Applied Mathematical Sciences, Vol. 1, 2007, no. 19, 931 - 937

A Functional Equation Related to a Multiple Cold Standby System

Edmond J. Vanderperre[1]

Department of Decision Sciences
University of South Africa
P.O. Box 392, Pretoria 2003, South Africa
evanderperre@yahoo.com

Abstract

We consider a basic multiple cold standby system attended by a single repairman. The system satisfies the usual conditions (i.i.d. random variables, perfect repair, instantaneous and perfect switch, queueing). We analyse the long-run availability of the system based on a functional equation valid in the complex plane. The solution is constructed by means of a Cauchy-type integral. As an example, we consider the case of deterministic repair.

Mathematics Subject Classification: 60K10

Keywords: Cold standby, functional equation, Cauchy integral, invariant measure, availability

1 Introduction

Statistical Reliability Engineering is a particular but important topic of Applied Mathematics, e.g. [1], [2].

Standby redundant systems are often employed to increase the reliability, availability, quality and safety of operational plants, e.g. [3].

A basic renewable cold standby system, attended by a single repairman, is the so-called **K**-system, [4]. The **K**-system consists of a single operative unit (the on-line unit) endowed with K *identical* units in cold standby. The notion of "cold" standby signifies that any unit in standby has a zero failure rate. The **K**-system acts as a *closed* queueing system evolving in time, i.e. the failed unit is immediately replaced by a standby spare (if available upon

---

[1] Correspondence Address: Dr E.J. Vanderperre, A. Ruzettelaan 183, Bus 158, 8370 Blankenberge, Belgium.

failure) and taken into repair provided that the repairman is idle. Otherwise, the failed unit has to queue for repair. On the other hand, a repaired unit lines-up in cold standby or becomes instantaneously operative if the K remaining units are down.

Our **K**-system satisfies the usual conditions, i.e. independent identically distributed random variables, perfect repair, instantaneous and perfect switch from standby into the operative state. An example of the so-called *duplex* system ($K = 1$) is the (internal) cooling device of the International Space Station (ISS) sustained by an (external) cooling device in cold standby. Note that astronauts can provide local repair.

The analysis of the **K**-system is far from complete. At present, we analyse the *long-run* availability of the basic **K**-system. Our analysis involves a functional equation valid in the complex plane. The solution is constructed by means of a Cauchy-type integral. Finally, as an example, we consider the case of *deterministic* repair.

2 Formulation

Consider the basic **K**-system, satisfying the usual conditions. The *operative* unit has a constant failure rate (e.g. [2]) $\lambda > 0$, a *zero* failure rate in standby and a repair time distribution $R(\cdot), R(0) = 0$ with finite mean. Without loss of generality (see forthcoming remark, Chapter 5), we may assume that $R(\cdot)$ has a density function of bounded variation on $[0, \infty)$. The failure-free time of the on-line unit and the repair time r are supposed to be *statistically* independent. Any repair is perfect and any switch from standby into the operative state is instantaneous.

Let N_t be the number of failed units in the repairshop at time t, so that $N_t \in \{0, \dots, K+1\}$. State $K+1$ is called the system downstate. A (vector) Markov characterization of the stochastic process $\{N_t, t \geq 0\}$ is piecewise and conditionally defined by:

$$\begin{aligned}&\{N_t\}, \quad \text{if } N_t = 0, \text{ i.e. if the event } \{N_t = 0\} \text{ occurs,}\\ &\{(N_t, X_t)\}, \quad \text{if } N_t = j;\ j = 1, \dots, K+1,\end{aligned}$$

where X_t denotes the *remaining* repair time of the failed unit being under progressive repair at time t. The state space of the underlying Markov process is given by

$$\{0\} \bigcup \{(j, x);\ j = 1, \dots, K+1; x \geq 0\}.$$

Next, we consider the **K**-system in *stationary* state (the so-called ergodic state) with *invariant* measure $\{p_j; j = 0, \dots, K+1\}$; $\sum_{j=0}^{K+1} p_j = 1$, where

$$p_j := \mathbf{P}\{N = j\} := \lim_{t\to\infty} \mathbf{P}\{N_t = j \mid N_0 = 0\}.$$

