Harmonic Morphisms from
Conformally Flat Spaces

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Abstract

In this note we give a method to construct non-trivial harmonic morphisms via conformal change of the metric of the domain generalizing a theorem previously only known in the case of start manifold to be an open subset of \( \mathbb{C}^2 \). As its application, we manufacture harmonic morphisms from conformally flat spaces.

Keywords: horizontally conformal map, harmonic morphism, conformally flat space

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1 Preliminaries

Harmonic morphisms between Riemannian manifolds are mappings, which preserve solutions of Laplace’s equation. They form a special class of harmonic maps, namely those that are horizontally conformal.

Call a smooth map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds is horizontally (weakly) conformal if for any point \( x \in M \) which is not contained in the critical set \( C_\phi = \{ x \in M \mid d\phi_x = 0 \} \) of \( \phi \), the restriction of \( d\phi_x \) to the orthogonal complement

\[
\mathcal{H}_x = \{ X \in T_xM \mid g(X, Y) = 0 \text{ for all } Y \in \text{Ker } d\phi_x \}
\]

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of $\text{Ker } d\phi_x$ is surjective and conformal onto the tangent space $T_{\phi(x)}N$.

Recall that a smooth map $f : M \to N$ between Riemannian manifold is harmonic if and only if it has vanishing tension field, equivalently, it is a critical point of its energy functional [1].

A smooth map $f : M \to N$ between Riemannian manifold is called a harmonic morphism if for any harmonic function $\psi : U \to \mathbb{R}$ defined on an open subset $U$ of $N$ with $f^{-1}(U)$ non-empty, $\psi \circ f : f^{-1}(U) \to \mathbb{R}$ is a harmonic function. The reader is referred to [2] for a detailed account of harmonic morphisms. Harmonic morphisms can be characterized as follows:

**Theorem 1.1** ([2, 3]) A map $\phi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.

### 2 Harmonic morphisms with respect to a conformally altered metric

In this section we extract a sufficient condition for $\varphi : (M^{2n}, e^{2n}g) \to (N^2, h)$ to be harmonic.

Recall that an almost Hermitian manifold $(M, g, J)$ with Kähler form $\omega$, is said to be cosymplectic if $d^*\omega = 0$ or equivalently, $\text{div} J = 0$.

**Theorem 2.1.** Assume that $(M^{2n}, g, J)$ is a cosymplectic manifold and that $\eta$ is a real valued function defined in $M^{2n}$. If $\varphi$ is a holomorphic map from $M^{2n}$ into some Riemann surface $(N^2, h, J^N)$ satisfying $d\varphi(\text{grad} \eta) = 0$, then $\varphi : (M^{2n}, e^{2n}g) \to (N^2, h)$ is a harmonic morphism.

**Proof.** The well-known result by Lichnerowicz tells us that holomorphic map from a cosymplectic manifold to a $(1, 2)$-symplectic manifold is harmonic [5]. Note that an almost Hermitian manifold with Kähler form $\omega$, is said to be $(1, 2)$-symplectic if the $(1, 2)$-part of $d\omega$ vanishes, and any Riemann surface is automatically $(1, 2)$-symplectic.

Recall from the Cauchy-Riemann equations that any holomorphic map from an almost Hermitian manifold to a Riemann surface is horizontally weakly conformal. Combining this with Lichnerowicz’s result and Fuglede-Ishihara’ characterization [2, 3], we obtain that a holomorphic map $\varphi$ from cosymplectic manifold $(M^{2n}, g, J)$ to Riemann surface $(N^2, h, J^N)$ is a harmonic morphism. Setting $\tilde{g} = e^{2n}g$. It is easy to verify that $\varphi : (M^{2n}, \tilde{g}) \to (N^2, h)$ is horizontally weakly conformal. Moreover, by Theorem 5.1 of [8], $\varphi : (M^{2n}, \tilde{g}) \to$
(N^2, h) is harmonic if and only if \(\text{grad} \left[ (e^{-\eta})^{n-2} \right] \) is vertical on \(M \setminus C_\varphi\) where \(\text{grad}\) denotes the gradient of function. This is equivalent to \(d\varphi(\text{grad}\eta) = 0\). On the other hand, if \(x\) is a interior point of \(C_\varphi\), then there is an open subset \(U\) of \(M\), such that \(x \in U \subset C_\varphi\), and \(\tau(\varphi)(x) = \text{Trace}_\tilde{g} \nabla d\varphi(x) = 0\) where \(\tau(\varphi)\) is the tension field with respect to \(\tilde{g}\). Suppose that \(x\) is a condensation point of \(C_\varphi\). Then there exists a sequence \(x_j\) \((j = 1, 2, \cdots)\) on \(M \setminus C_\varphi\) such that \(\lim_{j \to +\infty} x_j = x\). Because \(\tau(\varphi)\) is a smooth field along \(\varphi\), we get

\[0 = \lim \tau(\varphi)(x_j) = \tau(\varphi)(x)\]

