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Three-point Boundary Value Problems for Second-Order Functional Differential Equations[1]

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Abstract

This paper is concerned with the existence of extreme solutions of the three-point boundary value problem for a class of second order functional differential equations. We introduce new concept of lower and upper solutions. By using the method of upper and lower solutions and monotone iterative technique, we obtain the existence of extreme solutions.

Keywords: Functional differential equation; Monotone iterative technique; Three-point boundary value problem

Mathematics Subject Classification: Primary 34B37

1. Introduction

Theory of functional differential equations has became important aspect of differential equations. In recent years, many authors have paid attention to the research of boundary value problems for functional differential equations because of its potential applications, see [1,3,4,5,12]. In [7,8], J.J. Nieto and R. Rodriguez-Lopez introduced a new concept of lower and upper solutions, they consider the periodic boundary value problems for the following first order functional differential equation

$$\begin{cases} u'(t) = g(t, u(t), u(\theta(t))),\ t \in [0, T], \\ u(0) = u(T). \end{cases} \tag{1.1}$$

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Similar method has already succeeded in employing to nonlinear impulsive integro-differential equations [13] and impulsive functional differential equations [2].

Motivated by [2,7,8,13], we consider three-point boundary value problems for the functional differential equation

$$\begin{cases} -u''(t) = f(t, u(t), u(\theta(t))), & t \in J = [0,1], \\ u(0) = 0, \quad u(1) = au(\eta), \end{cases} \tag{1.2}$$

where $f \in C(J \times R^2, R)$, $0 \le \theta(t) \le t,\ t \in J$, $0 < a \le 1,\ 0 < \eta < 1$.

When $\theta(t) = t$, equation (1.2) reduces to three-point boundary value problems for ordinary differential equation which has been studied in many papers, see [6,9,10,11]. To our knowledged, only a few paper paid attention to multi-point boundary value problems for the functional differential equations. Recently, Tadeusz [14] has discussed solvability of three- point boundary value problems for a class of second order ordinary differential equations with deviating arguments by using the monotone iterative technique. Our method is different from that of [14].

In this paper, we are concerned with the existence of extreme solutions for equation (1.2). The paper is organized as follows. In Section 2, we establish a comparison principle. In section 3, we first introduce new concept of lower and upper solutions, and then give a proof for the existence theorem related to a linear problem associated to equation (1.2). In Section 4, by using the method of upper and lower solutions and monotone iterative technique, we obtain the existence of extreme solutions for equation (1.2).

2. A comparison principle

Let $E = C(J, R) \cap C^2(J, R)$ with norm

$$\| u \|_E = \max\{| u |_0,\ | u' |_0\},$$

where $| u |_0 = \sup_{t\in J} | u(t) |$, $| u' |_0 = \sup_{t\in J} | u'(t) |$, then E is a Banach space.

In the following, we denote

$$c(t) = \frac{t(1-t)}{a\eta(1-\eta)}, \quad r = \frac{2}{a\eta(1-\eta)}$$

We now present main results of this section.

Theorem 2.1 *Assume that* $u \in E$ *satisfies*

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) \le 0, & t \in J, \\ u(0) \le 0, \quad u(1) \le au(\eta), \end{cases} \tag{2.1}$$

where $0 \le a \le 1,\ 0 < \eta < 1,$ *constants* $M,\ N$ *such that*

$$M + N \le 2,\ \text{if } N > 0,\ M > 0, \tag{2.2}$$

$$M + N > 0, \quad \text{if } N \le 0. \tag{2.3}$$

Then $u(t) \le 0$ *for* $t \in J$.

Proof Suppose, to the contrary, that $u(t) > 0$ for some $t \in J$. Then from boundary conditions, we have that there exists $t^* \in (0,1)$ such that

$$u_0 = u(t^*) = \max_{t \in J} u(t) > 0, \tag{2.4}$$

$$u'(t^*) = 0, \quad u''(t^*) \le 0. \tag{2.5}$$

We consider the following two cases.

(1) $N \le 0$. From the first inequality of (2.1), (2.4) and (2.5), we have

$$0 \ge u''(t^*) \ge Mu(t^*) + Nu(\theta(t^*)) \ge (M+N)u_0,$$

which contradicts (2.3) and (2.4).