Finally, for $j = 1, \dots, K+1$ we introduce the measures

$$p_j(x)dx := \mathbf{P}\{N = j, X \in dx\} := \lim_{t\to\infty} \mathbf{P}\{N_t = j, X_t \in dx \mid N_0 = 0\}.$$

Note that

$$p_j = \int_0^\infty d\mathbf{P}\{N = j, X \le x\} = \int_0^\infty p_j(x)dx.$$

Notations

The complex plane is denoted by $\mathbf{C}$. The indicator of an event $\{N = \cdot\}$ is denoted by $\mathbf{1}\{N = \cdot\}$. Let D_z be a circle with centre at the origin of $\mathbf{C}$, with counter-clockwise orientation and with radius $\Re < \rho$, where ρ is the smallest root of the equation $z = \mathbf{E}e^{-\lambda(1-z)r}$. Note that $\rho < 1$, if $\lambda \mathbf{E}r > 1$ and that $\rho = 1$, if $\lambda \mathbf{E}r \le 1$. See Lemma of Takács, e.g. [5, Appendix].

We recall that the $\mathbf{K}$-system is only down in state $K+1$. Therefore, the long-run availability of the $\mathbf{K}$-system, denoted by $\mathcal{A}_K$, is given by $1 - p_{K+1}$.

3 Differential equations

In order to determine $\mathcal{A}_K$, we first construct a system of coupled Hokstad-type differential equations. Applying Hokstad's supplementary variable technique, e.g. [6], yields for $j = 1, \dots, K$ and $x > 0$,

$$\lambda p_0 = p_1(0),$$

$$(\lambda - \frac{d}{dx})p_j(x) = \lambda p_{j-1}(x) + p_{j+1}(0)\frac{dR}{dx}(x),$$

$$-\frac{d}{dx}p_{K+1}(x) = \lambda p_K(x),$$

where by convention

$$p_0(x) := p_0 \frac{dR}{dx}(x).$$

4 Functional equation

It should be noted that the differential equations are well-adapted to a Fourier-transformation. In fact, the integrability of the p-functions and the repair time density function on $[0,\infty)$ implies that the derivative of each p-function is also integrable. Consequently, each p-function vanishes at infinity. Applying a Fourier-transform technique to the equations entails that for $\operatorname{Im}\omega \geq 0$; $j = 1,\ldots,K$

$$(\lambda + i\omega)\mathbf{E}\left(e^{i\omega X}\mathbf{1}\{N=j\}\right) + p_j(0) = \lambda\mathbf{E}\left(e^{i\omega X}\mathbf{1}\{N=j-1\}\right) + p_{j+1}(0)\mathbf{E}e^{i\omega r}, \tag{1}$$

$$i\omega\mathbf{E}\left(e^{i\omega X}\mathbf{1}\{N=K+1\}\right) + p_{K+1}(0) = \lambda\mathbf{E}\left(e^{i\omega X}\mathbf{1}\{N=K\}\right). \tag{2}$$

Inserting $\omega = 0$ into equations (1), (2) reveals that for $j = 0,1,\ldots,K$

$$p_{j+1}(0) = \lambda p_j. \tag{3}$$

Substituting the relation (3) into equations (1), (2) and applying a generating function technique, yields the functional equation

$$(\lambda(1-z)+i\omega)\sum_{j=1}^{K} z^j\mathbf{E}\left(e^{i\omega X}\mathbf{1}\{N=j\}\right) + \lambda\left(z - \mathbf{E}e^{i\omega r}\right)\sum_{j=0}^{K} z^j p_j$$

$$+ z^{K+1}i\omega\mathbf{E}\left(e^{i\omega X}\mathbf{1}\{N=K+1\}\right) = \lambda(z-1)p_0\mathbf{E}e^{i\omega r}, \tag{4}$$

valid for $|z| < \infty$, $\operatorname{Im}\omega \geq 0$.

