

$$(2.4) \quad LSH \equiv \int_0^1 \left[ (1 + kx)v^2 + x^2 \left( \frac{\partial v}{\partial x} \right)^2 \right] (\tau, x) dx + 2 \int_{\Omega^\tau} \left[ kv^2 + x \left( \frac{\partial v}{\partial t} \right)^2 + x \left( \frac{\partial v}{\partial x} \right)^2 \right] (t, x) dt dx,$$

and

$$(2.5) \quad RSH \equiv 2 \left( \ell v, v + x \frac{\partial v}{\partial t} \right)_{L_2(\Omega^\tau)} + \int_0^1 x \phi^2 + k \int_0^1 x \phi^2 + \int_0^1 x^2 \left( \frac{d\phi}{dx} \right)^2.$$

We now apply the  $\varepsilon$ - inequality  $2 | ab | \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ ,  $\varepsilon > 0$  to the first term of (2.5) to obtain

$$(2.6) \quad 2 \left( \ell v, v + x \frac{\partial v}{\partial t} \right)_{L_2(\Omega^\tau)} \leq \frac{1}{\varepsilon_1} \int_{\Omega^\tau} f^2 + \frac{1}{\varepsilon_2} \int_{\Omega^\tau} x f^2 + \varepsilon_1 \int_{\Omega^\tau} v^2 + \varepsilon_2 \int_{\Omega^\tau} x \left( \frac{\partial v}{\partial t} \right)^2,$$

where  $\ell v = f$  and  $\varepsilon_i > 0$ ,  $i = 1, 2$

Since  $x \leq 1$  and  $\Omega^\tau \subset \Omega$ . Next choosing  $\varepsilon_i, i = 1, 2$  as  $2k - \varepsilon_1 = k_1 > 0$  and  $2 - \varepsilon_1 = k_2 > 0$ . Thus we have from (2.4), (2.5), (2.6) and the inequality of lemma 2

$$(2.7) \quad \int_0^1 \left[ (1 + x)v^2 + x^2 \left( \frac{\partial v}{\partial x} \right)^2 \right] (\tau, x) dx + \int_{\Omega^\tau} \left[ v^2 + x \left( \frac{\partial v}{\partial t} \right)^2 + x \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial}{\partial x} \left( x^2 \frac{\partial v}{\partial x} \right) \right)^2 \right] dt dx \leq c_2 \left[ \| f \|_{L_2(\Omega)}^2 + \| \phi \|_{L_2(\Omega)}^2 + \| \phi \|_{H^1(0,1)}^2 \right],$$

where  $\| \phi \|_{H^1(0,1)}^2 = \int_0^1 (\phi^2 + (\frac{d\phi}{dx})^2) dx$  and  $c_2$  is dependent on  $\varepsilon_i, i = 1, 2$  and  $k$ .

Now, as the right-hand side of (2.7) is independent of  $\tau$ , replacing the left-hand side by its upper bound with respect to  $\tau$  in the interval  $(0, T)$ , we obtain the desired inequality which  $c = c_2^{\frac{1}{2}}$ .

This completes the proof. □

## 3. EXISTENCE AND UNIQUENESS OF SOLUTION

WE shall now establish the existence of solution of  $(Pr)_2$ . For this we make use of the Fourier's method.

Consider the function  $v_n(t, x) = U_n(t)V_n(x)$  where  $V_n(x)$  is a eigenfunction of the BVP

$$\begin{cases} \frac{1}{x} \frac{d}{dx} \left[ x^2 \left( \frac{dV_n(x)}{dx} \right) \right] - kV_n(x) = \lambda_n V_n(x), \\ V_n(0) = V_n(1), \\ \frac{dV_n(1)}{dx} = 0, \end{cases}$$

$\lambda_n$ ,  $n = 1, 2, \dots$  is called the eigenvalue corresponding to the eigenfunction  $V_n(x)$ , and  $U_n(t)$  is satisfying the initial problem

$$\begin{cases} \frac{dU_n}{dt} - \lambda_n U_n(t) = f_n(t), \\ U_n(0) = \phi_n, \end{cases}$$

where the functions  $\phi(x)$ ,  $\phi'(x)$ , and  $f(t, x)$  are expanded in Fourier series in terms of the system  $V_1, V_2, \dots$  of eigenfunctions

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n V_n(x),$$

$$\phi'(x) = \sum_{n=1}^{\infty} \phi_n^* V_n(x),$$

$$f(t, x) = \sum_{n=1}^{\infty} f_n(t) V_n(x),$$

and by the Parseval-Steklov equality

$$\|\phi\|_{L_2(0,1)}^2 = \sum_{n=1}^{\infty} \phi_n^2,$$

$$\|\phi'\|_{L_2(0,1)}^2 = \sum_{n=1}^{\infty} (\phi_n^*)^2,$$

and

$$\int_0^1 f(t, x) dx = \sum_{n=1}^{\infty} f_n(t).$$

Hence

$$\sum_{n=1}^{\infty} \int_0^T f_n^2(t) dt = \int_{\Omega} f^2(t, x) dt dx.$$

Then direct computation yields

$$U_n(t) = \phi_n \exp(\lambda_n t) + \int_0^t f_n(\tau) \exp[\lambda_n(t - \tau)] d\tau,$$

$$\int_0^1 x V_n(x) V_m(x) dx = 0, \quad n \neq m,$$

and

$$\int_0^1 x \phi(x) V_n(x) dx = \phi_n \int_0^1 x V_n^2(x) dx.$$

By principle of superposition, the solution of (Pr)<sub>2</sub> is given by the series

$$(3.1) \quad v(t, x) = \sum_{n=1}^{\infty} U_n(t) V_n(x).$$

Then we have

**Theorem 2.** *Let  $f \in L_2(\Omega)$ , and  $\phi \in H^1(0, 1)$ . Then the solution  $v(t, x)$  of (Pr)<sub>2</sub> exists and its represented by series (3.1) which converges in  $E$ .*

*Proof.* Consider the partial sum  $S_N(t, x) = \sum_{n=1}^N U_n(t) V_n(x)$  of the series (3.1). Then by theorem 1

$$(3.2) \quad \left\| \sum_{n=1}^N U_n(t) V_n(x) \right\|_E^2 \leq c_1 \sum_{n=1}^N \left( \int_0^T f_n^2(t) dt + \phi_n^2 + (\phi_n^*)^2 \right).$$

The series  $\sum_{n=1}^N \int_0^T f_n^2(t) dt$ ,  $\sum_{n=1}^N \phi_n^2$ , and  $\sum_{n=1}^N (\phi_n^*)^2$  converge. Therefore it follows from (3.2) that the series (3.1) converges in  $E$  and accordingly its sum  $v \in E$ .  $\square$

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**Received: November 25, 2006**