On Riddled Sets and Bifurcations of Chaotic Attractors

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Abstract

Coupled systems provide us examples of blowout bifurcations and these maps have parameter values at which there are attractors which are a filled-in quadrilateral and, simultaneously, the synchronized state is a Milnor attractor. Riddled basins denote a characteristic type of fractal domain of attraction that can arise when a chaotic attractor is restricted to an invariant subspace of total phase space. The paper examines the conditions for the appearance of such basins for a system of two symmetrically coupled maps.

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1 Introduction

In this paper we consider how it can lead to the appearance of attractors with riddled basins and to know if riddling may help one to understand the nature of chaotic behavior. These basins appear because symmetries of dynamical systems force the presence of invariant submanifolds; the attractors within invariant manifolds may be only weakly attracting transverse to the invariant manifold and this leads to a basin structure that is, roughly speaking, full of
holes. Since the phenomena of riddled basins was uncovered [1], these basins have been found in a range of applications, for example learning dynamical systems [16], coupled chaotic oscillators [6, 21], mechanical systems [19] and electronic systems [12] and especially coupled maps, e.g. [10]. There has been some very interesting recent work looking at dynamical systems of globally delay pulse coupled oscillator systems, motivated by simple models of neural systems. These systems can possess attractors that are an extreme case of riddling; there is a neighborhood of the attractor that has zero measure intersection with the attractor’s basin. Such unstable attractors have been found to be quite widespread in the dynamics of certain systems [20].

An essential key to understand the origin of these riddled basins of attraction is provided by the mechanisms underlying the transverse destabilization of low periodic orbits embedded in chaotic attractors and by contact bifurcations (see [13-15]). Special interest is Mira’s analysis of the role of contact bifurcations (boundary crises) in which the boundary of the absorbing area for a chaotic set makes contact with the basin boundary. This type of global bifurcations are involved in the transition from local to global riddling. It is possible to use the theory of critical curves in sense of Mira but in the map we considered bifurcation values and critical curves seem difficult to be calculated explicitly.

It is a challenge to understand constraints on the appearance of such attractors and their characteristic properties. It has been recognised that coupled identical systems provide a good supply of examples. As a parameter is varied an asymptotically stable chaotic synchronized state loses asymptotic stability as some orbits in the attractor become transversely unstable. There is then an interval of parameter values for which typical synchronized orbits are transversely stable, but some are not, leading to riddled basins of attraction (either locally or globally). At a critical parameter value typical synchronized orbits lose transverse stability (the blowout bifurcation) although some synchronized orbits may remain transversely stable. Finally, the synchronized state becomes completely unstable. Here, these mechanisms have been investigated only via direct numerical simulation of the dynamics. The purpose of this paper is to check the conditions for the appearance of such basins by numerical results of a interesting model given in [9] according to [11,17].

Our map due to Ding and Yang [9] gives rise to intermingled basins defined as a coupled map on  \( (x, y) \in [-1, 1]^2 \) : \( T(x, y) = (x', y') \) and \( f(x) = ax(1 - x^2) \exp(-x^2) \)

\[
x' = f(x) + b \left[ f(y) - f(x) + f^3(y) - f^3(x) \right]
\]
\[
y' = f(y) + b \left[ f(x) - f(y) + f^3(x) - f^3(y) \right]
\]

This map has attractors \( A^\pm \) in the diagonal \( x = y \) on either side of the origin. Two transition parameters \( b_1 \) and \( b_2 \) are found to be at \( b_1 = 0.4701 \) and
$b_2 = 0.5940$, for $b_1 \leq b \leq b_2$ attractors $A^\pm$ are global attractors. In [9] the authors present evidence supporting the claim that their basins of attraction are intermingled in this interval $[b_1, b_2]$. For $a = 3.4$ the individual map $x_{n+1} = f(x_n)$ has two chaotic attractors, one in the region $x > 0$ which we denote $A^+$, and the other in the region $x < 0$ which we denote $A^-$. These results are essentially a restatement or simple elaboration of results in [9]. Figure 1 illustrates the basins of these attractors. $A^\pm$ are Milnor attractors (they attract all points from their respective neighborhoods except for sets of zero measure) for $T$ then

$$\bigcap_{m>n} \left( \bigcup_{p>m} A_p \right)$$

with $A_p = T^{-p}([-1,1] \times [-1,1])$ converge (apart from a set of zero measure) to $B(A^\pm)$ as $n \to \infty$.

The figure shows typical examples of the basins for different values of $b$ inside the interval $[b_1, b_2]$ and outside, we have four stable one-band chaotic attractors. This result as proved in [9], the basins are thoroughly intermingled. The effect appears to be a frothy mixture that has been subjected to lots of stirring and folding. One should stress however that these basins, not just the boundary has ‘fractal’ properties, but the whole set is inseparable from its boundary $\partial B$ and in fact $\partial B = B$ up to a set of zero measure.