5 Long-run availability

In order to determine $\{p_j\}$, we first convert the functional equation into an equation in z by eliminating the function

$$\sum_{j=1}^{K} z^j\mathbf{E}\left(e^{i\omega X}\mathbf{1}\{N=j\}\right), \quad \operatorname{Im}\omega \geq 0.$$

Substituting $\omega = i\lambda(1-z)$, $\operatorname{Re} z \leq 1$ into equation (4) yields

$$\left(z - \mathbf{E}e^{-\lambda(1-z)r}\right)\sum_{j=1}^{K} z^j p_j - z^{K+1}(1-z)\mathbf{E}\left(e^{-\lambda(1-z)X}\mathbf{1}\{N=K+1\}\right)$$

$$= p_0(z-1)\mathbf{E}e^{-\lambda(1-z)r}. \tag{5}$$

We recall that the function $z - \mathbf{E}e^{-\lambda(1-z)r}$ has no zeros inside the region $\{z \in \mathbf{C} : |z| < \rho\}$. From the relation

$$p_j = \frac{1}{2\pi i}\int_{D_z} \frac{1}{z^{j+1}} \sum_{j=0}^{K} z^j p_j dz,$$

we obtain by (5),

$$p_j - \frac{1}{2\pi i}\int_{D_z} \psi_j(z) dz = p_0 \frac{1}{2\pi i}\int_{D_z} \frac{(z-1)\mathbf{E}e^{-\lambda(1-z)r}}{z - \mathbf{E}e^{-\lambda(1-z)r}} \frac{dz}{z^{j+1}},$$

where

$$\psi_j(z) := \frac{z^{K-j}(1-z)\mathbf{E}\left(e^{-\lambda(1-z)X}\mathbf{1}\{N = K+1\}\right)}{z - \mathbf{E}e^{-\lambda(1-z)r}}, \quad j = 0, \ldots, K.$$

Note that the function $\psi_j(z)$ is analytic in the region $\{z \in \mathbf{C} : |z| < \Re\}$ and boundedly continuous on the closure $\{z \in \mathbf{C} : |z| \leq \Re\}$. Hence, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{D_z} \psi_j(z) dz = 0.$$

Consequently, we have for $j = 0, \ldots, K$

$$p_j = p_0 \frac{1}{2\pi i}\int_{D_z} \frac{(z-1)\mathbf{E}e^{-\lambda(1-z)r}}{z - \mathbf{E}e^{-\lambda(1-z)r}} \frac{dz}{z^{j+1}}. \tag{6}$$

In order to determine p_{K+1}, we use the identities $\sum_{j=0}^{K+1} p_j = 1$ and

$$(z-1)\sum_{j=0}^{K} \frac{1}{z^{j+1}} = 1 - \frac{1}{z^{K+1}}.$$

By (6) and Cauchy's theorem we obtain

$$1 - p_{K+1} = p_0 \frac{1}{2\pi i}\int_{D_z} \frac{\mathbf{E}e^{-\lambda(1-z)r}}{\mathbf{E}e^{-\lambda(1-z)r} - z} \frac{dz}{z^{K+1}}. \tag{7}$$

Finally, substituting $z = 1$ into the functional equation (4) yields for $\mathrm{Im}\,\omega \geq 0$,

$$\sum_{j=1}^{K+1} \mathbf{E}\left(e^{i\omega X}\mathbf{1}\{N = j\}\right) = \lambda \mathbf{E}r \frac{\mathbf{E}e^{i\omega r} - 1}{i\omega \mathbf{E}r} \sum_{j=0}^{K} p_j,$$

where

$$\left.\frac{\mathbf{E}e^{i\omega r}-1}{i\omega\mathbf{E}r}\right|_{\omega=0}:=1.$$

