

Throughout Positive Solutions for Delay Nonlinear Differential Equation¹

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Abstract

In this paper, we consider the delay differential equation

$$\left(a(t) (x(t) - p(t)x(t - \tau))' \right)' + f(t, x(g(t))) = 0, \quad t \geq t_0.$$

Using the Banach contraction mapping principle, we obtain the existence of throughout positive solutions for the above equations.

Mathematics Subject Classification: 34B15, 39A10

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1 Introduction

Recently, there has been an increasing interest in the study of the oscillation and nonoscillation of solutions of second-order ordinary and delay neutral differential and difference equations. Also eventually positive solutions and asymptotic behavior of nonoscillatory solutions have been investigated widely. Delay differential equations play a very important role in many practical problems. The papers [3, 4, 5] discuss the oscillation of second order differential and difference equations. The papers [1, 6] discuss the oscillation and non-oscillation criteria for second order differential equations. Of course there is also the discussion of the existence of eventually positive solutions, such as

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[7, 8]. The paper [9] studies the throughout positive solutions of the second-order nonlinear differential equations

$$\left(a(t) (x(t) - p(t)x(t-\tau))' \right)' + f(t, x(g(t))) = 0, \quad t \geq t_0$$

where $f : [t_0, \infty) \times R \rightarrow R$ is continuous and nonnegative. Motivated by [9], we have studied further and extended the results of Zhang, Wang and Li [9] to the delay differential equations. We obtain the existence of throughout positive solutions by introducing a weighted norm (see [2, 6, 9]) and using the Banach contraction mapping principle (see [2]).

This paper is concerned with existence of throughout positive solutions for the following delay nonlinear differential equation

$$\left(a(t) \Phi \left(x'(t) \right) + p(t) x'(t-\tau) \right)' + f(t, x(g(t))) = 0, \quad t \geq t_0 \quad (1.1)$$

where $a(t) > 0$ is continuous; $f(t, x)$, $\Phi(u)$ are continuous and satisfies $f(t, x)x > 0$ for $x \neq 0$; $g(t)$ is continuous, increasing and satisfies $g(t) \leq t$, $\lim_{t \rightarrow \infty} g(t) = \infty$.

Definitions. A solution of differential equation is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be non-oscillatory.

A solution of differential equation is said to be eventually positive solution if there exists some $T > t_0$ such that $x(t) > 0$ for all $t \geq T$.

A solution of differential equation is said to be throughout positive solution if $x(t) > 0$ for all $t \geq t_0$.

To prove our main results, we need the following lemma.

Lemma [9]. Assume $x(t)$ is bounded,

$$\lim_{t \rightarrow \infty} p(t) = p, p \neq 1, z(t) = x(t) - p(t)x(t-\tau), \quad \lim_{t \rightarrow \infty} z(t) = l,$$

then $\lim_{t \rightarrow \infty} x(t)$ exists and $\lim_{t \rightarrow \infty} x(t) = l/(1-p)$.

2 Main Results

In this section we give existence theorems of throughout positive solutions for equation (1.1). Throughout this paper, for any positive constant $M > 0$, $0 \leq u, v \leq M$ and $t \geq t_0$, we assume that

(H_1) There exists a continuous function $k(t) > 0$, satisfying

$$\int_{t_0}^{\infty} \frac{s}{\bar{a}(s)} k(s) ds < \infty, \quad (2.1)$$

such that the nonlinearity f fulfils a Lipschitz condition

$$|f(t, u) - f(t, v)| \leq k(t) |u - v|, \tag{2.2}$$

where $\bar{a}(s) = \min \{a(\theta) : \min \{t_0 - \tau, g(t_0)\} \leq \theta \leq s\}$.

(H_2) There exists a positive constant L such that

$$|\Phi^{-1}(u) - \Phi^{-1}(v)| \leq L |u - v|. \tag{2.3}$$

For sake of simple, we will use $\psi(t)$ denote

$$\psi(t) = \frac{1}{a(t)} \int_t^\infty f(s, x(g(s))) ds, \quad t \geq t_0. \tag{2.4}$$

and $G(t)$ denote

$$G(t) = \exp \left(NL \int_t^\infty \frac{s}{\bar{a}(s)} k(s) ds \right) \quad \text{for } t \geq t_0, \tag{2.5}$$

where N is a positive integer. From $N > 0$, $a(t) > 0$ and $k(t) > 0$, we know $G'(t) < 0$ and

$$L \frac{t}{\bar{a}(t)} k(t) G(t) = -\frac{1}{N} G'(t), \tag{2.6}$$

which implies that $G(t)$ is decreasing.

