Applied Mathematical Sciences, Vol. 1, 2007, no. 2, 59 - 67

The Characteristic Polynomial of Some Perturbed Tridiagonal k-Toeplitz Matrices[1]

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Abstract. We generalize some recent results on the spectra of tridiagonal matrices, providing explicit expressions for the characteristic polynomial of some perturbed tridiagonal k-Toeplitz matrices. The calculation of the eigenvalues (and associated eigenvectors) follows straightforward.

Mathematics Subject Classification: 15A18, 42C05, 33C45

Keywords: characteristic polynomial; eigenvalues; eigenvectors; tridiagonal k-Toeplitz matrices; orthogonal polynomials; Chebyshev polynomials of second kind

1. PRELIMINARIES

An $n \times n$ matrix $A = [a_{ij}]$ is said tridiagonal if $a_{ij} = 0$, whenever $|i-j| > 1$, i.e.,

$$A = \begin{pmatrix} a_1 & c_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix}, \tag{1.1.1}$$

with non-mentioned entries equal to zero. Since our aim will be the study of the characteristic polynomial of some tridiagonal matrices, the cases when $b_1 \cdots b_{n-1} = 0$ or $c_1 \cdots c_{n-1} = 0$ (i.e., when A is reducible) can be reduced to small order cases. In fact, throughout we will assume that $b_\ell c_\ell > 0$, for $\ell = 1, \ldots, n-1$.

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[1] This work was supported by CMUC (Centro de Matemática da Universidade de Coimbra).

Tridiagonal matrices have many applications on pure and applied mathematics, engineering, physics, etc. Since the study of the eigenvalues of these kind of matrices is somehow elementary, they have been thoroughly studied on several fields for a long time, yet with technics and purposes quite different with researchers ignoring each others.

There is a very special kind of these matrices, known as tridiagonal k-Toeplitz matrices.

Definition 1.1. Given two positive integer numbers k and N such that $k \leq N$, a *tridiagonal k-Toeplitz matrix* is a matrix of the form

$$\begin{pmatrix} a_1 & c_1 & & & & & & & \\ b_1 & \ddots & \ddots & & & & & & \\ & \ddots & a_k & c_k & & & & & \\ & & b_k & a_1 & c_1 & & & & \\ & & & b_1 & \ddots & \ddots & & & \\ & & & & \ddots & a_k & c_k & & \\ & & & & & b_k & a_1 & c_1 & \\ & & & & & & b_1 & \ddots & \ddots \\ & & & & & & & \ddots & \end{pmatrix} \in \mathbb{C}^{N \times N} . \tag{1.1.2}$$

Notice that if $k = 1$, then we have a tridiagonal Toeplitz matrix.

These particular kind of matrices has many applications such as in the so-called *chain models* on finding the solutions of a stationary Schrödinger equation (cf., e.g., [1, 8]), or in Maxwell's equations for guided waves of some fibers (cf. [9]). Our aim here is to study the characteristic polynomial and the eigenvalues of tridiagonal k-Toeplitz matrices, with some perturbations on the extreme entries $(1, 1)$ and (N, N), generalize some recent related results (cf., e.g., [10, 18]) and bring them together in one place. Our approach is based on orthogonal polynomials theory. We provide some numerical examples.

2. Orthogonal polynomials

An orthogonal polynomial sequence (OPS) $\{P_n\}_{n\geq 0}$ is characterized by a three-term recurrence relation

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) , \quad n = 0, 1, 2, \ldots \tag{2.2.1}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = \text{const.} \neq 0$, where $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are sequences of complex numbers such that $\alpha_n\gamma_{n+1} \neq 0$ for all $n = 0, 1, 2, \ldots$. We can give a matrix form to this three-term recurrence relation:

$$x \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} = J_{n+1} \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} + \alpha_n P_{n+1}(x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} ,$$

where J_{n+1} is a tridiagonal matrix of order $n+1$, defined by

$$J_{n+1} := \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \gamma_1 & \beta_1 & \alpha_1 & & \\ & \gamma_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \alpha_{n-1} \\ & & & \gamma_n & \beta_n \end{pmatrix} , \qquad \text{for } n = 0, 1, 2, \dots .$$

Hence each zero of the polynomial P_n, say λ_{nj} (for $j = 1, \dots , n$), is an eigenvalue of the corresponding tridiagonal matrix J_n of order n, and an associated eigenvector is

$$(2.2.2) \qquad (P_0(\lambda_{nj}), P_1(\lambda_{nj}), \dots , P_{n-1}(\lambda_{nj}))^t .$$

The (monic) characteristic polynomial of J_n is precisely P_n, with $\alpha_n = 1$, i.e.,

$$P_n(x) = \det (xI_n - J_n) \ , \quad n = 1, 2, \dots ,$$

where I_n denotes the identity matrix of order n.