(2) $N > 0$. Suppose that $u(t) \ge 0$ for $t \in J$. (2.1) implies that $u(0) = 0$ and $u''(t) \ge 0$ for $t \in J$. From $u(0) = 0$ and $u(t) \ge 0$ for $t \in J$, we get that $u'(0) \ge 0$. Therefore, $y'(t) \ge y'(0) \ge 0$. It follows that $u(1) = \max_{t \in J} u(t) > 0$.

If $a = 1$, then $u(1) \le u(\eta) \le u(1)$. It follows that $u(t) \equiv c > 0$(c is a constant) for $t \in [\eta, 1]$. Let $t \in [\eta, 1]$, from the first inequality of (2.1), we obtain that $Mc \le 0$. This is a contradiction.

If $0 < a < 1$, then it is easy to obtain that $u(\eta) > u(1)$, which contradicts $u(1) = \max_{t \in J} u(t)$.

If $a = 0$, then $u(1) \le 0$, which contradicts $u(1) > 0$.

Suppose that there exists a $t_1,\ t_2 \in J$ such that $u(t_1) > 0$ and $u(t_2) < 0$. Let $t_* \in [0, t^*)$ such that $u(t_*) = \min_{t \in [0,t^*)} u(t) \le 0$. From the first inequality of (2.1), we have

$$u''(t) \ge (M+N)u(t_*), \quad t \in [0, t^*).$$

Integrating the above inequality from $s(t_* \le s \le t^*)$ to t^*, we obtain

$$-u'(s) \ge (t^* - s)(M+N)u(t_*), \quad t_* \le s \le t^*,$$

and then integrate from t_* to t^* to obtain

$$\begin{aligned} -u(t_*) &< u(t^*) - u(t_*) \\ &\le \int_{t_*}^{t^*} (s - t^*)(M+N)u(t_*)ds \\ &\le -\frac{M+N}{2} u(t_*)(t^* - t_*)^2 \\ &\le -\frac{M+N}{2} u(t_*). \end{aligned}$$

Hence,

$$u(t_*)(2-M-N)>0.$$

This is a contradiction. The proof is complete.

Corollary 2.1 *Assume that $u\in E$ satisfies*

$$\begin{cases} -u''(t)+Mu(t)+Nu(\theta(t))+[Mc(t)+Nc(\theta(t))+r](u(1)-au(\eta))\le 0, & t\in J,\\ u(0)\le 0, \quad u(1)>au(\eta), \end{cases}$$

where $0<a\le 1,\ 0<\eta<1$, constants $M,\ N$ satisfying (2.2) or (2.3), then $u(t)\le 0$ for $t\in J$.

Proof Put

$$y(t)=u(t)+c(t)(u(1)-au(\eta)),\quad t\in J,$$

then $y(t)\ge u(t)$ for all $t\in J$. Noting that $y''(t)=u''(t)-r(u(1)-au(\eta))$, $t\in J$, we have

$$\begin{aligned} -y''(t)+My(t)+Ny(\theta(t)) &= -u''(t)+Mu(t)+Nu(\theta(t))\\ &\quad +[Mc(t)+Nc(\theta(t))+r](u(1)-au(\eta))\le 0, \end{aligned}$$

$$y(0)=u(0)\le 0,$$

$$ay(\eta)=au(\eta)+ac(\eta)(u(1)-au(\eta))=u(1)=y(1).$$

Hence by Theorem 2.1, $y(t)\le 0$ for all $t\in J$, which implies that $u(t)\le 0$ for $t\in J$. This ends the proof.

3. Linear problem

In this section, we first give the following definition.

Definition 3.1 A function $\alpha\in E$ is called a lower solution of equation (1.2) if

$$\begin{cases} -\alpha''(t)\le f(t,\alpha(t),\alpha(\theta(t)))-a(t), & t\in J,\\ \alpha(0)\le 0, \end{cases}$$

where

$$a(t)=\begin{cases} 0, & \alpha(1)\le a\alpha(\eta),\\ (Mc(t)+Nc(\theta(t))+r)(\alpha(1)-a\alpha(\eta)), & \alpha(1)>a\alpha(\eta). \end{cases}$$