Theorem 1. Assume that $\lim_{t \rightarrow \infty} p(t) = p$, where $p \in (-1, 0)$ and $p \leq p(t) < 0, p'(t) \leq 0$. Define the set

$$X = \left\{ \begin{array}{l} u \in C^1[t_0, \infty), 0 \leq u(t) \leq M(1-p) \text{ for } t \geq t_0; \\ u(t) = u(t_0) \text{ for } \min \{g(t_0), t_0 - \tau\} \leq t < t_0 \end{array} \right\}$$

where M is a positive constant. In addition to (2.1) and (2.3) for $0 \leq u, v \leq M(1-p)$, assume that

$$0 < \int_{t_0}^\infty \Phi^{-1}(\psi(s)) ds - \int_{t_0}^\infty p'(s) x(s - \tau) ds < M(1-p^2) \tag{2.7}$$

hold for $x \in X$. Assume further that there exists a positive integer $N > 1$ such that $0 < (\frac{1}{N} - 2p)l < 1$, where $l = \max_{t \geq t_0} \left\{ \frac{G(t-\tau)}{G(t)}, \frac{G(g(t))}{G(t)} \right\}$. Then equation (1.1) has a positive solution $x(t)$ satisfying $\lim_{t \rightarrow \infty} x(t) = M$.

Proof. Define a mapping \mathcal{T} on X as follows

$$(\mathcal{T}x)(t) = \begin{cases} M(1-p) + p(t)x(t-\tau) - \int_t^\infty \Phi^{-1}(\psi(s)) ds \\ \quad + \int_t^\infty p'(s)x(s-\tau) ds, \text{ for } t \geq t_0; \\ (\mathcal{T}x)(t_0), \text{ for } \min\{g(t_0), t_0 - \tau\} \leq t < t_0. \end{cases} \quad (2.8)$$

For $t \geq t_0$, from (2.8) and $p(t) < 0$, we have $0 \leq (\mathcal{T}x)(t) \leq M(1-p)$, so $\mathcal{T}X \subseteq X$. In view of the definition $G(t)$, we introduce the norm $\|\cdot\|$ on X , $\|x\| = \sup_{t \geq t_0} |x(t)/G(t)|$. Note that X is closed with respect to this norm, and therefore we have a complete metric space.

Next, we show that \mathcal{T} is a contraction mapping on X . For any $x_1, x_2 \in X$, from the assumptions we have

$$\begin{aligned} & \frac{1}{G(t)} |(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)| \\ \leq & |p(t)| \frac{1}{G(t)} |x_1(t-\tau) - x_2(t-\tau)| \\ & + \frac{1}{G(t)} \int_t^\infty |\Phi^{-1}(\psi_1(s)) - \Phi^{-1}(\psi_2(s))| ds \\ & + \frac{1}{G(t)} \int_t^\infty |p'(s)| |x_1(s-\tau) - x_2(s-\tau)| ds \\ \leq & |p(t)| \frac{G(t-\tau)}{G(t)} \frac{1}{G(t-\tau)} |x_1(t-\tau) - x_2(t-\tau)| \\ & + \frac{1}{G(t)} \int_t^\infty L |\psi_1(s) - \psi_2(s)| ds \\ & + \frac{1}{G(t)} \int_t^\infty |p'(s)| G(s) \frac{G(s-\tau)}{G(s)} \frac{1}{G(t-\tau)} |x_1(s-\tau) - x_2(s-\tau)| ds \\ \leq & |p|l \|x_1 - x_2\| + l \|x_1 - x_2\| \frac{1}{G(t)} \int_t^\infty |p'(s)| G(s) ds \\ & + \frac{1}{G(t)} \int_t^\infty L |\psi_1(s) - \psi_2(s)| ds \end{aligned} \quad (2.9)$$

In view of (2.5) and $p'(t) \leq 0$, we know

$$\frac{1}{G(t)} \int_t^\infty |p'(s)| G(s) ds \leq \int_t^\infty |p'(s)| ds = \left| \int_t^\infty p'(s) ds \right| \leq |p|. \quad (2.10)$$