It is important to point out that, since we are interested on the eigenvalues of tridiagonal matrices, the eigenvalues of J_n are the same of the matrix

$$\begin{pmatrix} \beta_0 & 1 & & & \\ \gamma_1\alpha_0 & \beta_1 & 1 & & \\ & \gamma_2\alpha_1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \gamma_{n-1}\alpha_{n-2} & \beta_{n-1} \end{pmatrix}$$

or

$$\begin{pmatrix} \beta_0 & \sqrt{\gamma_1\alpha_0} & & & \\ \sqrt{\gamma_1\alpha_0} & \beta_1 & \sqrt{\gamma_2\alpha_1} & & \\ & \sqrt{\gamma_2\alpha_1} & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\gamma_{n-1}\alpha_{n-2}} \\ & & & \sqrt{\gamma_{n-1}\alpha_{n-2}} & \beta_{n-1} \end{pmatrix} .$$

Since all these matrices have the same characteristic polynomial, we will simply consider

$$\tilde{J}_n = \begin{pmatrix} \beta_0 & 1 & & & \\ \gamma_1 & \beta_1 & 1 & & \\ & \gamma_2 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \gamma_{n-1} & \beta_{n-1} \end{pmatrix} ,$$

with positive γ_i's, according to our original assumption. It follows from this fact the uselessness of the separation in three "different" families of tridiagonal matrices made in [10].

3. CHEBYSHEV POLYNOMIALS OF SECOND KIND

The Chebyshev polynomials of second kind, denoted by $\{U_n\}_{n\geq 0}$, are a very useful example of a (monic) OPS. These polynomials satisfy the three-term recurrence relations

$$U_{n+1}(x) = xU_n(x) - U_{n-1}(x)\,, \quad \text{for all } n = 1, 2, \ldots \tag{3.3.1}$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. Since each U_n satisfies

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}\,, \quad \text{with } x = \cos\theta \quad (0 \leq \theta < \pi), \tag{3.3.2}$$

for all $n = 0, 1, 2\ldots$, we can deduce the orthogonality relations

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,\mathrm{d}x = \tfrac{\pi}{2}\,\delta_{n,m}$$

(cf. [2], e.g.). It is also well known the explicit formula for Chebyshev polynomials of second kind

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-)^k \frac{(n-k)!}{k!(n-2k)!}\,(2x)^{n-2k}.$$

Consider now the $n \times n$ real tridiagonal Toeplitz matrix T_n defined by

$$T_n = \begin{pmatrix} a & 1 & & \\ b & a & \ddots & \\ & \ddots & \ddots & 1 \\ & & b & a \end{pmatrix} \in \mathbb{R}^{n\times n}\,, \tag{3.3.3}$$

with $b > 0$. The characteristic polynomial of T_n is $P_n(\lambda) = (\sqrt{b})^n U_n\left(\frac{\lambda - a}{2\sqrt{b}}\right)$ and it is follows immediately that the eigenvalues of T_n are

$$\lambda_\ell = a + 2\sqrt{b}\cos\left(\frac{\ell\pi}{n+1}\right)\,,$$

for $\ell = 1, 2\ldots, n$, since they are the zeros of P_n, with associated eigenvectors

$$\left(\sin\left(\frac{\ell\pi}{n+1}\right), \sqrt{b}\sin\left(\frac{2\ell\pi}{n+1}\right), \ldots, (\sqrt{b})^{n-1}\sin\left(\frac{n\ell\pi}{n+1}\right)\right)^t,$$

from (2.2.2).

Notice that if $b < 0$, then $P_n(\lambda) = (i\sqrt{-b})^n U_n\left(\frac{\lambda - b}{2i\sqrt{-a}}\right)$.

4. EIGENVALUES OF TRIDIAGONAL 2-TOEPLITZ MATRIX

In 1966, Rózsa held a seminar at University of Hamburg on tridiagonal k-Toeplitz matrices, called at that time as "periodic continuants", motivated mainly by some problems of lattice dynamics, of ladder networks and of structural analysis. In this occasion, Elsner and Redheffer had the idea to use Chebyshev polynomials of the second kind to derive formulas to the characteristic polynomial and the eigenvectors of that matrices in some particular cases. They published some new results one year later in [3] and independently, but slightly later, Rózsa in [16] presented similar propositions using some tools from theory of matrices. Later, these results were generalized to periodic block-tridiagonal matrices by Rózsa and Romani [17].

Recently the study of general tridiagonal k-Toeplitz matrices was considered again by the author and Petronilho in [6], after the characteristic polynomial of a tridiagonal 2-Toeplitz matrix had been considered by Gover [7], Marcellán and Petronilho [14] and Fonseca and Petronilho [5]. Based on this last paper, recently Álvarez-Nodarse, Petronilho and Quintero [1] studied the spectra of tridiagonal 2 and 3-Toeplitz matrices motivated by some Quantum Physics problems already mentioned.