Definition 3.2 A function $\beta\in E$ is called an upper solution of equation (1.2) if

$$\begin{cases} -\beta''(t)\ge f(t,\beta(t),\beta(\theta(t)))+b(t), & t\in J,\\ \beta(0)\ge 0, \end{cases}$$

where

$$b(t)=\begin{cases} 0, & \beta(1)\ge a\beta(\eta),\\ (Mc(t)+Nc(\theta(t))+r)(a\beta(\eta)-\beta(1)), & \beta(1)<a\beta(\eta). \end{cases}$$

Theorem 3.1 *Let $\sigma \in C(J)$, $0 < a \leq 1$, $0 < \eta < 1$, constants M, N satisfying (2.2) or (2.3), and $M + |N| < 8$. Consider the problem*

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J, \\ u(0) = 0, \quad u(1) = au(\eta). \end{cases} \tag{3.1}$$

Suppose that there exist $\alpha, \beta \in E$ such that
(h_1) $\alpha \leq \beta$ on J.
(h_2)

$$\begin{cases} -\alpha''(t) + M\alpha(t) + N\alpha(\theta(t))) \leq \sigma(t) - a^*(t), \ t \in J, \\ \alpha(0) \leq 0, \end{cases}$$

where

$$a^*(t) = \begin{cases} 0, & \alpha(1) \leq a\alpha(\eta), \\ (Mc(t) + Nc(\theta(t)) + r)(\alpha(1) - a\alpha(\eta)), & \alpha(1) > a\alpha(\eta). \end{cases}$$

(h_3)

$$\begin{cases} -\beta''(t) + M\beta(t) + N\beta(\theta(t))) \geq \sigma(t) + b^*(t), \quad t \in J, \\ \beta(0) \geq 0, \end{cases}$$

where

$$b^*(t) = \begin{cases} 0, & \beta(1) \geq a\beta(\eta), \\ (Mc(t) + Nc(\theta(t)) + r)(a\beta(\eta) - \beta(1)), & \beta(1) < a\beta(\eta). \end{cases}$$

Then, there exists a unique solution u for problem (3.1). Moreover, $\alpha \leq u \leq \beta$

Proof We first show that the solution of equation (3.1) is unique. Let $u_1,\ u_2$ be the solution of (3.1) and set $v = u_1 - u_2$. Thus

$$\begin{cases} -v''(t) + Mv(t) + Nv(\theta(t)) = 0, \quad t \in J, \\ v(0) = 0, \quad v(1) = av(\eta). \end{cases}$$

By Theorem 2.1, we have that $v \leq 0$ for $t \in J$, that is, $u_1 \leq u_2$ on J. Similarly, one can obtain $u_2 \leq u_1$ on J. Hence $u_1 = u_2$.

Next, we prove that if u is a solution of equation (3.1), then $\alpha \leq u \leq \beta$. Let $m = \alpha - u$.

If $\alpha(1) \leq a\alpha(\eta)$, then $a^*(t) = 0$ on J. So we have

$$\begin{cases} -m''(t) + Mm(t) + Nm(\theta(t)) \leq 0, \quad t \in J, \\ m(0) \leq 0, \quad m(1) \leq am(\eta). \end{cases}$$

By Theorem 2.1, we have that $m = \alpha - u \leq 0$ on J.

If $\alpha(1) > a\alpha(\eta)$, then $a^*(t) = (Mc(t) + Nc(\theta(t)) + r)(\alpha(1) - a\alpha(\eta))$. Thus

$$\begin{aligned} -m''(t) + Mm(t) + Nm(\theta(t)) &= -\alpha''(t) + M\alpha(t) + N\alpha(\theta(t)) \\ &\quad +u''(t) - Mu(t) - Nu(\theta(t))) \\ &\leq \sigma(t) - a^*(t) - \sigma(t) \\ &\leq -a^*(t) \\ &= -(Mc(t) + Nc(\theta(t)) + r)(m(1) - am(\eta)). \end{aligned}$$

It is easy to see that $m(0) \leq 0, \quad m(1) > am(\eta)$. By Corollary 2.1, we have that $m = \alpha - u \leq 0$ on J. Analogously, $u \leq \beta$ on J.

Finally, we show that equation (3.1) has a solution by five step as follows.