The map $T$ is smooth generating a dynamical system on iteration. As is well-known, there are three ways in which a fixed point $p$ of a discrete map $T$ may fail to be hyperbolic: $DT(p)$ may have an eigenvalue $+1$, an eigenvalue $-1$, or a pair of complex eigenvalues, say $\lambda$ and $\bar{\lambda}$, with $|\lambda| = 1$. The first two situations are familiar from period-doubling cascades and appear much more frequently than the last one. But it is precisely this last situation, characteristic of a Hopf bifurcation, that is of great interest for coupled systems and that we consider here. Eq. (1-2) has fixed points diagonal or/and non-diagonal. The stability of these points is ruled by the equation $|J - \lambda I| = 0$, where $\lambda$ is the eigenvalue, $I$ is the identity matrix and $J$ is the Jacobian matrix of the mapping. Destabilization of the fixed point $(0,0)$ happens in a pitchfork bifurcation.

The stability domains for this coupled map can be generated numerically, displaying the richness of the several coexisting attractors in parameter space. We describe a specific family of bifurcations in a region of real parameter $(a, b)$ plane for which the mappings were expected to have simple dynamics. We compute the first few bifurcation curves in this family and we study the bifurcation diagram which consists to these bifurcation curves in the parameter $(a, b)$ plane together with representative phase portraits. The Figure 2(a,b) presents then information on stability region for the fixed point (blue domain), and the existence region for attracting cycles of order $k$ exists ($k \leq 14$).
Figure 1: The basin structure for the coupled map (1) for $a = 3.4$ and $b = 0.42, 0.4709, 0.50, 0.52, 0.69, 0.70$ the intermingled character of the basin is apparent for $b = 0.5, 0.52, 0.69, 0.70$. We have four one-band chaotic attractors, two on the main diagonal and two on the other diagonal.
The black regions ($k = 15$) corresponds to the existence of bounded iterated sequences. We can recognize on the diagram period doubling bifurcation on Figure (2.b) [13]. The bifurcation structure is of complex type. The borderline seen between two domains of different colors is defined by Mira as bifurcation curve.

2 General properties

We assume that a closed and invariant set $A$ is called an attracting set if some neighborhood $U$ of $A$ may exist such that $T(U) \subset U$, and $T^n(x, y) \to A$ as $n \to \infty, \forall (x, y) \in U$. The set $B = \cup_{n \geq 0} T^{-n}(U)$ is called the total basin of $A$.

**Definition 2.1** A chaotic area $A$, is an invariant absorbing area ($T(A) = A$) that exhibits chaotic dynamics, and the points of which give rise to iterated sequences having the property of sensitivity to initial conditions.

**Definition 2.2** We say that $a = a^*$ is a bifurcation of contact of $A$, if a contact between the frontier of $A$ and the frontier its basin of attraction takes place.

**Proposition 2.1** When a bifurcation of contact of a chaotic area $A$ arises for a value $a = a^*$, the crossing of this value leads to the destruction of $A$ or to a qualitative modification of properties of $A$.

**Definition 2.3** Let $T$ be a coupled map of $\mathbb{R}$, the blowout bifurcation is associated with the loss of transverse stability of an asymptotically stable chaotic synchronized state at a critical parameter value $a = a^*$. 
3 Basins and attractors

In this section we show the birth and evolution of the Hopf bifurcation in phase space when we walk along the horizontal line \( b = 1 \) in parameter space (see Figure 3).

For several values of \( a \), a quasiperiodic transition to chaos takes place. Periodic orbits, visible in Figure 3, appear in this route to chaos via quasiperiodicity. We demonstrated that the map can show quasiperiodic motion arising from a Hopf bifurcation of period-2 orbits. This was shown and corroborated through a numerical simulation, as may be seen from the similarity between Figures 3 and 4.

According to [17], in the case of two-dimensional phase space and, hence, the invariant subspace is a line. Before the bifurcation, the chaotic attractor attracts all points in some of its neighborhood, and all the periodic orbits embedded in the chaotic attractor are saddles. At the bifurcation, one of the periodic orbits, usually of low period, becomes transversely unstable. Since this periodic orbit is already unstable in the attractor, it becomes a repeller in the two-dimensional phase space. This phenomenon was been described in detail in [17], the authors suggested that the loss of transverse stability is induced by the collision, at a critical value of one parameter, of two repellers \( r_1 \) and \( r_2 \), located symmetrically with respect to the invariant subspace, with the saddle (a so called a saddle-repeller pitchfork bifurcation). Trajectories in the chaotic attractor, however, remain there even after bifurcation, since the subspace in which the chaotic attractor lies is invariant and each tongue has a zero width there. The basin structure of the attractor not in the invariant subspace is made up of an open and dense set of tongues. For more complicated systems as our case, it is difficult to determine which unstable periodic orbits would lose transverse stability first. In all examples studied until now, it is a low-period periodic orbit, but no proof is given that this is the generic case.