Considering the tridiagonal 2-Toeplitz matrix

$$T_N^{(2)} = \begin{pmatrix} a_1 & 1 & & & \\ b_1 & a_2 & 1 & & \\ & b_2 & a_1 & 1 & \\ & & b_1 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}_{N\times N}, \tag{4.4.1}$$

let us define the polynomials

$$\pi_2(x) = (x-a_1)(x-a_2)$$

and

$$P_k^*(x) = \left(\sqrt{b_1b_2}\right)^k U_k\left(\frac{x-b_1-b_2}{2\sqrt{b_1b_2}}\right).$$

It is well known (cf. [5, 7, 14]) that the eigenvalues of the matrix (4.4.1) are the solutions of the polynomial Q_N defined by

$$Q_{2k+1}(x) = (x-a_1)P_k^*(\pi_2(x))$$

and

$$Q_{2k}(x) = P_k^*(\pi_2(x)) + b_2P_{k-1}^*(\pi_2(x)).$$

In the case of $N = 2n+1$, eigenvalues of $T_N^{(2)}$ are a_1 and

$$\frac{a_1+a_2}{2} \pm \sqrt{\frac{(a_1-a_2)^2}{4} + b_1^2 + b_2^2 + 2\sqrt{b_1b_2}\cos\left(\frac{k\pi}{n+1}\right)},$$

for $k = 1, \dots, n$. If $N = 2n$, the eigenvalues are

$$\frac{a_1 + a_2}{2} \pm \sqrt{\frac{(a_1 - a_2)^2}{4} + b_1^2 + b_2^2 + 2\sqrt{b_1 b_2} \cos\theta_{nk}},$$

for $k = 1, \dots, n$, where each θ_{nk} is a nonzero solution of the trigonometric equation

$$\sqrt{b_1} \sin((n+1)\theta) + \sqrt{b_2} \sin(n\theta) = 0 \ , \quad \text{with } 0 < \theta < \pi.$$

Both spectral formulas generalize the particular cases considered in [10, 18].

5. EIGENVALUES OF PERTURBED TRIDIAGONAL 2-TOEPLITZ MATRIX

Next we will consider the case when to the $(1,1)$ and (N,N) entries of the tridiagonal 2-Toeplitz matrix (4.4.1) we add real numbers α and β, respectively.

First suppose that $N = 2n+1$. Using the linear property of the determinant on columns (or rows), the characteristic polynomial of the tridiagonal matrix

$$\begin{pmatrix} \alpha + a_1 & 1 & & & & & \\ b_1 & a_2 & 1 & & & & \\ & b_2 & a_1 & 1 & & & \\ & & b_1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & a_2 & 1 \\ & & & & & b_2 & \beta + a_1 \end{pmatrix}_{N \times N} \tag{5.5.1}$$

is

$$P_{2n+1}(x) = (x - a_1 - \alpha - \beta) P_n^*(\pi_2(x)) + (\alpha\beta(x - a_2) - \alpha b_1 - \beta b_2)\, P_{n-1}^*(\pi_2(x)).$$

If N is even, say $N = 2n$, then the characteristic polynomial of the tridiagonal matrix

$$\begin{pmatrix} \alpha + a_1 & 1 & & & & & \\ b_1 & a_2 & 1 & & & & \\ & b_2 & a_1 & 1 & & & \\ & & b_1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & a_1 & 1 \\ & & & & & b_1 & \beta + a_2 \end{pmatrix}_{N \times N}$$

is $P_{2n}(x) =$

$$P_n^*(\pi_2(x)) + (\alpha(a_2 - x) + \beta(a_1 - x) + \alpha\beta + b_2)\, P_{n-1}^*(\pi_2(x)) \alpha\beta b_1\, P_{n-2}^*(\pi_2(x))).$$

The expressions for the characteristic polynomials of these matrices are quite more general than the formulas one can find in the main results of [10, 18]. From them, the determination of the eigenvalues (and associated eigenvectors), for some particular α and β, as considered in the references just cited follows straightforward. For example, if

$$\alpha\beta = \pm\sqrt{b_1 b_2} \quad \text{and} \quad \alpha\left(1 \mp \sqrt{\frac{b_1}{b_2}}\right) + \beta\left(1 \mp \sqrt{\frac{b_2}{b_1}}\right) = a_2 - a_1 \ ,$$

then $a_1 + \alpha + \beta$ and, respectively,

$$\frac{a_1 + a_2}{2} \pm \sqrt{\frac{(a_1 - a_2)^2}{4} + b_1^2 + b_2^2 + 2\sqrt{b_1 b_2}\cos\left(\frac{2k\pi}{2n+1}\right)},$$