Step 1 Let

$$\bar{\alpha}(t) = \begin{cases} \alpha(t), & \alpha(1) \leq a\alpha(\eta), \\ \alpha(t) + c(t)[\alpha(1) - a\alpha(\eta)], & \alpha(1) > a\alpha(\eta). \end{cases}$$

$$\bar{\beta}(t) = \begin{cases} \beta(t), & \beta(1) \geq a\beta(\eta), \\ \beta(t) - c(t)[a\beta(\eta) - \beta(1)], & \beta(1) < a\beta(\eta). \end{cases}$$

We shall show that $\bar{\alpha}$, $\bar{\beta}$ are the lower and upper solutions of (3.1) respectively, and

$$\alpha \leq \bar{\alpha} \leq \bar{\beta} \leq \beta, \quad \text{for } t \in J. \tag{3.2}$$

Obviously, $\alpha(0) = \bar{\alpha}(0),\ \alpha(1) = \bar{\alpha}(1),\ \beta(0) = \bar{\beta}(0),\ \beta(1) = \bar{\beta}(1)$ and

$$a\bar{\alpha}(\eta) = \begin{cases} a\alpha(\eta), & \alpha(1) \leq a\alpha(\eta), \\ \alpha(1), & \alpha(1) > a\alpha(\eta). \end{cases}$$

$$a\bar{\beta}(\eta) = \begin{cases} a\beta(\eta), & \beta(1) \geq a\beta(\eta), \\ \beta(1), & \beta(1) < a\beta(\eta). \end{cases}$$

Hence

$$\bar{\alpha}(0) \leq 0,\ \bar{\alpha}(1) \leq a\bar{\alpha}(\eta), \tag{3.3}$$

$$\bar{\beta}(0) \geq 0,\ \bar{\beta}(1) \geq a\bar{\beta}(\eta). \tag{3.4}$$

If $\alpha(1) \leq a\alpha(\eta)$, then $\alpha = \bar{\alpha}$ on J. So

$$-\bar{\alpha}''(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t))) \leq \sigma(t),\ t \in J.$$

If $\alpha(1) > a\alpha(\eta)$, then $\bar{\alpha}(t) = \alpha(t) + c(t)[\alpha(1) - a\alpha(\eta)]$. Thus

$$\begin{aligned} -\bar{\alpha}''(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) &= -\alpha''(t) + M\alpha(t) + N\alpha(\theta(t)) \\ &\quad +(Mc(t) + Nc(\theta(t)) + r)(\alpha(1) - a\alpha(\eta)) \\ &\leq \sigma(t). \end{aligned}$$

Combining the above two cases and (3.3), we see that $\bar{\alpha}$ is a lower solution of (3.1). Similarly, $\bar{\beta}$ is an upper solution of (3.1).

It is easy to see that $\alpha \le \bar{\alpha},\ \bar{\beta} \le \beta$ on J. We show that $\bar{\alpha} \le \bar{\beta}$ on J. We need consider the following four cases.

Case 1 $\alpha(1) \le a\alpha(\eta)$ and $\beta(1) \ge a\beta(\eta)$.

Case 2 $\alpha(1) \le a\alpha(\eta)$ and $\beta(1) < a\beta(\eta)$.

Case 3 $\alpha(1) > a\alpha(\eta)$ and $\beta(1) \ge a\beta(\eta)$.

Case 4 $\alpha(1) > a\alpha(\eta)$ and $\beta(1) < a\beta(\eta)$.

Here we only consider Case 4. Let $m = \bar{\alpha} - \bar{\beta}$ for $t \in J$, then $m(0) \le 0$, $m(1) \le am(\eta)$ and

$$\begin{aligned}
-m''(t) + Mm(t) + Nm(\theta(t)) &= -\bar{\alpha}''(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) \\
&\quad + \bar{\beta}''(t) - M\bar{\beta}(t) - N\bar{\beta}(\theta(t)) \\
&= -\alpha''(t) + M\alpha(t) + N\alpha(\theta(t)) - a^*(t) \\
&\quad + \beta''(t) - M\beta(t) - N\beta(\theta(t))) + b^*(t) \\
&\le \sigma(t) - \sigma(t) \le 0.
\end{aligned}$$

By Theorem 2.1, we obtain that $m \le 0$ on J, that is, $\bar{\alpha} \le \bar{\beta}$ on J. Thus (3.2) holds.

Step 2 Put $\lambda \in [a\bar{\alpha}(\eta),\ a\bar{\beta}(\eta)]$, we consider the equation

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J, \\ u(0) = 0, \quad u(1) = \lambda. \end{cases} \tag{3.5}$$

Next, we show the equation (3.5) has a unique solution $u(t, \lambda)$.