To sum up, we have \(\tau(\varphi) = 0\), hence, \(\varphi\) is harmonic if \(d\varphi(\text{grad}\eta) = 0\). Therefore \(\varphi\) is a harmonic morphism by Fuglede-Ishihara’ result [2, 3]. \(\square\)

**Remark 2.2** Theorem 2.1 is a natural extension of Theorem 3.1 of [10].

### 3 Harmonic morphisms from conformly flat spaces

Two Riemannian metrics \(g\) and \(\bar{g}\) on \(M\) are said to be **conformally equivalent**, if there exists a function \(\psi\) on \(M\) such that \(\bar{g} = e^{2\psi} g\). A map \(\varphi : (M, g) \to (N, h)\) between Riemannian manifolds is said to be **conformal** if there exists a function \(\psi\) on \(M\) such that \(\varphi^* h = e^{2\psi} g\). Two Riemannian manifolds \((M, g)\) and \((N, h)\) are said to be **conformally diffeomorphic**, if there exists a conformal diffeomorphism \(\varphi : (M, g) \to (N, h)\). An \(n\)-dimensional Riemannian manifold \((M, g)\) is called a **conformal flat space** if for any point of \(M\) there is a neighborhood which is conformally diffeomorphic to the Euclidean space \(R^n\).

We shall construct a harmonic morphism from \(\mathbb{R}^{2m}\), with a suitable conformally flat metric, to \(\mathbb{R}^2\). Let \(k_1, \cdots, k_m\) be non-negative integers which are not all zero, and let \(\varphi : \mathbb{R}^{2m} \to \mathbb{R}^2\) be the polynomial map, homogeneous of degree \(k_1 + \cdots + k_m\), defined in complex coordinates by

\[
\varphi(z) = z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m} \quad (3.1)
\]

\((z = (z_1, \cdots, z_m) \in \mathbb{C} \times \cdots \times \mathbb{C} = \mathbb{R}^{2m})\)

For any \(i\), \(\frac{\partial \varphi}{\partial \bar{z}_i} = 0\) implies that \(\varphi\) is holomorphic.