From (2.4), (2.5) and (2.6), we have

$$\frac{1}{G(t)} \int_t^\infty L |\psi_1(s) - \psi_2(s)| ds$$

$$\begin{aligned}
 &= \frac{1}{G(t)} \int_t^\infty L \left| \frac{1}{a(s)} \int_s^\infty [f(r, x_1(g(r))) - f(r, x_2(g(r)))] dr \right| ds \\
 &\leq \frac{1}{G(t)} \int_t^\infty \frac{L}{a(s)} \int_s^\infty |f(r, x_1(g(r))) - f(r, x_2(g(r)))| dr ds \\
 &\leq \frac{1}{G(t)} \int_t^\infty \frac{L}{a(s)} \int_s^\infty k(r) |x_1(g(r)) - x_2(g(r))| dr ds \\
 &\leq \frac{1}{G(t)} \int_t^\infty \frac{L}{a(s)} \int_s^\infty \frac{1}{G(g(r))} k(r) |x_1(g(r)) - x_2(g(r))| G(g(r)) dr ds \\
 &\leq \frac{\|x_1 - x_2\|}{G(t)} \int_t^\infty \frac{L}{\bar{a}(s)} \int_s^\infty k(r) G(r) \frac{G(g(r))}{G(r)} dr ds \\
 &\leq \frac{l \|x_1 - x_2\|}{G(t)} \int_t^\infty \frac{L}{\bar{a}(s)} \int_s^\infty k(r) G(r) dr ds \\
 &= \frac{l \|x_1 - x_2\|}{G(t)} \int_t^\infty \frac{sLk(s)G(s)}{\bar{a}(s)} ds \\
 &= \frac{l \|x_1 - x_2\|}{G(t)} \int_t^\infty \left(-\frac{1}{N}\right) G'(s) ds \\
 &= l \frac{G(t) - 1}{NG(t)} \|x_1 - x_2\| \\
 &\leq \frac{l}{N} \|x_1 - x_2\|. \tag{2.11}
 \end{aligned}$$

Integrating (2.9), (2.10) and (2.11), we obtain

$$\begin{aligned}
 &\frac{|(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)|}{G(t)} \\
 &\leq 2|p|l \|x_1 - x_2\| + \frac{l}{N} \|x_1 - x_2\| = \left(\frac{1}{N} - 2p\right) l \|x_1 - x_2\|.
 \end{aligned}$$

Since $0 < \left(\frac{1}{N} - 2p\right) l < 1$, \mathcal{T} is contraction mapping on X . By the Banach fixed point theorem, we deduce the existence of a unique fixed point in X , for $t \geq t_0$

$$\begin{aligned}
 x(t) &= (\mathcal{T}x)(t) = M(1 - p) + p(t)x(t - \tau) \\
 &\quad - \int_t^\infty \Phi^{-1} \left(\frac{1}{a(s)} \int_s^\infty f(r, x(g(r))) dr \right) ds + \int_t^\infty p'(s)x(s - \tau) ds.
 \end{aligned}$$

From $x \in X$ and $p \leq p(t) < 0$ we have $p(t)x(t - \tau) \geq pM(1 - p)$. From the inequality and the condition (2.7), we obtain

$$x(t) > M(1 - p) + pM(1 - p) - M(1 - p^2) = 0.$$

Hence $x(t) > 0$ for $t \geq t_0$. Substituting $x(t)$ into equation (1.1), we know that $x(t)$ is a positive solution of (1.1) and

$$\lim_{t \rightarrow \infty} (x(t) - p(t)x(t - \tau)) = M(1 - p).$$

In view of the Lemma, we obtain $\lim_{t \rightarrow \infty} x(t) = M$, which completes the proof.

Theorem 2. Assume that $\lim_{t \rightarrow \infty} p(t) = p$, where $p \in (-\infty, -1)$ and $p(t) \leq p$. Define the set

$$X = \left\{ \begin{array}{l} u \in C^1[t_0, \infty), 0 \leq u(t) \leq \frac{M(1+|p|)}{|p|} \text{ for } t \geq t_0; \\ u(t) = u(t_0) \text{ for } g(t_0) \leq t < t_0 \end{array} \right\}$$

where M is a positive constant. In addition to (2.1) and (2.3) for $0 \leq u, v \leq \frac{M(1+|p|)}{|p|}$, suppose that

$$\frac{M(1 - p^2)}{p(t)p} < \int_t^\infty \frac{1}{p(s)} \Phi^{-1}(\psi(s)) ds + \int_t^\infty \frac{(M(1 - p) - x(s))p'(s)}{p^2(s)} ds < 0 \tag{2.12}$$

hold for $x \in X, t \geq t_0$, and there exists a positive integer $N > 1$ such that $0 < \frac{1}{|p|} (2 + \frac{l}{N}) < 1$, where $l = \max_{t \geq t_0} \left\{ \frac{G(g(t))}{G(t)} \right\}$. Then equation (1.1) has a positive solution $x(t)$ satisfying $\lim_{t \rightarrow \infty} x(t) = M$.