It is easy to check that equation (3.5) is equivalent to the integral equation

$$u(t) = \lambda t + \int_0^1 G(t,s)[\sigma(s) - Mu(s) - Nu(\theta(s))]ds$$

where

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Define a mapping $A : C(J) \to C(J)$ by

$$Au(t) = \lambda t + \int_0^1 G(t,s)[\sigma(s) - Mu(s) - Nu(\theta(s))]ds.$$

For any $x,\ y \in C(J)$, we have

$$(Ax)(t) - (Ay)(t) = \int_0^1 G(t,s)[M(y(s) - x(s)) + N(y(\theta(s)) - x(\theta(s)))]ds.$$

Noting that $\max_{t\in J}\int_0^1 G(t,s)ds = 1/8$, $0 < M + |N| < 8$, it is easy to verify that $A : C(J) \to C(J)$ is a contraction mapping. Thus there exists a $u \in C(J)$ such that $Au = u$. The equation (3.5) has a unique solution.

Step 3 We show that for any $t \in J$, the unique solution $u(t,\lambda)$ of (3.5) is continuous in λ. Let $u(t,\lambda_1)$, $u(t,\lambda_2)$ be the solution of

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J, \\ u(0) = 0, \quad u(1) = \lambda_1. \end{cases} \tag{3.6}$$

and

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J, \\ u(0) = 0, \quad u(1) = \lambda_2, \end{cases} \tag{3.7}$$

respectively. Then

$$u(t,\lambda_i) = \lambda_i t + \int_0^1 G(t,s)[\sigma(s) - Mu(s,\lambda_i) - Nu(\theta(s),\lambda_i)]ds, \quad i = 1,2. \tag{3.8}$$

From (3.8), we have

$$\begin{aligned} |u(t,\lambda_1) - u(t,\lambda_2)|_0 &\le \;| \lambda_1 - \lambda_2 | + (M + |N|)|u(t,\lambda_1) - u(t,\lambda_2)|_0 \max_{t\in J}\int_0^1 G(t,s)ds \\ &\le \;| \lambda_1 - \lambda_2 | + \frac{M + |N|}{8}|u(t,\lambda_1) - u(t,\lambda_2)|_0. \end{aligned}$$

Hence

$$|u(t,\lambda_1) - u(t,\lambda_2)|_0 \le \frac{8}{8 - M - |N|}\, | \lambda_1 - \lambda_2 | \,.$$

Step 4 We show that

$$\bar{\alpha}(t) \le u(t,\lambda) \le \bar{\beta}(t) \tag{3.9}$$

for any $t \in J$ and $\lambda \in [a\bar{\alpha}(\eta),\ a\bar{\beta}(\eta)]$, where $u(t,\lambda)$ is unique solution of (3.5).

Let $m(t) = \bar{\alpha}(t) - u(t,\lambda)$. From $\lambda \in [a\bar{\alpha}(\eta),\ a\bar{\beta}(\eta)]$ and (3.3), we have $m(1) = \bar{\alpha}(1) - \lambda \le a\bar{\alpha}(\eta) - \lambda \le 0$ and $m(0) = \bar{\alpha}(0) \le 0$. And

$$\begin{aligned} -m''(t) + Mm(t) + Nm(\theta(t)) &= -\bar{\alpha}''(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) \\ &\quad + u''(t,\lambda) - Mu(t,\lambda) - Nu(\theta(t),\lambda) \\ &\le \sigma(t) - \sigma(t) \le 0. \end{aligned}$$

By Theorem 2.1, we obtain that $m \le 0$ on J, that is, $\bar{\alpha}(t) \le u(t,\lambda)$ on J. Similarly, $u(t,\lambda) \le \bar{\beta}(t)$ on J.

Step 5 Let

$$D = [a\bar{\alpha}(\eta),\ a\bar{\beta}(\eta)], \quad P(\lambda) = au(\eta,\lambda),$$

where $u(t,\lambda)$ is unique solution of (3.5). From step 4, we have

$$P(D) \subset D.$$

Since D is a compact convex set and P is continuous, it follows by Schaefer's fixed point theorem that P has a fixed point λ_0 in D such that $au(\eta, \lambda_0) = \lambda_0$. Obviously, $u(t, \lambda_0)$ is unique solution of (3.1). This ends the proof.