Note that

$$\frac{\mathbf{E}e^{i\omega r}-1}{i\omega\mathbf{E}r}=\frac{1}{\mathbf{E}r}\int_0^\infty e^{i\omega x}\,(1-R(x))\,dx,\quad \operatorname{Im}\omega\geq 0.$$

Hence,

$$1-p_0=\lambda\mathbf{E}r(1-p_{K+1}). \tag{8}$$

From (7) and (8), we finally obtain:

$$\mathcal{A}_K=\frac{\frac{1}{2\pi i}\int_{D_z}\frac{\mathbf{E}e^{-\lambda(1-z)r}}{\mathbf{E}e^{-\lambda(1-z)r}-z}\frac{dz}{z^{K+1}}}{1+\lambda\mathbf{E}r\frac{1}{2\pi i}\int_{D_z}\frac{\mathbf{E}e^{-\lambda(1-z)r}}{\mathbf{E}e^{-\lambda(1-z)r}-z}\frac{dz}{z^{K+1}}}.$$

Remark

Note that the integral

$$\mathbf{E}e^{-\lambda(1-z)r}=\int_0^\infty e^{-\lambda(1-z)x}dR(x),$$

exists as a Lebesgue-Stieltjes integral for an arbitrary R. Consequently, our initial assumption concerning the existence of a repair time density function is superfluous to ensure the existence of an invariant measure, provided that r has a *finite* mean.

6 Deterministic repair

As an example, we consider the case of deterministic repair (constant replacement). Let

$$R(x):=\begin{cases}1, & \text{if } x\geq t_0>0,\\ 0, & \text{if } x<t_0.\end{cases}$$

Clearly, $\mathbf{E}e^{-\lambda(1-z)r}=e^{-\lambda(1-z)t_0}$. An application of Mertens' theorem [7, page 204] entails that

$$\frac{1}{2\pi i}\int_{D_z}\frac{e^{-\lambda(1-z)t_0}}{e^{-\lambda(1-z)t_0}-z}\frac{dz}{z^{K+1}}=\sum_{j=0}^{K}(-1)^j\frac{(\lambda t_0(K-j))^j}{j!}e^{\lambda(K-j)t_0}.$$

Hence,

$$\mathcal{A}_K=\frac{\sum_{j=0}^{K}(-1)^j\,\frac{(\lambda t_0(K-j))^j}{j!}e^{\lambda(K-j)t_0}}{1+\lambda t_0\sum_{j=0}^{K}(-1)^j\frac{(\lambda t_0(K-j))^j}{j!}e^{\lambda(K-j)t_0}}.$$

7 Conclusion

The long-run availability of the **K**-system can be analysed by elegant methods of complex analysis based on a functional equation.

The analysis of the **K**-system, subjected to a general failure and repair time distribution, constitutes an open (harsh) problem in Probabilistic Reliability Engineering.

References

[1] Gnedenko, B., Ushakov, I.A., *Probabilistic Reliability Engineering*, John Wiley & Sons, New York, (1995) .

[2] Shaked, M., Shanthikumar I.G., *Reliability and Maintainability, In : Handbook in Operations Research and Management Science: Stochastic Models* **2**, Eds.: Heyman, D.P. and Sobel, M.J., North-Holland, Amsterdam, (1990).

[3] Birolini, A., *Reliability and Quality of Technical Systems.* Springer-Verlag, Berlin, (1994).

[4] Vanderperre, E.J., Reliability analysis of a renewable multiple cold standby system. *Operations Research Letters* **32**, (2004) 288–292.

[5] Cohen, J.W., *The Single Server Queue.* North-Holland, Amsterdam, (1982).

[6] Alfa, A.S., Sriniva Rao, T.S.S., Supplementary variable technique in stochastic models, *Probability in the Engineering and Informational Sciences* **14**, (2000) 203–218.

[7] Apostol, T.M., *Mathematical Analysis*, Addison-Wesley, Amsterdam, (1998).

Received: September 13, 2006