According to [11], a point cycle embedded in the synchronized state may lose its transverse stability through a period-doubling bifurcation. This situation plays also a significant role. In both cases, i.e., for the pitchfork as well as for the period-doubling bifurcation, the transversely destabilized orbit will be surrounded by saddle points with unstable manifolds along the invariant subspace of the synchronized state. This leads to the phenomenon of local riddling.

Next, we examine the bifurcation effects that appear in the coupled system on varying parameters; notably we put in evidence the route to creation of riddled basins via a riddling bifurcation that may well be common than that described in references therein.

It is observed that the basins of these attractors are intermingled very complexly (on the figure, you can observe the beige and pink area by expanding
Figure 3: Chaotic attractors and riddled basins for $a = 2.085, 2.09, 2.11, 2.18, 2.24, 2.26019$. 
the any region in the blue area, there are several attractors (exactly five)). These are also called “Riddled Basins” because each basin is riddled with holes. For the existence of the riddled basin, it is known that the following conditions are required:

- There exists an invariant submanifold, and its internal dynamics is chaotic.
- The lyapunov exponent normal to the invariant submanifold is negative, i.e., the submanifold is an attractor.
- There exist other attractors.

As argued in [9], the two attractors (red and yellow) are symmetric with respect to $x = 0$. Clearly, if $x = y$ is plugged into Eq(1-2), we can have synchronized chaos. The synchronization manifold, defined by $x = y$, is one dimensional ($n = 1$), and is invariant under the dynamics. $A^+$ and $A^-$ are also attractors for the full two dimensional space. Since the symmetry is preserved between $A^+$ and $A^-$, the two transversal Lyapunov exponents are identical.

Fujisaka and Yamada [20] showed how two identical chaotic systems under variation of the coupling strength can attain a state of chaotic synchronization in which the motion of the coupled system takes place on an invariant subspace of total phase space. For two coupled identical one-dimensional maps, for instance, the synchronized chaotic motion is one dimensional, and occurs along the main diagonal in phase plane, and the transverse Lyapunov exponent provides a measure of the average stability of the chaotic attractor perpendicularly to this direction.

Now, we show the basin structure for the coupled map, where the intermingled character of the basins is apparent.

Since $f(x)$ is bimodal, $f(x)$ has two intervals over which it is unimodal and Figure (2.b) demonstrates that it produces a transition to chaos in accordance with Mirberg scenario. By virtue of its symmetry with respect to the main diagonal, the map displays two two-band chaotic attractors. Figure 4 displays the basins for each of these attractors for several values of $b$ in Figure 4. For both of these sets the basin is riddled with holes that belong to basins of closed curves of period 3 which become chaotic, have contact (contact bifurcation) with their basin boundary and then disappear with their basins. In the Figure 5, we can see on the second diagonal appearance of two one-band chaotic sets, which become chaotic attractors restricted to their absorbing areas. With further increase of $b$ (at $b = 0.55$) a crisis take place, eight basins present holes (basins of period-2 chaotic attractors). These basins suddenly disappear and we have an inverse cascade of doubling period.

There are several ways that global riddling can happen, and brings the following cases together: An attractor in an invariant subspace can lose asymptotic stability. The basin of attraction of an attractor can undergo a crisis leading to leakage from the basin. This will occur if the attractor is contained inside a weak attractor. The structure of basins of attraction seems to be very
Figure 4: Chaotic attractors emergence of riddling and intermingled basins for \( a = 3 \) and \( b = 0.050, 0.075, 0.160, 0.36, 0.425, 0.430, 0.450, 0.454 \)
Figure 5: Basins of attraction for the synchronous chaotic states with holes that belong to the basin of a coexisting cycle or an invariant closed curve $a = 3$ and $b = 0.4582, 0.4585, 0.5, 0.52, 0.55, 0.55, \text{(zoom)}, 0.58, 0.59$. 
complicated. The structure and properties of the basins (including bifurcations) are examined. The formation of riddled basins of attraction is shown, but the bifurcation curves for the transverse destabilization of low-periodic orbits embedded in the chaotic attractor are too difficult to be determined. For one-band, and two-band chaotic attractors we follow the changes in the attractor and its basin of attraction that take place under passage of the riddling and the blowout bifurcations. We illustrate the phenomenon of intermingled basins of attraction for a situation where the system has two coexisting one or two-band attractors, each displaying a riddled basin. We can see also two superpersistent chaotic transients in which the behavior initially resembles the chaotic motion before the blowout bifurcation.

4 Conclusion

It has been shown that there are chaotic attractors whose basins are such that any point in the basin has pieces of another attractor basin, this basin is "riddled" with holes. Here we consider the dynamics as a parameter is varied. Using a simple model, we obtain characteristic behaviors. Numerical tests are consistent, it seems that these results are universal for the class of coupled systems. We can show that riddling bifurcation have occurred.

References


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