4. Main results

Our main result is the following theorem.

Theorem 4.1 *Suppose that* $0 < a \leq 1,\ 0 < \eta < 1$ *and the following conditions are satisfied*

(i) $\alpha,\ \beta$ *are the lower and upper solutions for boundary value problem (1.1) respectively,* $\alpha(t) \leq \beta(t)$ *on* J.

(ii) There exists $0 < a \leq 1,\ 0 < \eta < 1$, *constants* $M,\ N:\ M + |N| < 8$ *satisfying (2.2) or (2.3) such that*

$$f(t, x, y) - f(t, \bar{x}, \bar{y}) \geq -M(x - \bar{x}) - N(y - \bar{y}),$$

for $\alpha(t) \leq \bar{x} \leq x \leq \beta(t),\ \alpha(\theta(t)) \leq \bar{y} \leq y \leq \beta(\theta(t)),\ t \in J$.

Then, there exist monotone sequence $\{\alpha_n\},\ \{\beta_n\}$ *with* $\alpha_0 = \alpha,\ \beta_0 = \beta$ *such that* $\lim_{n\to\infty} \alpha_n(t) = \rho(t),\ \lim_{n\to\infty} \beta_n(t) = r(t)$ *uniformly on* J, *and* $\rho,\ r$ *are the minimal and the maximal solutions of (1.2) respectively, such that*

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \alpha_n \leq \rho \leq x \leq r \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0$$

on J, *where* x *is any solution of (1.2) such that* $\alpha(t) \leq x(t) \leq \beta(t)$ *on* J.

Proof Let $[\alpha, \beta] = \{u \in E : \alpha(t) \leq u(t) \leq \beta(t), t \in J\}$. For any $\gamma \in [\alpha, \beta]$, we consider the equation

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) = f(t, \gamma(t), \gamma(\theta(t))) + M\gamma(t) + N\gamma(\theta(t)), & t \in J, \\ u(0) = 0, \quad u(1) = au(\eta). \end{cases} \tag{4.1}$$

Since α is a lower solution of (1.2), from (ii), we have that and

$$\begin{aligned} -\alpha''(t) + M\alpha(t) + N\alpha(\theta(t))) &\leq f(t, \alpha(t), \alpha(\theta(t))) + M\alpha(t) + N\alpha(\theta(t))) - a(t) \\ &\leq f(t, \gamma(t), \gamma(\theta(t))) + M\gamma(t) + N\gamma(\theta(t)) - a^*(t) \\ \alpha(0) &\leq 0. \end{aligned}$$

Similarly, we have that

$$\begin{aligned} -\beta''(t) + M\beta(t) + N\beta(\theta(t))) &\geq f(t, \gamma(t), \gamma(\theta(t))) + M\gamma(t) + N\gamma(\theta(t)) + b^*(t) \\ \beta(0) &\geq 0, \end{aligned}$$

where $a^*,\ b^*$ are defined in Theorem 3.1.

By Theorem 3.1, the equation (4.1) has a unique solution $u \in E$. We define an operator A by $u = A\gamma$, then A is an operator from $[\alpha, \beta]$ to E.

We shall show that

(a) $\alpha_0 \le A\alpha_0, \quad A\beta_0 \le \beta_0$.

(b) A is nondecreasing in $[\alpha_0, \beta_0]$.

To prove (a), let $p = \alpha_0 - \alpha_1$, where $\alpha_1 = A\alpha_0$. We finish (a) by two cases.

Case 1. $\alpha_0(1) \le a\alpha_0(\eta)$, then

$$-\alpha''(t) \le f(t, \alpha(t), \alpha(\theta(t))).$$

As α_0 is a lower solution of (1.1), then for $t \in J$,

$$\begin{aligned} -p''(t) + Mp(t) + Np(\theta(t))) &\le -\alpha_0''(t) + \alpha_1''(t) + M\alpha_0(t) - M\alpha_1(t) \\ &\quad +N\alpha_0(\theta(t))) - N\alpha_1(\theta(t))) \\ &\le f(t, \alpha_0(t), \alpha_0(\theta(t))) - f(t, \alpha_0(t), \alpha_0(\theta(t))) \\ &\le 0. \end{aligned}$$

It is easy to verify that

$$p(0) \le 0, \quad p(1) \le ap(\eta).$$

By Theorem 2.1, $p(t) \le 0$, which implies $\alpha_0 \le A\alpha_0$.