Consider real valued function in \(\mathbb{C}^m\). Then

\[
\text{grad}\eta = \Sigma_i \left( \frac{\partial \eta}{\partial \bar{z}_i} \frac{\partial}{\partial \bar{z}_i} + \frac{\partial \eta}{\partial z_i} \frac{\partial}{\partial z_i} \right).
\]
Note that \( \varphi \) is holomorphic,
\[
d\varphi(\text{grad} \eta) = \sum_i \left( \frac{\partial \eta}{\partial z_i} \frac{\partial \varphi}{\partial \bar{z}_i} + \frac{\partial \eta}{\partial \bar{z}_i} \frac{\partial \varphi}{\partial z_i} \right)
\]
\[
= \sum_i \frac{\partial \eta}{\partial z_i} \frac{\partial \eta}{\partial z_i}
\]
\[
= \left( \prod_{i=1}^m z_i^{k_i-1} \right) \left( \sum_{j=1}^m k_j z_1 \cdots \hat{z}_j \cdots z_m \frac{\partial \eta}{\partial \bar{z}_j} \right).
\]
Thus the equation \( d\varphi(\text{grad} \eta) = 0 \) is equivalent to
\[
\left( \prod_{i=1}^m z_i^{k_i-1} \right) \left( \sum_{j=1}^m k_j z_1 \cdots \hat{z}_j \cdots z_m \frac{\partial \eta}{\partial \bar{z}_j} \right) = 0.
\]
A solution to this equation is given by
\[
\eta(z) = \begin{cases} 
\sum_{i=1}^{m/2} (k_{2i} |z_{2i-1}|^2 - k_{2i-1} |z_{2i}|^2) & \text{if } m \text{ is even} \\
2k_3 |z_1|^2 - \frac{k_1}{2} (k_3 |z_2|^2 + k_2 |z_3|^2) + \sum_{i=2}^{(m-1)/2} (k_{2i+1} |z_{2i}|^2 - k_{2i} |z_{2i+1}|^2) & \text{if } m \text{ is odd.}
\end{cases} \tag{3.2}
\]
In fact, when \( m \) is even, then
\[
\eta(z_1, \cdots, z_m) = \sum_{i=1}^{m/2} (k_{2i} z_{2i-1} \bar{z}_{2i-1} - k_{2i-1} z_{2i} \bar{z}_{2i}).
\]
It follows that
\[
\frac{\partial \eta}{\partial \bar{z}_{2i-1}} = k_{2i} z_{2i-1}, \quad \frac{\partial \eta}{\partial z_{2i}} = -k_{2i-1} \bar{z}_{2i}.
\]
Thus we have
\[
\sum_{j=1}^m k_j z_1 \cdots \hat{z}_j \cdots z_m \frac{\partial \eta}{\partial \bar{z}_j} = \sum_{j=1}^{m/2} k_{2j} z_1 \cdots \hat{z}_{2j-1} \cdots z_m \frac{\partial \eta}{\partial \bar{z}_{2j-1}} + \sum_{j=1}^{m/2} k_{2j-1} z_1 \cdots \hat{z}_{2j-1} \cdots z_m \frac{\partial \eta}{\partial \bar{z}_{2j}}
\]
\[
= \sum_{j=1}^{m/2} k_{2j} z_1 \cdots \hat{z}_{2j-1} \cdots z_m (-k_{2j-1} z_{2j}) + \sum_{j=1}^{m/2} k_{2j-1} z_1 \cdots \hat{z}_{2j-1} \cdots z_m (k_{2j} z_{2j-1})
\]
\[
= -\sum_{j=1}^{m/2} k_{2j} k_{2j-1} \prod_{i=1}^m z_i + \sum_{j=1}^{m/2} k_{2j} k_{2j-1} \prod_{i=1}^m z_i = 0.
\]
It follows that \( d\varphi(\text{grad} \eta) = 0 \). If \( m \) is odd, then
\[
\eta(z_1, \cdots, z_m) = k_2 k_3 z_1 \bar{z}_1 - \frac{k_1}{2} (k_3 z_2 \bar{z}_2 + k_2 z_3 \bar{z}_3)
\]
\[
+ \sum_{i=2}^{(m-1)/2} (k_{2i+1} z_{2i} \bar{z}_{2i} - k_{2i-1} z_{2i+1} \bar{z}_{2i+1}).
\]
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It follows that
\[ \frac{\partial \eta}{\partial \z_1} = -k_2 k_3 z_1, \quad \frac{\partial \eta}{\partial \z_2} = -\frac{k_1 k_3}{2} z_2, \quad \frac{\partial \eta}{\partial \z_3} = -\frac{k_1 k_2}{2} z_3 \]
and when \( i \geq 2 \),
\[ \frac{\partial \eta}{\partial \z_{2i}} = k_{2i+1} z_{2i}, \quad \frac{\partial \eta}{\partial \z_{2i+1}} = -k_{2i} z_{2i+1}. \]

Thus we have
\[ \Sigma_{j=1}^{m} k_j \hat{z}_j \cdots \hat{z}_m \frac{\partial \eta}{\partial \hat{z}_j} = k_1 \hat{z}_2 \cdots \hat{z}_m \frac{\partial \eta}{\partial \hat{z}_1} + k_2 \hat{z}_1 \hat{z}_3 \cdots \hat{z}_m \frac{\partial \eta}{\partial \hat{z}_2} + k_3 \hat{z}_1 \hat{z}_2 \hat{z}_4 \cdots \hat{z}_m \frac{\partial \eta}{\partial \hat{z}_3} + \sum_{j=2}^{(m-1)/2} k_{2j} \hat{z}_1 \cdots \hat{z}_{2j} \frac{\partial \eta}{\partial \hat{z}_{2j}} + \sum_{j=2}^{(m-1)/2} k_{2j+1} \hat{z}_1 \cdots \hat{z}_{2j+1} \frac{\partial \eta}{\partial \hat{z}_{2j+1}} = 0. \]

We obtain that \( d\varphi(\text{grad} \eta) = 0 \) if \( m \) is odd.

By using Theorem 2.1, we obtain the following

**Proposition 3.1** Let \( \varphi : (\mathbb{R}^m, g_0) \to \mathbb{R}^2 \) be the polynomial map defined in (3.1) where \( g_0 \) is the standard Riemannian metric on \( \mathbb{R}^m \). Then \( \varphi \) is a harmonic morphism from conformally flat space \( (\mathbb{R}^m, e^\eta g_0) \) to \( \mathbb{R}^2 \) where \( \eta \) is defined in (3.2).

**Remark 3.2** For a different approach to the same problem where \( m = 2 \), using isoparametric functions, see [1], page 404.
References


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