Proof. Define a mapping \mathcal{T} on X as follows

$$(\mathcal{T}x)(t) = \begin{cases} \frac{1}{-p(t+\tau)} [M(1 - p) - x(t + \tau)] + \int_{t+\tau}^\infty \frac{\Phi^{-1}(\psi(s))}{p(s)} ds \\ \quad + \int_{t+\tau}^\infty \frac{p'(s)}{p^2(s)} (M(1 - p) - x(s)) ds, & \text{for } t \geq t_0; \\ (\mathcal{T}x)(t_0), & \text{for } g(t_0) \leq t < t_0. \end{cases} \tag{2.13}$$

From (2.12) and (2.13) we have $0 \leq (\mathcal{T}x)(t) \leq \frac{M(1+|p|)}{|p|}$, so $\mathcal{T}X \subseteq X$. In view of the definition $G(t)$, we introduce the norm $\|\cdot\|$ on $X, \|x\| = \sup_{t \geq t_0} |x(t)/G(t)|$. Note that X is closed with respect to this norm, and therefore we have a complete metric space. We now show that \mathcal{T} is a contraction mapping on X . For any $x_1, x_2 \in X$, from the assumptions we have

$$\begin{aligned} & \frac{1}{G(t)} |(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)| \\ & \leq \frac{-1}{p(t + \tau)G(t + \tau)} |x_1(t + \tau) - x_2(t + \tau)| + \\ & \quad \frac{1}{G(t)} \int_{t+\tau}^\infty \left| \frac{1}{p(s)} (\Phi^{-1}(\psi_1(s)) - \Phi^{-1}(\psi_2(s))) \right| ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{G(t)} \int_{t+\tau}^{\infty} \frac{1}{p^2(s)} |p'(s)| |x_1(s) - x_2(s)| ds \\
 \leq & \frac{1}{|p|} \|x_1 - x_2\| + \frac{1}{|p|} \frac{l}{N} \|x_1 - x_2\| + \frac{1}{|p|} \|x_1 - x_2\| \\
 = & \frac{1}{|p|} \left(2 + \frac{l}{N} \right) \|x_1 - x_2\|. \tag{2.14}
 \end{aligned}$$

Since $0 < \frac{1}{|p|} \left(2 + \frac{l}{N} \right) < 1$, (2.14) implies that \mathcal{T} is contraction mapping on X . By the Banach fixed point theorem, we deduce the existence of a unique fixed point in X , for $t \geq t_0$,

$$\begin{aligned}
 x(t) & = (\mathcal{T}x)(t) \\
 & = \frac{1}{-p(t+\tau)} (M(1-p) - x(t+\tau)) + \int_{t+\tau}^{\infty} \frac{\Phi^{-1}(\psi(s))}{p(s)} ds \\
 & \quad + \int_{t+\tau}^{\infty} \frac{p'(s)}{p^2(s)} (M(1-p) - x(s)) ds.
 \end{aligned}$$

From $x \in X$ we have $x(t+\tau) \leq \frac{M(1+|p|)}{|p|}$. From the inequality and the condition (2.12), we obtain

$$x(t) > \frac{1}{-p(t+\tau)} \left(M(1-p) + \frac{M(1-p)}{p} \right) - \frac{M(p^2-1)}{p(t+\tau)p} = 0.$$

Hence $x(t) > 0$ for $t \geq t_0$. Substituting $x(t)$ into equation (1.1), we know that $x(t)$ is a positive solution of (1.1) and $\lim_{t \rightarrow \infty} (x(t) - p(t)x(t-\tau)) = M(1-p)$. In view of the Lemma, we obtain $\lim_{t \rightarrow \infty} x(t) = M$, which completes the proof.