Case 2. $\alpha_0(1) > a\alpha_0(\eta)$, which implies that

$$a(t) = [Mc(t) + Nc(\theta(t)) + r](\alpha_0(1) - a\alpha_0(\eta)).$$

Hence, $p(0) \le 0,\ p(1) > ap(\eta)$ and

$$\begin{aligned} -p''(t) + Mp(t) + Np(\theta(t))) &\le -\alpha_0''(t) + \alpha_1''(t) + M\alpha_0(t) - M\alpha_1(t) \\ &\quad +N\alpha_0(\theta(t))) - N\alpha_1(\theta(t))) \\ &\le f(t, \alpha_0(t), \alpha_0(\theta(t))) - a(t) - f(t, \alpha_0(t), \alpha_0(\theta(t))) \\ &\le -a(t) \\ &= -(Mc(t) + Nc(\theta(t)) + r)(p(1) - ap(\eta)). \end{aligned}$$

It follows by Corollary 2.1 that $p(t) \le 0$, which implies $\alpha_0 \le A\alpha_0$. Similarly, $A\beta_0 \le \beta_0$.

To prove (b). We show that $A\mu_1 \le A\mu_2$ if $\alpha_0 \le \mu_1 \le \mu_2 \le \beta_0$.

Let $\mu_1^* = A\mu_1,\ \mu_2^* = A\mu_2$ and $p = \rho_1^* - \rho_2^*$, then by (ii), we have

$$\begin{aligned} -p''(t) + Mp(t) + Np(\theta(t))) &= f(t, \mu_1(t), \mu_1(\theta(t))) + M\mu_1(t) + N\mu_1(\theta(t)) \\ &\quad -f(t, \mu_2(t), \mu_2(\theta(t))) - M\mu_2(t) - N\mu_2(\theta(t)) \\ &\le 0. \end{aligned}$$

And

$$p(0) = 0, \quad p(1) = ap(\eta).$$

By Theorem 2.1, $p(t) \le 0$, which implies $A\rho_1 \le A\rho_2$.

It is easy to define the sequence $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\alpha_{n+1} = A\alpha_N$, $\beta_{n+1} = A\beta_n$ for $n = 0, 1, 2, \cdots$. From (a) and (b), we have

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \alpha_n \le \beta_n \le \cdots \le \beta_2 \le \beta_1 \le \beta_0$$

on $t \in J$, and each α_n, $\beta_n \in E$ satisfies

$$\begin{cases} -\alpha_n''(t) + M\alpha_n(t) + N\alpha_n(\theta(t)) = f(t, \alpha_{n-1}(t), \alpha_{n-1}(\theta(t))) \\ \qquad\qquad +M\alpha_{n-1}(t) + N\alpha_{n-1}(\theta(t)), \quad t \in J, \\ \alpha_n(0) = 0, \quad \alpha_n(1) = a\alpha_n(\eta). \end{cases}$$

$$\begin{cases} -\beta_n''(t) + M\beta_n(t) + N\beta_n(\theta(t)) = f(t, \beta_{n-1}(t), \beta_{n-1}(\theta(t))) \\ \qquad\qquad +M\beta_{n-1}(t) + N\beta_{n-1}(\theta(t)), \quad t \in J, \\ \beta_n(0) = 0, \quad \beta_n(1) = a\beta_n(\eta). \end{cases}$$

Therefore there exist ρ, r such that $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly on J. Clearly, ρ, r are solutions of (1.2).

Finally, we prove that if $x \in [\alpha_0, \beta_0]$ is any solution of (1.2), then $\rho(t) \le x(t) \le r(t)$ on J. To this end, we assume, without loss of generality, that $\alpha_n(t) \le x(t) \le \beta_n(t)$ for some n. Since $\alpha_0(t) \le x(t) \le \beta_0(t)$, from property (b), we can get

$$\alpha_{n+1}(t) \le x(t) \le \beta_{n+1}(t), \quad t \in J.$$

Hence we can conclude that

$$\alpha_n(t) \le x(t) \le \beta_n(t), \quad \text{for all } n,$$

Passing to the limit as $n \to \infty$, we obtain

$$\rho(t) \le x(t) \le r(t), \quad t \in J.$$

This ends the proof.

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