for $k = 1, \ldots, n$, or

$$\frac{a_1 + a_2}{2} \pm \sqrt{\frac{(a_1 - a_2)^2}{4} + b_1^2 + b_2^2 + 2\sqrt{b_1 b_2}\cos\left(\frac{k\pi}{2n+1}\right)},$$

for $k = 1, \ldots, n$, are the eigenvalues of the matrix $T_N^{(2)}$ with $N = 2n + 1$. According to the above formulas, the eigenvalues of

$$\begin{pmatrix} 3/2 & 1 & & & & & \\ 1 & -1 & 1 & & & & \\ & 4 & 1/2 & 1 & & & \\ & & 1 & -1 & 1 & & \\ & & & 4 & 1/2 & 1 & \\ & & & & 1 & -1 & 1 \\ & & & & & 4 & 5/2 \end{pmatrix}$$

are

$$\frac{7}{2} \quad \text{and} \quad -\frac{1}{4} \pm \sqrt{\frac{89}{16} + 4\cos\left(\frac{2k\pi}{7}\right)},$$

for $k = 1, 2, 3$. We leave the details to the reader.

6. EIGENVALUES OF TRIDIAGONAL k-TOEPLITZ MATRIX

For $k \geq 3$, we consider now the general k-Toeplitz matrix

$$T_N^{(k)} = \begin{pmatrix} a_1 & 1 & & & & & & & & \\ b_1 & \ddots & \ddots & & & & & & & \\ & \ddots & a_k & 1 & & & & & & \\ & & b_k & a_1 & 1 & & & & & \\ & & & b_1 & \ddots & \ddots & & & & \\ & & & & \ddots & a_k & 1 & & & \\ & & & & & b_k & a_1 & 1 & & \\ & & & & & & b_1 & \ddots & \ddots & \\ & & & & & & & \ddots & & \end{pmatrix}_{N\times N} . \tag{6.6.1}$$

For $j > i$, define the polynomial in x of degree $j - i + 1$

$$\Delta_{i,j}(x) := \begin{vmatrix} x-a_i & 1 & & & \\ b_i & x-a_{i+1} & 1 & & \\ & b_{i+1} & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & b_{j-1} & x-a_j \end{vmatrix},$$

and for $j \leq i$, set

$$\Delta_{i,j}(x) := \begin{cases} 0 & \text{if } \; j < i-1 \\ 1 & \text{if } \; j = i-1 \\ x-a_i & \text{if } \; j = i\,. \end{cases}$$

We consider also the polynomial of degree k

$$\varphi_k(x) := \frac{1}{2\mu}\left\{ D_k(x) + (-1)^k\left(b_k + \frac{\mu^2}{b_k}\right)\right\},$$

where $\mu^2 = b_1 \cdots b_k$, and

$$D_k(x) := \begin{vmatrix} x-a_1 & 1 & & & & 1 \\ b_1 & x-a_2 & 1 & & & \\ & b_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ b_k & & & & b_{k-1} & x-a_k \end{vmatrix}.$$

Theorem 6.1. [6] *The characteristic polynomial P_N of the tridiagonal k-Toeplitz matrix $T_N^{(k)}$, is*

$$\mu^{-\lfloor N/k \rfloor} P_N(x) = \Delta_{1,r}(x) U_{\lfloor N/k \rfloor}(\varphi_k(x)) + \frac{b_1 \ldots b_r b_k}{\mu}\, \Delta_{r+2,k-1}(x) U_{\lfloor (N-k)/k \rfloor}(\varphi_k(x)) \, .$$

for $0 \leq r \leq k-1$ and $N \equiv r \bmod k$.

Notice that this theorem is in fact a generalization the results presented by Elsner and Redheffer in [3] and by Rózsa in [16] for the cases when the residue mod k of N is equal to 0 or $k - 1$.

Notice that if $N \equiv (k-1) \bmod k$, then

$$P_N(x) = \mu^{\lfloor N/k \rfloor}\, \Delta_{1,k-1}(x)\, U_{\lfloor N/k \rfloor}(\varphi_k(x)) \,,$$

and the eigenvalues of $T_N^{(k)}$ are the $k-1$ zeros of $\Delta_{1,k-1}(x)$ and all the solutions of the $\lfloor N/k \rfloor$ algebraic equations of degree k

$$\varphi_k(x) = 2\mu \cos \frac{jk\pi}{N+1}\,, \quad j = 1, 2, \cdots, \lfloor N/k \rfloor\,,$$

generalizing the main results on eigenvalues in [1].

The procedure used in the section 5 can be used to determine the eigenvalues (and naturally the eigenvectors by (2.2.2) of any perturbed tridiagonal k-Toeplitz matrix, with a perturbation similar to the one occurred in the tridiagonal 2-Toeplitz matrix.

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Received: June 27, 2006