Theorem 3. Assume that $\lim_{t \rightarrow \infty} p(t) = p$, where $p \in (1, +\infty)$ and $p(t) \geq p$. Define the set

$$X = \left\{ \begin{array}{l} u \in C^1[t_0, \infty), 0 \leq u(t) \leq \frac{M(1+p)}{p} \text{ for } t \geq t_0; \\ u(t) = u(t_0) \text{ for } g(t_0) \leq t < t_0, \end{array} \right\}$$

where M is a positive constant. In addition to (2.1) and (2.3) for $0 \leq u, v \leq \frac{M(1+p)}{p}$, suppose that for $x \in X$

$$0 < \int_{t_0}^{\infty} \frac{1}{p(s)} \Phi^{-1}(\psi(s)) ds - \int_{t_0}^{\infty} \frac{(M(1-p) - x(s)) p'(s)}{p^2(s)} ds < \frac{M(p-1)}{p^2} \tag{2.15}$$

hold and there exists a positive integer $N > 1$ such that $0 < \frac{1}{p} \left(2 + \frac{l}{N} \right) < 1$, where $l = \max_{t \geq t_0} \left\{ \frac{G(g(t))}{G(t)} \right\}$, Then equation (1.1) has a positive solution $x(t)$ satisfying $\lim_{t \rightarrow \infty} x(t) = M$.

Proof. Define a mapping \mathcal{T} on X as follows

$$(\mathcal{T}x)(t) = \begin{cases} \frac{1}{p(t+\tau)} (M(p-1) + x(t+\tau)) + \int_{t+\tau}^{\infty} \frac{\Phi^{-1}(\psi(s))}{p(s)} ds \\ \quad - \int_{t+\tau}^{\infty} \frac{p'(s)}{p^2(s)} (M(p-1) + x(s)) ds & \text{for } t \geq t_0; \\ (\mathcal{T}x)(t_0) & \text{for } g(t_0) \leq t < t_0. \end{cases} \quad (2.16)$$

From (2.15) and (2.16) we have $0 \leq (\mathcal{T}x)(t) \leq \frac{M(1+p)}{p}$, so $\mathcal{T}X \subseteq X$. In view of the definition $G(t)$, we introduce the norm $\|\cdot\|$ on X , $\|x\| = \sup_{t \geq t_0} |x(t)/G(t)|$. Note that X is closed with respect to this norm, and therefore we have a complete metric space. Similar to the proof of Theorem 2, we can show that \mathcal{T} is a contraction mapping on X . For any $x_1, x_2 \in X$, from the assumptions we have

$$\begin{aligned} & \frac{1}{G(t)} |(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)| \\ & \leq \frac{1}{p(t+\tau)G(t+\tau)} |x_1(t+\tau) - x_2(t+\tau)| \\ & \quad + \frac{1}{G(t)} \int_{t+\tau}^{\infty} \left| \frac{\Phi^{-1}(\psi_1(s)) - \Phi^{-1}(\psi_2(s))}{p(s)} \right| ds \\ & \quad + \frac{1}{G(t)} \int_{t+\tau}^{\infty} \frac{|p'(s)|}{p^2(s)} |x_1(s) - x_2(s)| ds \\ & \leq \frac{1}{p} \|x_1 - x_2\| + \frac{1}{pN} \|x_1 - x_2\| + \frac{1}{p} \|x_1 - x_2\| \\ & = \frac{1}{p} \left(2 + \frac{l}{N} \right) \|x_1 - x_2\|. \end{aligned} \quad (2.17)$$

Since $0 < \frac{1}{p} \left(2 + \frac{l}{N} \right) < 1$, (2.17) implies that \mathcal{T} is contraction mapping on X . By the Banach fixed point theorem, we deduce the existence of a unique fixed point in X ,

$$\begin{aligned} x(t) &= (\mathcal{T}x)(t) \\ &= \frac{1}{p(t+\tau)} (M(p-1) + x(t+\tau)) + \int_{t+\tau}^{\infty} \frac{\Phi^{-1}(\psi(s))}{p(s)} ds \\ & \quad - \int_{t+\tau}^{\infty} \frac{p'(s)}{p^2(s)} (M(p-1) + px(s)) ds, \end{aligned}$$

for $t \geq t_0$. The condition $p > 1$ implies $M(p-1) > 0$, and all the other terms which are in the expression of $x(t)$ are negative, we easily know that $x(t) > 0$ for $t \geq t_0$. Substituting $x(t)$ into equation (1.1), we know that $x(t)$

is a throughout positive solution of (1.1) and $\lim_{t \rightarrow \infty} (x(t) - p(t)x(t - \tau)) = M(1 - p)$. In view of the Lemma we have $\lim_{t \rightarrow \infty} x(t) = M$. The proof is complete.

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